PROBLEM SET 2
due March 14, 2017

1. The maximal ideal $\langle x_1 + x_2 + 3, x_1 + 5x_2 + 7\rangle \subseteq \mathbb{C}[x_1^{\pm 1}, x_2^{\pm 1}]$ defines a point in the plane. Find a tropical basis for this ideal.

2. Let I be a homogeneous ideal in $\mathbb{Q}[x, y, z]$ generated by the set

$$\mathcal{G} = \{x + y + z, x^2y + xy^2, x^2z + xz^2, y^2z + yz^2\}.$$

   Show that $\mathcal{G}$ is a universal Gröbner basis, that is $\mathcal{G}$ is a Gröbner basis for I for all $w \in \mathbb{R}^n$. Also show that $\mathcal{G}$ is not a tropical basis.

3. Draw $trop(V(f))$ for
   - $f = t^3x^2 + xy + ty^2 + tx + y + 1$
   - $f = tx^2 + 4xy - 7y^2 + 8$.

4. Suppose that $I = \langle f_1, \ldots, f_r\rangle \subset \mathbb{C}\{\{t\}\}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$, where each $f_i \in \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ is homogeneous with respect to each vector v in a linear space $L \subset \mathbb{R}^n$.
   - Show that for each $a \in \mathbb{C}\{\{t\}\}$, $v = (v_1, \ldots, v_n) \in L$ and $p = (p_1, \ldots, p_n) \in V(I)$ the point $a^v p = (a^{v_1}p_1, \ldots, a^{v_n}p_n)$ also lies on $V(I)$.
   - Prove that if $w \in trop(V(I))$, then $trop(V(I))$ contains the affine space $w + L$.

5. (bonus) Let $K = \mathbb{Q}$ with the 2-adic valuation. Compute a tropical basis for the ideal $\langle 2x + y - 4, x + 2y + z - 1\rangle \subset \mathbb{Q}[x, y, z]$.