PROBLEM SET 3
due April 28, 2017

1. Compute the multiplicities of all rays in the one dimensional fan *trop* $(V(f))$, where

$$f(x,y) = x^3y^2 - x^2y^3 - 5x^2y^2 - 2x^2y - 4xy^2 - 33xy + 16y^2 + 72y + 81.$$

2. What is the largest multiplicity of any edge in the tropicalization of any plane curve of degree d?

3. Let V be the row space of the following matrix

$$A = \begin{bmatrix} 1 & 1 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 1 & 0 \\ 0 & -1 & 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 & -1 & -1 \end{bmatrix},$$

defined over $K = \mathbb{C}\{\{t\}\}$. Let M be the matroid of columns of A with ground set $[6] = \{1,\dots,6\}$.

(a) Let $I(V)$ be the linear ideal in $K[x_1,\dots x_6]$, defining $V \subset \mathbb{P}^5$. Compute $I(V)$, list the circuits of $I(V)$ and M, respectively.

(b) Draw the Hasse diagram of the lattice of flats of M and show that the flats of M are in correspondence with partitions of the set $\{2,3,4,5\}$.

(c) For each $i = 1,\dots,6$ let H_i be the hyperplane in $\mathbb{P}^3_K$ with normal vector $a_i =$ the i-th column of A. Let X be $\mathbb{P}^3_K \setminus \bigcup_{i=1}^6 H_i$. Show that the map

$$\phi : X \to \mathbb{P}^5_K, \quad \boldsymbol{x} = [x_1 : \cdots : x_4] \mapsto [a_1 \cdot \boldsymbol{x} : \cdots : a_6 \cdot \boldsymbol{x}] \in (K^*)^6/K^* \cong (K^*)^5.$$

is injective and identifies its image with V.

4. Given the lattice of flats of a matroid M (not necessarily realizable), describe a method to recover the circuits of M, the independent sets of M and the bases of M.

5. (bonus) Let I be a homogeneous ideal in $K[x_1,\dots,x_n]$, and let $\boldsymbol{w}$ be in the relative interior of a maximal cell σ of *trop* $(V(I))$. Let P be the toric ideal associated to the lattice $\{\boldsymbol{u} \in \mathbb{Z}^n : in_{\boldsymbol{u}}(in_{\boldsymbol{w}}(I)) = in_{\boldsymbol{w}}(I)\}$. [1] Show that the multiplicity of σ can be computed by the formula

$$mult(\sigma) = degree(in_{\boldsymbol{w}}(I))/degree(P).$$

[1] cf Sturmfels - "Gröbner basis and convex polytopes"