SOLUTIONS TO MATH 225 NOTES AND SOLUTIONS
LINEAR ALGEBRA AND MATRIX THEORY

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MIDTERM REVIEW I

Basics of Linear Transformation. Are these true or false? Let $T : V \to W$ where V, W are finite dimensional vector spaces over F.

(a) If T is linear, then T preserves sums and scalar products.
(b) If $T(x+y) = T(x) + T(y)$, then T is linear.
(c) T is one-to-one if and only if the only vector x such that $T(x) = 0$ is $x = 0$.
(d) If T is linear, then $T(0) = 0$.
(e) If T is linear, then $\dim\ker(T) + \mathrm{rk}(T) = \dim W$.
(f) If T is linear, then T carries linearly independent subsets of V onto linearly independent subsets of W.
(g) If $T, U : V \to W$ are both linear and agree on a basis for V, then $T = U$.
(h) Given $x_1, x_2 \in V$ and $y_1, y_2 \in W$, there exists a linear transformation $T : V \to W$ such that $T(x_1) = y_1$ and $T(x_2) = y_2$.

Examples of Linear Transformation.

(a) Consider the linear transformation $T : P_2(\mathbb{R}) \to P_3(\mathbb{R})$ defined by $T(f(x)) = xf(x) + f'(x)$.
  - Compute the nullity and rank of T.
  - Verify the dimension theorem.
  - Determine if T is one-to-one or onto.

(b) Consider the linear transformation $M_{n\times n}(F) \to F$ defined by $T(A) = \mathrm{tr}(A)$. Recall that
$$\mathrm{tr}(A) = \sum_{i=1}^{n} A_{ii}.$$
  - Compute the nullity and rank of T.
  - Verify the dimension theorem.
  - Determine if T is one-to-one or onto.

(c) Prove that there exists a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ such that $T(1,1) = (1,0,2)$ and $T(2,3) = (1,-1,4)$. What is $T(8,11)$?
(d) Let $T : P(\mathbb{R}) \to P(\mathbb{R})$ be defined by $T(f(x)) = f'(x)$. Recall that T is linear. Prove that T is onto, but not one-to-one.

Characteristic of Fields. Linear independence or dependence of a set depends on the characteristic of the field.

(a) Is the set $\{(1\,0\,1), (1\,1\,1), (0\,1\,0)\}$ a linearly independent set in $\mathbb{F}_2^3 = (\mathbb{Z}/2\mathbb{Z})^3$?