Review for Math 225

1. Consider the matrix:
$$A = \begin{bmatrix} 1 & 0 & 3 \\ 5 & 2 & -1 \\ 1 & 1 & 1 \end{bmatrix}.$$
   (a) What is $\det(A)$?
   (b) What are the rank and nullity of A?
   (c) Is L_A an isomorphism?
   (d) Find the eigenvalues and eigenvectors of A.
   (e) Let $\mathbb{B}$ be the set of eigenvectors of A. Is this a basis for $\mathbb{R}^3$? If so find the matrix of L_A in this basis.
   (f) Compute A^5.

2. (a) Find two different orthonormal bases for the image of A, where A is the matrix in Problem 1.
   (b) Find an orthonormal basis for the vector space $V = span\{(1,0,2),(1,-1,3)\} \subseteq \mathbb{R}^3$.

3. Consider the matrix $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.
   (a) How many different A^k are there for k a non-negative integer?
   (b) Find a matrix A such that there are only 6 different matrices that A^k can be, for k a non-negative integer.

4. Let A be a 2×2 matrix such that $A^2 = 0$.
   (a) What is the relation between the image and the kernel of A?
   (b) Is A invertible?
   (c) What is the rank of A?
   (d) What is an example of such matrix?

5. Let $T : \mathbb{R}^3 \to \mathbb{R}$, such that $T(a,b,c) = \int_0^1 (ax^2 + bx + c)dx$.
   (a) Show T is linear.
   (b) Find a basis for the kernel T.

6. Decide if the following statements are true or false. If true, prove it. If false, give a counterexample.
   (a) If two matrices are similar, then they share the same reduced row reduced echelon form.
   (b) If two matrices share the same reduced row echelon form, they are similar.

7. Let $A \in M_{n\times m}(\mathbb{R})$ with m linearly independent columns. Is $A^T A$ invertible? Is it an isomorphism? Does that depend on whether $n \geq m$ vs $m \geq n$?

8. Let A be an $n \times n$ matrix with real entries.
   (a) Show that $T(A) = A - A^T$ is linear.
   (b) What is the dimension of the kernel of T?

9. Let $V \subseteq \mathbb{R}^n$, $\dim V = m$. Let A be the matrix of the orthogonal projection onto V. What can you say about the eigenvalues of A and their multiplicities. (Hint: Look at an example).

10. Consider the $n \times n$ matrix $A = \begin{bmatrix} n-1 & -1 & \dots & -1 \\ -1 & n-1 & \dots & -1 \\ \dots & & & \\ -1 & -1 & \dots & n-1 \end{bmatrix}$.
    (a) Find the eigenvalues of A and their algebraic multiplicities.
    (b) Find a basis of eigenvectors for A.

11. Let T be a linear operator on a finitely dimensional vector space V. Assume that T is a diagonalizable. Let $\mathcal{B}$ and $\mathcal{B}'$ be two different basis for V. How are the eigenvalues of $[T]_{\mathcal{B}}$ and $[T]_{\mathcal{B}'}$ related?

12. Denote by $C(\mathbb{R}; \mathbb{R})$ the set of all continuous functions from $\mathbb{R}$ to $\mathbb{R}$.
    (a) Let P_n denote the set of all polynomials of degree n or less. Show that P_n is a subspace of $C(\mathbb{R}; \mathbb{R})$.
    (b) Let P denote the set of all polynomials (of all degrees). Show that P is a subspace of $C(\mathbb{R}; \mathbb{R})$. What is the dimension of P? How do you know?

13. Consider the set of vectors $S = \{v_1, \dots v_5\}$, where

$$\begin{aligned} v_1 &= (0, 1, 2, 3) \\ v_2 &= (1, 2, 3, 4) \\ v_3 &= (3, 2, 1, 0) \\ v_4 &= (0, 0, 0, 0) \\ v_5 &= (1, 1, 1, 1). \end{aligned}$$

    What is $\dim S$?

14. From the book:
    (a) p. 117 - 6 d)
    (b) p. 181 - 7 d)
    (c) p. 196 - 6
    (d) p. 221 - 4
    (e) all True/False (problem 1 after each section)