

PROBLEM SET 4  
due April 14, 2020

Solve at least three problems:

1. Find infinitely many distinct primary decompositions of  $(x^2, xy) \subseteq \mathbb{R}[x, y]$ .
2. Find the primary decomposition of  $(x^2 - (y+1)^3, (y^2 - 1)^2) \subseteq \mathbb{Q}[x, y]$ . (Warning: This requires patience and determination!)
3. Find the primary decomposition of  $(x^2 - yz, x(z - 1)) \subseteq \mathbb{Q}[x, y, z]$ .
4. Let  $I$  be an ideal of  $R$  and  $x \in R$  such that both  $I + (x)$  and  $(I : x)$  are finitely generated. Show that  $I$  is also finitely generated.
5. Show that if the nilradical of  $R$  is trivial then the Jacobson radical of  $R[x]$  is trivial. (Hint: find a condition for the invertibility of a polynomial)
6. Show that if the nilradical is a maximal ideal then  $(0)$  is a primary ideal.
7. Let  $I, J$  be two ideals of a Noetherian ring. Show that there exists an integer  $n$  such that  $I \cap J^n \subseteq IJ$ .