PROBLEM SET 2
due October 2, 2018

Notation and conventions:
We assume that varieties are irreducible.
We denote by $\mathbb{I}(S)$ the set $\{f \in k[x_1, \dots, x_n] : f(s) = 0, \ \forall s \in S\}$.

1. Use Buchberger's Algorithm to compute (by hand) a Gröbner basis for the ideal $I \in \mathbb{C}[x, y, z]$, where
$$I = \left\langle z - x^5, \ y - x^3 \right\rangle,$$
with respect to either the lexicographic order or graded reverse lexicographic order. Use a computer algebra system[1] to check which of these two orderings gives you a smaller Gröbner basis (that is, which basis has fewer elements).

2. Let $X \subset \mathbb{A}^n_{\mathbb{C}}$ be the variety of the ideal $J = \left\langle x^2 + y^2 - 1, \ x - 1 \right\rangle$. Determine the ideal $\mathbb{I}(X)$. Is it true that $\mathbb{I}(X) = J$?

3. Let $X \subset \mathbb{A}^n_k$ be a closed algebraic set (reducible variety) which consist of two (distinct) points. What's the coordinate ring of X?

4. Prove that $\mathbb{A}^n_k$ is compact in the Zariski topology, that is, prove that from any covering of $\mathbb{A}^n_k$ by Zariski open sets one can extract a finite subcovering.

5. A curve C in $\mathbb{A}^n_k$ is a one dimensional closed subvariety. Show that if $X \subset \mathbb{A}^n_k$ is a closed subvariety, containing infinitely many points of C, then $C \subseteq X$.

[1] I recommend using `Macaulay2`. You can learn about it here: http://www2.macaulay2.com/Macaulay2/ and you can use it online: http://habanero.math.cornell.edu:3690/