Addendum to Higher Grassmann Codes II

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Let $\mathbb{F}_q$ denote the finite field with q elements, where q is a power of a prime number. For positive integers l and m such that $l \leq m$, let V denote the vector space $\overline{\mathbb{F}}_q^m$, where $\overline{\mathbb{F}}_q$ stands for an algebraic closure of $\mathbb{F}_q$. Let $Gr(l, V)$ denote the Grassmann variety of l-dimensional subspaces of V. The r-th order (higher) Grassmann code, denoted by $C_{Gr(l,V)}(r)$, is an AG code that is obtained by evaluating the sections of the r-th Serre twist of the structure sheaf of $Gr(l, V)$ on the $\mathbb{F}_q$-rational points of $Gr(l, V)$. The theory of higher Grassmann codes is developed in the articles [1, 2]. They have important specializations. For $r = 1$, the higher Grassmann codes are precisely the Grassmann codes that are originally introduced by Ryan and Ryan [9] and studied by many authors including Nogin [7], Lachaud and Ghorpade [4], and Ghorpade and Kaipa [3]. In the special cases $l \in \{1, m-1\}$, the r-th order higher Grassmann code is precisely the r-th order Projective Reed-Müller code, introduced by Manin and Vladut [11, Example 2.b), pg. 2619] and investigated by Lachaud [5] and Sørensen [10]. In particular, Sørensen established a dimension formula for the r-th order Projective Reed-Müller code. Furthermore, the article [10] presented numerous insightful ideas that have proven remarkably valuable in our work. The purpose of this note is to reassess a hasty remark of ours, as documented in [2], pertaining to the formula of Sørensen.

Let $m \in \mathbb{Z}_+$. For $\nu \in \mathbb{Z}_+$, we denote by $\mathbb{F}_q[x_0, \dots, x_m]_h^\nu$ the set of all homogeneous polynomials of degree ν from $\mathbb{F}_q[x_0, \dots, x_m]$. By adding 0 to $\mathbb{F}_q[x_0, \dots, x_m]_h^\nu$, we obtain a vector space of dimension $\binom{m+\nu}{\nu}$. The *projective Reed-Müller code of degree ν over $\mathbb{F}_q$*, denoted $PC_\nu(m, q)$, is defined as follows:

$$PC_\nu(m,q) := \{(F(P_1), \dots, F(P_n)) \in \mathbb{F}_q^n : \ F(x_0, \dots, x_n) \in \mathbb{F}_q[x_0, \dots, x_m]_h^\nu\}. \tag{0.1}$$

In [2, Page 9], we wrote

> “For $\nu \in [q-2]$, the parameter of the projective Reed-Müller codes of degree ν were first determined by Lachaud in [5]. For $\nu \geq q-1$, the parameters are

determined by Sorensen [10] except for the dimension formula which we proceed to explain.

In [10, Theorem 1], Sørensen stated the following formula for the dimension of $PC_r(m,q)$:

$$\dim PC_\nu(m,q) = \sum_{\substack{0 < t \leq \nu \text{ s.t.} \\ t \equiv \nu \mod q-1}} \left(\sum_{j=0}^{m+1} (-1)^j \binom{m+1}{j} \binom{t-jq+m}{t-jq} \right). \tag{0.2}$$

We claim that this dimension formula is not correct.."

In the subsequent paragraphs of the relevant section of our paper, we embarked on elucidating the reasons behind our skepticism towards the validity of formula (0.2). Ironically, we were oblivious to the fact that we were unwittingly falling into the snare of capricious binomial coefficients. Our arguments refuting the formula, along with our proposal of an alternative correct formula, were logically grounded. However, to our surprise, it came to light that Sørensen's formula was indeed accurate. This apparent paradox perplexed us until Sudhir Ghorpade enlightened us through a detailed explanation provided in a personal communication. The crux of the matter revolved around the imperative need to establish a clear and unambiguous definition for the binomial coefficient function.

Our purported refutation proceeded as follows.

.. We will use a well-known identity that is obtained by the "finite differences" formalism.

Lemma 0.3. *Let K be a field. Let $P(x) \in K[x]$ be a polynomial of the form $P(x) := a_0 + a_1x + \cdots + a_ex^e$. Then we have*

$$\sum_{j=0}^{e} (-1)^j \binom{e}{j} P(j) = (-1)^e e! a_e.$$

Proof. Our assertion is stated (in a different notation) in [**?**, pg 190]. □

Let $P(x) \in \mathbb{Q}[x]$ be the polynomial defined by

$$P(x) := \frac{(t-xq)(t-xq-1)\cdots(t-xq-(m-1))}{m!}.$$

For every integer $j \in \mathbb{N}$, we have

$$P(j) = \binom{t-jq+m}{m} = \binom{t-jq+m}{t-jq}.$$

We now apply Lemma 0.3 to $P(x)$. Since $\deg P(x) = m$, we find

$$\sum_{j=0}^{m+1} (-1)^j \binom{m+1}{j} P(j) = 0. \tag{0.4}$$

The identity in (0.4) shows that the r.h.s. of Søresen's formula (0.2) is always 0.

The issue with our previous lines, which are logically correct in every step, is that our polynomial $P(x)$ does not agree everywhere with the combinatorial binomial coefficients. To be more precise, let us recall the definition of the well-known *binomial coefficients*. For $(x, m) \in \mathbb{Z} \times \mathbb{Z}$, we have

$$\binom{x}{m} := \begin{cases} \frac{x(x-1)\cdots(x-m+1)}{m!} & \text{if } m > 0, \\ 1 & \text{if } m = 0, \\ 0 & \text{if } m < 0. \end{cases} \tag{0.5}$$

This definition gives us a well-defined function $\binom{\cdot}{\cdot} : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$, which restricts to give a polynomial function of x whenever m is fixed. The crucial point here is the part of the definition for $m < 0$. Indeed, by requiring that $\binom{x}{m} = 0$ for $m < 0$, we are essentially removing the polynomial nature of the function $x \mapsto \binom{x}{m}$, $x \in \mathbb{Z}$. In other words, if we do not impose the requirement that $\binom{x}{m} = 0$ for $m < 0$, then indeed we are working with a polynomial function leading to (0.4). Unfortunately, both of the papers [10] and [2] did not define the binomial coefficients as in (0.5). This explains the fact that there was no mistake in [10] regarding (0.2) but both of the papers [10] and [2] had a shortcoming on properly defining the binomial coefficients. Fortunately, the second reference contains numerous additional findings that do not duplicate the content of the first reference. This episode has served as a valuable lesson, reminding us of the importance of exercising utmost care, even when dealing with seemingly simple and well-known mathematical tools. In hindsight, we deeply regret any distress or confusion that may have been caused by our initial assessment of (0.2). We sincerely apologize to Sørensen for mistakenly doubting the accuracy of his formula, as it has become evident that his formulation is precise.

Finally, we want to mention that, in [2, Proposition 3.9], we gave another dimension formula,

$$\dim PC_\nu(m, q) = \sum_{\substack{t \in [\nu] \\ t \equiv \nu \mod q-1}} \sum_{e=1}^{\min\{t, m+1\}} \binom{m+1}{e} \left(\sum_{j=1}^{\lfloor \frac{t-e}{q-1} \rfloor} (-1)^j \binom{e}{j} \binom{t-1-(q-1)j}{e-1} \right).$$

This particular reformulation turned out to be useful for finding a formula for $\dim C_{Gr(l,V)}(\nu)$. The details are given in [2, Theorem 5.3].

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