NOTES ON DEMAZURE AND WEYL CHARACTER FORMULAS

1. Introduction

Let $G = SL_n(k)$ and fix a maximal torus $T \subset G$. Let $B$ be a Borel subgroup containing $T$, and let $S$ be the set of simple roots corresponding to $(B, T)$.

Let $s_i \in S$ be a simple root, and let $P_i$ be the maximal parabolic subgroup corresponding to $S - \{s_i\}$. The homogenous space $G/P_i$ has the Plücker embedding into a suitable projective space $\mathbb{P}(V)$. Let $L_i$ be the pull back of the hyperplane bundle from $\mathbb{P}(V)$. By abuse of notation we denote by $L_i$ the line bundle on $G/P_i$ as well as its pull back to $G/B$.

Let $a = (a_1, \ldots, a_{n-1})$ be a sequence of integers, and let $L^a$ be the line bundle

$$L^a := L_1^{a_1} \otimes \cdots \otimes L_{n-1}^{a_{n-1}}$$

on $G/B$.

Since the global sections of $L^a$ is a $B$-module, its restriction to a Schubert variety $X(w) \subseteq G/B$ is a $B$-module, also. The purpose of this note is to derive a formula for the $T$-character on the space of global sections of $L^a$ (assuming that $a_i \geq 0$ for all $i = 1, \ldots, n - 1$) on $X(w) \subseteq G/B$.

2. Borel-Weil-Bott

It is customary to denote the group of all characters of $T$, namely, the regular homomorphisms (or algebraic group morphisms) $T \to \mathbb{C}^*$ by $X^*(T)$, and similarly, by $X_*(T)$ the group of algebraic group homomorphism $\mathbb{C}^* \to T$. There is a nondegenerate bilinear pairing

$$\langle \cdot, \cdot \rangle : X^*(T) \times X_*(T) \to \mathbb{Z}$$

Let $T \subseteq G$ be a maximal torus in a reductive group $G$. Let $V$ be a $T$ module, and let $V = \bigoplus \chi V_\chi$ be the weight space decomposition. In other words, $V_\chi = \{v \in V : t \cdot v = \chi(t)v \text{ for all } t \in T\}$. The characters $\chi : T \to \mathbb{C}^*$ (which has to be a regular homomorphism) are called the weights of $V$. When $V = g$ is the Lie algebra of $G$, $T$ acts by the adjoint representation

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on $\mathfrak{g}$. The set of weights of $\mathfrak{g}$ is denoted by $\Phi = \Phi(G, T) \subset X^*(T)$. Let $E$ be the $\mathbb{R}$-span of $\Phi$.

**Lemma 2.1.** The pair $(E, \Phi)$ is a root system such that for each $\alpha, \beta \in \Phi$ the ratio

$$\langle \beta, \alpha \rangle = \frac{2(\beta, \alpha)}{(\alpha, \alpha)}$$

is an integer, where $(,)$ is a $W$-invariant real inner product on $E$.

Fix a Borel subgroup containing $T$. A slick way of defining the positive roots in $\Phi$ is to choose those weights $\chi \in \Phi$ for which the root subgroup $U_\chi$ is contained in $B$. For the definition of the root subgroups see Humphreys or Springer or Borel or Carrell. Once we know the positive roots, we can define the simple roots as those positive roots which cannot be written as a sum of two positive roots.

Fundamental dominant weights $\{\omega_1, \ldots, \omega_\ell\}$ are defined so that $\langle \omega_i, \alpha_j \rangle = \delta_{i,j}$. The fundamental dominant weights form a basis for the weight space $E$. Dominant weights are those weights $\alpha = \sum_i a_i \omega_i$ such that $a_i \geq 0$ for all $i = 1, \ldots, \ell$. It is called regular if $a_i > 0$ for all $i = 1, \ldots, \ell$. The set of all dominant weights is denoted by $X^*(T)^+$. There is a useful partial ordering on the set of weights: let $\lambda, \mu \in X^*(T)$, then

$$\lambda \leq \mu \iff \mu - \lambda = \sum_i a_i \alpha_i, \text{ with } a_i \geq 0.$$ 

It is well known that the Picard group $\text{Pic}(G/B)$ is isomorphic to the free abelian group $X^*(T)$. The isomorphism is given by $\lambda \sim L(\lambda)$, where $L(\lambda)$ is the line bundle whose total space is the quotient of $G \times k$ by the equivalence relation

$$(g, x) \sim (g \cdot b, -\lambda(b)x),$$

where $g \in G$, $b \in B$ and $x \in k$. Here, $\lambda$ can be viewed as a character of the Borel subgroup containing $T$, because $B = U T$ and $U$ is affine. There is a minus sign to indicate that we are using the additive inverse of the root $\lambda \in X^*(T)$.

**2.0.1. Consequences of the Borel-Weil theorem.** Observe that by construction $V^* = H^0(G/B, L(\lambda))$ is a rational $G$-module. When $\text{char}(k) = 0$, the Borel-Weil theorem says that the dual module $V = (H^0(G/B, L(\lambda)))^*$ is an irreducible $G$-module, whenever $\lambda$ is dominant. Furthermore, if $\lambda$ is dominant, then there exists unique line $\ell_\lambda \subseteq V$ such that the stabilizer of $\ell_\lambda$ contains $B$, and

1. $T$ acts on $\ell_\lambda$ with weight $\lambda$,
2. $G \cdot \ell_\lambda$ has the minimal dimension for the action of $G$ on the projective space $\mathbb{P}(V)$,
(3) if \( \lambda \) is regular, then the stabilizer of \( \ell_\lambda \) is precisely \( B \),
(4) every weight in \( V \) is of the form \( \lambda - \sum a_i \alpha_i \), where \( a_i \geq 0 \) are
nonnegative integers. In other words every weight of \( V \) is \( \leq \lambda \).

3. Demazure’s Character Formula

3.0. Bott-Samelson varieties. Let \( w \in W \) and let \( w = s_{\alpha_{i_1}} \cdots s_{\alpha_{i_r}} \) be
a reduced word of \( w \). Let \( m_w = (i_1, \ldots, i_r) \) be the associated sequence
of positive integers. Since \( BwB = (Bs_{\alpha_{i_1}} B) \cdot (Bs_{\alpha_{i_2}} B) \cdots (Bs_{\alpha_{i_r}} B) \), it
follows that

\[
X(w) = P_{i_1} \cdots P_{i_r}/B,
\]

where \( P_{i_j} = Bs_{\alpha_{i_j}} B \). The product map \( P_{m_w} := P_{i_1} \times \cdots \times P_{i_r} \rightarrow X(w) \)
is invariant under the action of \( B^n \) on the right:

\[
(p_1, \ldots, p_r) \cdot (b_1, \ldots, b_r) = (p_1b_1, b_1p_2b_2^{-1}, \ldots, b_{r-1}^{-1}p_rb_r).
\]

The Bott-Samelson variety is defined as the orbit space

\[
Z_{m_w} = P_{m_w}/B^n.
\]

It turns out that \( Z_{m_w} \) is a smooth projective variety and the induced map
\( Z_{m_w} \rightarrow X(w) \) is birational.

Recall that both \( X^*(T) \) and \( X_*(T) \) are free abelian groups of rank \( \ell \),
where \( \ell \) is the number of simple roots, or the dimension of the torus. From
now on we are going to use the multiplicative notation for the group \( X^*(T) \): \( \lambda \in X^*(T) \)
is denoted by \( e^\lambda \) so that

\[
e^\lambda \cdot e^\mu = e^{\lambda+\mu}.
\]

Let \( A = \mathbb{Z}[X^*(T)] \) be the integral group ring of \( X^*(T) \). The ring \( A = \mathbb{Z}[X^*(T)] \)
has a canonical involution: \( e^{\lambda} = e^{-\lambda} \). For any simple reflection
\( s_i \in S \), define the \( \mathbb{Z} \)-linear (Demazure) operator

\[
D_{s_i}(e^\lambda) = \frac{e^\lambda - e^{s_i\lambda - \alpha_i}}{1 - e^{-\alpha_i}}.
\]

Lemma 3.1.

\[
D_{s_i}(e^\lambda) = \begin{cases} 
e^\lambda + e^{\lambda - \alpha_i} + \cdots + e^{s_i\lambda} & \text{if } \langle \lambda, \alpha_i^+ \rangle \geq 0 \\
0 & \text{if } \langle \lambda, \alpha_i^+ \rangle = -1 \\
-(e^{\lambda + \alpha_i} + \cdots + e^{s_i\lambda - \alpha_i}) & \text{if } \langle \lambda, \alpha_i^+ \rangle < -1 \end{cases}
\]

Let \( w \in W \) and let \( w = s_{\alpha_{i_1}} \cdots s_{\alpha_{i_r}} \) be a reduced word of \( w \). Let \( m_w =
(i_1, \ldots, i_r) \) be the associated sequence of positive integers. We define

\[
D_{m_w} := D_{s_{\alpha_{i_1}}} \cdots D_{s_{\alpha_{i_r}}}.
\]

We denote by \( \theta_{m_w} : Z_{m_w} \rightarrow X(w) \) the Bott-Samelson resolution. For
any finite dimensional \( B \)-module \( M \), let \( L_{m_w}(M) \) denotes the pull-back
It turns out that $M \rightsquigarrow L_{m_w}(M)$ is an exact functor.

**Theorem 3.2.** If $\lambda \in X^*(T)^+$ is a dominant weight, $w \in W$ with $m_w = (i_1, \ldots, i_r)$, then the character of the $T$–module $H^0(X(w), \mathcal{L}^\lambda)$ is given by

$$\text{ch} H^0(X(w), \mathcal{L}^\lambda) = \overline{D_{m_w}(e^\lambda)}.$$

**Sketch of the proof.** For a finite dimensional $B$–module $M$ we denote by $\chi(Z_{m_w}, \mathcal{L}_{m_w}(M))$ the Euler characteristic

$$\chi(Z_{m_w}, \mathcal{L}_{m_w}(M)) := \sum_i (-1)^p \text{ch} H^i(Z_{m_w}, \mathcal{L}_{m_w}(M)).$$

For any exact sequence

$$0 \to M_1 \to M \to M_2 \to 0$$

of finite dimensional $B$–modules, from the corresponding long exact cohomology sequence we have

$$\chi(Z_{m_w}, \mathcal{L}_{m_w}(M)) = \chi(Z_{m_w}, \mathcal{L}_{m_w}(M_1)) + \chi(Z_{m_w}, \mathcal{L}_{m_w}(M_2)).$$

This is true because $\mathcal{L}_{m_w}$ is an exact functor. We are going to prove, by induction on the length $r$ of $m_w$ that

$$\chi(Z_{m_w}, \mathcal{L}_{m_w}(M)) = \overline{D_{m_w}(\text{ch} M)}.$$  \hfill (3.3)

in the group ring $A$.

Requires some easy work: The case of $r = 1$ with $\dim M = 1$ is somewhat easy to see. The case that $n = 1$ and $\dim M > 1$ follows from the Borel fixed point theorem. Now assume that (3.3) is true for any $w$ with $\ell(w)$ (the length of $m_w$ equals $n − 1$.

The Bott–Samelson resolutions have the following (inductive) property: there are $P_{i_r}/B$–fibrations of the form

$$\phi : Z_{m_w} \longrightarrow Z_{m_w'},$$

where $w'$ is the permutation obtained from $w$ by deleting the simple reflection $s_{i_r}$ from its reduced expression.

Using Leray spectral sequence associated with this fibration one can compute:

$$\sum_{p, q} (-1)^{p+q} \text{ch} H^p(Z_{m_w}, \mathcal{L}_{m_w'}(H^q(P_{i_r}/B, \mathcal{L}_{i_r}(M)))) = \chi(Z_{m_w}, \mathcal{L}_{m_w}(M)).$$

\[1\] Here $\mathcal{L}(M)$ denotes the $G$–equivariant vector bundle

$$G \times_B M \longrightarrow G/B$$

associated to the locally trivial principal $B$–bundle $\pi : G \to G/B$. (Its construction is similar to the construction of the line bundles $\mathcal{L}(\lambda)$ on $G/B$.)
By the induction hypotheses and the case \( r = 1 \), the left hand side of the above equation simplifies to
\[
\sum_q (-1)^q \chi(Z_{m_w}, L_{m_w}(H^q(P_i/B, \mathcal{L}_i(M)))) = D_{m_w}(D_{i*}(\text{ch } M)) = D_{m_w}(\text{ch } M).
\]
In other words,
\[
(3.4) \quad \chi(Z_{m_w}, L_{m_w}(M)) = D_{m_w}(\text{ch } M).
\]

Requires some easy work: By the general properties of the Bott-Samelson resolutions we know that
\[
(3.5) \quad \theta^*_{m_w} : H^i(X(w), S) \to H^i(Z_{m_w}, \theta^*_{m_w}(S))
\]
is an isomorphism for any locally free sheaf \( S \) on \( X(w) \) and for any \( i \geq 0 \). (This can be proven by induction.) Thus, we see that
\[
(3.6) \quad \chi(X(w), L(M)) = D_{m_w}(\text{ch } M).
\]

Now, let \( \lambda \in X^*(T)^+ \). By the Frobenius splitting methods, we know that
\[
H^i(X(w), L(\lambda)) = 0
\]
for all \( i > 0 \).

4. Weyl’s Character Formula

The simple reflection \( s_\alpha \) associated with a root \( \alpha \in S \) acts on \( X^*(T) \) (hence, on \( E = X^*(T) \otimes \mathbb{R} \)) by
\[
s_\alpha(v) = v - \frac{2(v, \alpha)}{(\alpha, \alpha)} \alpha.
\]
Obviously, the action descends onto \( A = \mathbb{Z}[X^*(T)] \).

**Theorem 4.1.** The invariants \( A^W \) has an integral basis given by the “symmetric functions” \( \{ A_{\lambda+\rho}/A_\lambda : \lambda \in X^*(T)^+ \} \), where
\[
A_\alpha := \sum_{w \in W} (-1)^{\ell(w)} e^{w(\alpha)}.
\]
Here \( \rho \) is the half of the sum of the positive roots. Furthermore,
\[
S_\lambda := A_{\lambda+\rho}/A_\lambda
\]
is the irreducible character of the highest weight \( \lambda \).

With some elementary computation, one can deduce the following formula from the Weyl’s character formula:
Theorem 4.2. The value $S_{\lambda}(1)$, dimension of the irreducible representation with highest weight $\lambda$, is

$$\prod_{\alpha \in \Phi^+} \frac{(\lambda + \rho, \alpha)}{(\rho, \alpha)} = \prod_{\alpha \in \Phi^+} 1 + \frac{(\lambda, \alpha)}{(\rho, \alpha)}.$$ 

REFERENCES


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