Ansatz for $(-1)^{n-1}\nabla p_n$

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June 14, 2018

Abstract

We construct a special family of equivariant coherent sheaves on the Hilbert scheme on \( n \)-points in the affine plane. The equivariant Euler characteristic of these sheaves are closely related to the symmetric functions \((-1)^{n-1}\nabla p_n\). We prove a higher cohomology vanishing result of these sheaves. It follows from the Bridgeland-King-Reid correspondence that there is an effective \( S_n \) module underlying the aforementioned family of symmetric functions.

Keywords: Hilbert scheme of points, Atiyah-Bott-Lefschetz formula, nabla operator, symmetric functions, Macdonald polynomials

MSC: 14L30, 14F99, 14C05, 05E05

1 Introduction

Let \( S_n \) denote the symmetric group of permutations of the set \( \{1, \ldots, n\} \) and let \( \{x_1, \ldots, x_n\} \cup \{y_1, \ldots, y_n\} \) be a set of \( 2n \) algebraically independent variables. We consider the diagonal action of \( S_n \) on \( \mathbb{C}[x,y] := \mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n] \) given by

\[
\sigma \cdot x_i = x_{\sigma(i)} \quad \text{and} \quad \sigma \cdot y_i = y_{\sigma(i)} \quad \text{for} \quad i = 1, \ldots, n \quad \text{and} \quad \sigma \in S_n.
\]

Let \( I_+ \) denote the ideal generated by the \( S_n \)-invariant homogeneous polynomials of positive degree in \( \mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n] \) and set

\[
R_n := \mathbb{C}[x,y]/I_+.
\]

It is known for sometime that the bi-graded Frobenius character of \( R_n \) is given by the transformation of the elementary symmetric function \( e_n \) under Bergeron-Garsia’s "nabla" operator, \( \nabla e_n \). In other words, the symmetric function \( \nabla e_n \) has an underlying \( S_n \)-representation.
Roughly stated, $\nabla$ is a $\mathbb{Q}$-linear operator defined on the ring of symmetric functions $\Lambda$ in such a way that the modified Macdonald symmetric functions are the eigenfunctions of $\nabla$ with prescribed eigenvalues (see §2 for more details).

A natural question that arises in this context is that: if $f$ is any well known other well-known family of symmetric functions, say $f = h_\lambda, p_\lambda, m_\lambda, \ldots$, etc., does there exist an underlying representation for the family $\nabla f$ or its “suitably modified” transformations. In general such questions are difficult to answer. A prerequisite for this to happen is that the family under consideration is Schur-positive, that is to say that it admits expansion in the Schur basis with nonnegative integral coefficients.

It is known from the work of Loehr and Warrington, see [LW07], that the symmetric function family $(-1)^{n-1}\nabla p_n$ is Schur positive. The purpose of our note is to show that there is an $\mathfrak{S}_n$-module underlying this family. This module will be constructed by using geometric techniques that have been pioneered by M.Haiman to prove the $n!$ conjecture, see [Hai03] for an overview.

The fundamental geometric object in this story is the Hilbert scheme of $n$ points in the affine plane, hence denoted by $H_n$. The scheme $H_n$ is a smooth $2n$-dimensional (fine) moduli space parametrizing ideals of colength $n$ in the coordinate ring $\mathbb{C}[x,y]$ of the affine plane. The closed points of $H_n$ are given by

$$H_n(\mathbb{C}) := \{I : I \subset \mathbb{C}[x,y] \text{ is an ideal such that } \dim_\mathbb{C} \mathbb{C}[x,y]/I = n\}.$$  \hspace{1cm} (1)

The Hilbert-Chow map

$$\sigma : H_n \longrightarrow \mathbb{C}^{2n}/\mathfrak{S}_n$$  \hspace{1cm} (2)

describes $H_n$ as a relative projective scheme over $\mathbb{C}^{2n}/\mathfrak{S}_n$; moreover, this turns out to be a crepant resolution of the isolated singularity. Consequently, the Bridgeland-King-Reid theorem provides a categorical equivalence

$$\Phi : \mathcal{D}^b(H_n) \longrightarrow \mathcal{D}^b_{\mathfrak{S}_n}(\mathbb{C}^{2n}),$$ \hspace{1cm} (3)

between the bounded derived category of $\mathfrak{S}_n$-equivariant sheaves on $\mathbb{C}^{2n}$ and the bounded derived category of coherent sheaves on $H_n$.

The algebraic torus $T := \mathbb{C}^* \times \mathbb{C}^*$, with coordinates $(t,q)$ acts on the plane $\mathbb{C}^2$ and this induces an action on $H_n$. This $T$-action along with the equivalence $\Phi$ above induces an isomorphism, also denoted by $\Phi$, between the equivariant Grothendieck groups,

$$\Phi : K^0_T(H_n) \cong K^0_{\mathfrak{S}_n \times T}(\mathbb{C}^{2n}).$$ \hspace{1cm} (4)

The above isomorphism (4) translates questions about $\mathfrak{S}_n$-modules into geometric questions about equivariant sheaves on $H_n$. Furthermore, under this isomorphism, the $T$-equivariant Euler-characteristic of a sheaf $\mathcal{F}$, computed with the canonical polarization coming from (2), equals the Frobenius characteristic of the $\mathfrak{S}_n$-representation $\Phi(\mathcal{F})$. Indeed, $K^0_{\mathfrak{S}_n \times T}(\mathbb{C}^{2n})$ is freely generated, as a $\mathbb{Z}[q,t,q^{-1},t^{-1}]$-module, by the free $\mathbb{C}[x,y]$-modules
$V^\lambda \otimes \mathbb{C}[x, y]$, where $\lambda$ runs over all partitions of $n$ and $V^\lambda$ is the irreducible $S_n$-module corresponding to $\lambda$. By applying the Frobenius characteristic map, it is not difficult to see that $K^0_T(H_n)$ is isomorphic to the algebra of symmetric functions $f$ with the property that $f[(1 - q)(1 - t)Z]$ has coefficients in $\mathbb{Z}[q, t, q^{-1}, t^{-1}]$. Here, $(1 - q)(1 - t)Z$ is the symmetric function $(1 - q)(1 - t)(z_1 + z_2 + \cdots)$ and the brackets in $f[(1 - q)(1 - t)Z]$ indicate that we are applying the “plethystic substitution.”

The isospectral Hilbert scheme, denoted by $X_n$, is the reduced fiber product

$$X_n := H_n \times_{\mathbb{C}^{2n}/S_n} \mathbb{C}^{2n},$$

which is obtained from the maps in Figure 1, where $\pi$ is the quotient map and $\sigma$ is the Hilbert-Chow morphism. (The map $\rho$ is the canonical projection map, which is defined after $X_n$.)

![Figure 1: The isospectral Hilbert scheme](image-url)

A deep result of Haiman in [Hai01] shows that the coherent sheaf $\mathcal{P} := \rho_*(\mathcal{O}_{X_n})$ is a locally free sheaf of rank $n!$ on $H_n$. The resulting vector-bundle, called the Procesi bundle, turns out to be of fundamental importance. In particular, the class of $[\mathcal{P}|_{Z_n}]$ in $K^0_T(H_n)$, where $Z_n$ is the reduced fiber $\sigma^{-1}(\pi(0))$ in $H_n$, corresponds to the representation underlying $\nabla e_n$ under the isomorphism $\Phi$, see [Hai03] for details.

Now we are ready to state the main results of our paper. We construct two $T$-equivariant coherent sheaves $\mathcal{N}$ and $\mathcal{N}'$ on $H_n$. The first sheaf is an object of $D^b_T(H_n, \mathbb{Z})$ and the second sheaf lives in $D^b_T(H_n, \mathbb{Z}[1/n])$. Roughly speaking, these sheaves are constructed by twisting some standard sheaves on $H_n$, considered by Haiman, with suitable étale local-systems coming from the torus $T$. Using a version of the Atiyah-Bott-Lefschetz theorem we calculate the $T$-equivariant Euler-characteristics of the sheaves $\mathcal{O}_{Z_n} \otimes \mathcal{P} \otimes \mathcal{N}$ and $\mathcal{O}_{Z_n} \otimes \mathcal{P} \otimes \mathcal{N}'$. For the first sheaf, our calculation yields $(-1)^{n-1}n^3 \nabla p_n$, and for the second sheaf it gives $(-1)^{n-1}\nabla p_n$. By proving the vanishing of the higher cohomology, we show that these $T$-equivariant Euler characteristics are indeed the bi-graded Frobenius series of the $S_n$-modules associated with the spaces of global sections on $H_n$ of $\mathcal{O}_{Z_n} \otimes \mathcal{P} \otimes \mathcal{N}$ and $\mathcal{O}_{Z_n} \otimes \mathcal{P} \otimes \mathcal{N}'$, respectively.

Let us mention in passing that the constructions of both of the vector bundles $\mathcal{N}$ and $\mathcal{N}'$ follow from similar ideas and the proofs carry over verbatim. For brevity, we will omit repetition and only emphasize the differences between $\mathcal{N}$ and $\mathcal{N}'$.

The structure of our paper is as follows. In §2 we set our notation and briefly review the known formula for $\nabla p_n$; we state our main result as Theorem 2.1. Also in this section
we recall some results from Haiman’s work. The proof of Theorem 2.1 is presented in the subsequent §3.

2 Notation and Preliminaries

A partition $\mu$ of $n$ is a nonincreasing sequence of nonnegative integers $\mu = (\mu_1, \mu_2, \ldots)$, with $\sum_{i \geq 1} \mu_i = n$. We will use the French convention to represent the diagram of partitions as in Figure 2. Therefore, there are $\mu_1$ squares in the bottom row, there are $\mu_2$ squares in the second row, and so on. The squares in the diagram of $\mu$ will be called the cells of the partition.

Let $\mu$ be a partition of $n$ with parts $\mu = (\mu_1, \mu_2, \ldots)$. In this case, we write $\mu \vdash n$. The partition that is conjugate to $\mu$ is denoted by $\mu'$. We will use $n(\mu)$ to denote the following sum:

$$n(\mu) = \sum_i (i - 1)\mu_i.$$ 

The dominance order on partitions, denoted by $\leq$, is defined by

$$(\mu_1, \mu_2, \ldots) \leq (\lambda_1, \lambda_2, \ldots) \iff \mu_1 + \cdots + \mu_i \leq \lambda_1 + \cdots + \lambda_i \text{ for all } i \geq 1.$$ 

For a cell $x \in \mu$, we have the following statistics:

1. $a(x) = \text{arm}$: the number of cells that are directly above $x$;
2. $a'(x) = \text{coarm}$: the number cells that are directly below $x$;
3. $l(x) = \text{leg}$: the number of cells to the right of $x$;
4. $l'(x) = \text{coleg}$: the number of cells to the left of $x$.

For example, the arm, coarm, leg, and the coleg of $x$ in Figure 2 are 4, 3, 3, and 2, respectively. In this notation, we identify the cells $x$ in the partition $\mu$ with their “cartesian coordinates” $(a'(x), l'(x))$. We denote by $\Box_n$ the partition with $n$ equal parts all of which equals $n$,

$$\Box_n := \{(r, s) \in \mathbb{Z} \times \mathbb{Z} | 0 \leq r, s \leq n - 1\}.$$
In addition, we use the following standard notations.

1. \([t]_n = 1 + t + \ldots + t^{n-1}\) and \([q]_n = 1 + q + \ldots + q^{n-1}\);

2. \(B_\mu = \sum_{x \in \mu} q^{a'(x)} t^{l'(x)}\);

3. \(T_\mu = q^{n(\mu')} t^{n(\mu)}\);

4. \(\Pi_\mu = \prod_{x \in \mu \setminus \{0,0\}} (1 - q^{a'(x)} t^{l'(c)})\).

Let \(\Lambda\) denote the algebra of symmetric functions in infinitely many variables \(z = z_1, z_2, \ldots\) with coefficients in \(\mathbb{Q}(q, t)\). This is a graded algebra, graded by the degree of the symmetric functions, \(\Lambda = \bigoplus_{n \geq 0} \Lambda_n\). It is well known, see Chapter I of [Mac79], that the elementary symmetric functions, power-sum symmetric functions, and Schur functions, are \(\mathbb{Q}\)-bases for \(\Lambda\). The modified Macdonald symmetric functions is a \(\mathbb{Q}(q, t)\)-basis for \(\Lambda\) uniquely determined by the following three properties:

1. \(\tilde{H}_\mu(z; q, t) \in \mathbb{Q}(q, t)\{s_\lambda[Z/(1-q)] : \mu \leq \lambda\}\),

2. \(\tilde{H}_\mu(z; q, t) \in \mathbb{Q}(q, t)\{s_\lambda[Z/(1-q)] : \mu' \leq \lambda\}\),

3. \(\tilde{H}_\mu[1; q, t] = 1\),

where the brackets indicate the plethystic substitution, \(Z/(1-q)\) is the symmetric function \(\frac{1}{1-q}(z_1 + z_2 + \cdots)\), and \(s_\lambda(z)\) denotes a Schur function. The Bergeron-Garsia operator on \(\Lambda\) is explicitly given by setting

\[
\nabla(\tilde{H}_\mu(z; q, t)) = t^{n(\mu)} q^{n(\mu')} \tilde{H}_\mu(z; q, t).
\]

It turns out that \(\nabla e_n\) is the Frobenius series of the ring of diagonal coinvariants \(R_n\) and its expansion in the (modified) Macdonald basis is given by

\[
\nabla e_n = \sum_{\mu \vdash n} \left( \frac{(1-q)(1-t)\Pi_\mu B_\mu}{\prod_{x \in \mu} (1-t^{1+l(x)} q^{-a(x)}) (1-t^{-l(x)} q^{1+a(x)})} \right) \tilde{H}_\mu(z; q, t); \tag{5}
\]

see Proposition 3.5.26 in [Hai03].

In [LW07], Loehr and Warrington obtained an expansion of the symmetric functions \((1)^{n-1} \nabla p_n\) in the (modified) Macdonald basis. It is given by

\[
(-1)^{n-1} \nabla p_n = \sum_{\mu \vdash n} \left( \frac{(1-t^n)(1-q^n)\Pi_\mu T_\mu}{\prod_{x \in \mu} (q^{a(x)} - t^{1+l(x)})(t^{l(x)} - q^{1+a(x)})} \right) \tilde{H}_\mu(z; q, t). \tag{6}
\]

Using the identities

\[
n(\mu) = \sum_{x \in \mu} l(x) \tag{7}
\]

\[
n(\mu') = \sum_{x \in \mu'} l(x) = \sum_{x \in \mu} a(x) \tag{8}
\]
we can rewrite (6) in a form closer to (5). It follows that

\[
(-1)^{n-1} \nabla p_n = \sum_{\mu \vdash n} \left( \frac{(1 - t)(1 - q) \Pi_{x \in \mu} (t_1^{l(x)} q^{-a(x)})}{(1 - t^{1 + l(x)} q^{-a(x)}) (1 - t^{-l(x)} q^{1 + a(x)})} \right) \tilde{H}_\mu(z; q, t). \tag{9}
\]

The purpose of this note is to prove the following theorem.

**Theorem 2.1.** There exists an \( S_n \)-module \( N \) (resp. \( N' \)) such that its Frobenius series is given by \((-1)^{n-1} n^3 \nabla p_n\) (resp. \((-1)^{n-1} \nabla p_n\)).

The proof of this proposition is presented in the next section. We summarize the major steps of the proof.

1. We construct the \( T \)-equivariant coherent sheaves on \( H_n \).

2. Using Atiyah-Bott-Lefschetz formula we calculate the equivariant Euler characteristic of the sheaves constructed in the above step.

3. We establish a vanishing result that proves that the module we have constructed, which is apriori an object in the derived category, is indeed concentrated on degree zero. This shows that the representation we obtain is indeed an object of the abelian category of \( S_n \)-representation and not in the derived category.

### 2.1 Basic facts about \( H_n \).

We will recall some basic facts about \( H_n \) which will be used in the proof of the theorem. We refer to §2 of [Hai98] for detailed proofs.

The two-dimensional torus \( T \) with coordinates \((q, t)\) acts on \( \mathbb{C}^2 \); explicitly on the co-ordinate ring \( \mathbb{C}[x, y] \) this action is given by \((t, q) \cdot x = tx \) and \((t, q) \cdot y = qy \). The induced action on closed points of \( H_n \) is given by pulling back ideals; explicitly if \( I = \langle p(x, y) \rangle \) then \((t, q) \cdot I = \langle p(t^{-1}x, q^{-1}y) \rangle \).

By a well known construction, due to Gotzmann and Grothendieck, we know that \( H_n \) has a closed embedding into a high dimensional Grassmann variety. By pulling back the standard open covering from the Grassmann variety, we get an open cover \( \{U_\mu\} \) of \( H_n \) indexed by the partitions of \( n \). Explicitly, the closed points of any \( U_\mu \) are described as follows

\[
U_\mu := \{ I \in H_n : \mathbb{C}[x, y]/I \text{ admits a vector-space basis of monomials } x^h y^k \text{ for } (h, k) \in \mu \}. \tag{10}
\]

The co-ordinate ring \( \mathcal{O}_{U_\mu} \) is generated by functions \( c_{hk}^{rs} \). On an ideal \( I \in U_\mu \), these functions take values

\[
x^r y^s = \sum_{(h, k) \in \mu} c_{hk}^{rs}(I) x^h y^k \mod (I)
\]
for all \( r, s \geq 0 \). As a result, the \( T \)-action on \( c_{hk}^{rs} \) is given by \((t, q) c_{hk}^{rs} = t^{-h} q^{s-k} c_{hk}^{rs}\). It follows from definition that the above open covering is \( T \)-stable for the \( T \)-action on \( H_n \) described above. The \( T \)-fixed points of \( H_n \) are precisely the monomial ideals

\[ I_\mu := \langle x^h y^k : (h, k) \notin \mu \rangle. \]

The maximal ideal \( \mathfrak{m}_\mu \) at \( I_\mu \) is described by

\[ \mathfrak{m}_\mu = \{ c_{hk}^{rs} | (r, s) \notin \mu \}. \]

Let \( F_n \) denote the universal family over \( H_n \) and denoting the canonical projection by \( \pi \) we set

\[ B := \pi_* \mathcal{O}_{F_n}. \tag{11} \]

The sheaf \( B \) is locally free and it has rank \( n \). The trace map

\[ \text{tr} : B \to \mathcal{O}_{H_n} \]

splits the canonical inclusion of \( \mathcal{O}_{H_n} \to B \) and hence we obtain a splitting of vector bundles

\[ B = B' \oplus \mathcal{O}_{H_n}. \]

This splitting is a \( T \)-equivariant splitting of vector bundles.

**Remark 2.1.** The vector bundle \( B \) is often called the tautological bundle on \( H_n \). The Hilbert-Chow map \( \sigma : H_n \to \mathbb{C}^{2n}/S_n \) realizes \( H_n \) as a blow-up of \( \mathbb{C}^{2n}/S_n \). This provides \( H_n \) with a canonical ample line bundle \( \mathcal{O}(1) \). It turns out that \( \wedge^n B = \mathcal{O}(1) \).

Let \([0]\) denote the image of origin under the canonical projection \( \mathbb{C}^{2n} \to \mathbb{C}^{2n}/S_n \). The reduced zero fiber, denoted by \( Z_n \) is the closed subset \( \sigma^{-1}([0]) \) with the reduced closed subscheme structure. Classical results, due to Briançon in [Bri77], show that \( Z_n \) is an irreducible scheme of dimension \( n - 1 \). In [Hai98], Haiman shows that the structure sheaf \( \mathcal{O}_{Z_n} \) is \( T \)-equivariantly isomorphic to a perfect complex in derived category of coherent sheaves in \( H_n \).

**Theorem 2.2.** In the derived category of \( T \)-equivariant coherent sheaves on \( H_n \), the sheaf \( \mathcal{O}_{Z_n} \) is isomorphic to

\[ 0 \to B \otimes \wedge^n (B' \oplus \mathcal{O}_t \oplus \mathcal{O}_q) \to \cdots \to B_n \otimes (B'_n \oplus \mathcal{O}_t \oplus \mathcal{O}_q) \to B_n \to 0; \tag{12} \]

where all tensor products are over \( \mathcal{O}_{H_n} \) and \( \mathcal{O}_t \) (resp. \( \mathcal{O}_q \)) denote trivial line bundles on \( H_n \) with \( T \)-characters \( t \) (resp. \( q \)).
3 Proof of Theorem 2.1

3.1 Step 1.

Let $p_1$ and $p_2$ denote the canonical projections as shown in Figure 3.1.

\[ \begin{diagram}
  H_n \times T & \xrightarrow{p_2} & T \\
  \downarrow{p_1} & & \\
  H_n
\end{diagram} \]

Figure 3:

Lemma 3.1. If $\mathcal{F}$ is a quasi-coherent sheaf on $H_n \times T$, then we have $R^i p_\ast(\mathcal{F}) = 0$ for all $i > 0$.

Proof. This follows directly from Corollary I.3.2 Chapter III of [Gro61].

We will interchangeably use the same letter to denote a vector bundle and the associated sheaf of sections. Suppose $\chi$ is any character of the torus and $\mathcal{O}_\chi$ is the associated line bundle on $T$. We set

\[ L_\chi := p_1 \ast \circ p_2 \ast(\mathcal{O}_\chi). \]

Proposition 3.1. The quasi-coherent sheaf $L_\chi$ is a $T$-equivariant line bundle on $H_n$.

Proof. We will first show that indeed $L_\chi$ is a line bundle. This is clear because for any open subscheme $U \subset H_n$ we have sections

\[ L_\chi(U) := \mathcal{O}_{H_n}(U) \otimes \mathbb{C} \cdot \chi. \]

This shows that indeed $L_\chi$ is a line bundle.

To show that it is $T$-equivariant we will construct an isomorphism $\varphi : p_1^\ast L_\chi \to a^\ast L_\chi$ where $a : H_n \times T \to H_n$ is the action map. We use the $T$-stable affine-covering of $\{U_\mu\}$ of $H_n$ constructed above. Restricted to the open set $U_\mu$, we have canonical identifications $\varphi_\mu : p_1^\ast L_\chi(U_\mu) \cong a^\ast L_\chi(U_\mu)$. These sections agree on the intersections $U_\mu \cap U_\lambda$ and hence glue to a global isomorphism of sheaves. This proves the assertion.

For a cell $(r, s)$ in the square grid $\square_n$, we let $\chi_{r,s}$ denote the character $\chi_{r,s} : T \to \mathbb{C}^\ast$ defined by

\[ \chi_{r,s}(t, q) := t^r q^s. \]
Definition 3.1. Consider the $T$-equivariant coherent sheaf

$$\mathcal{V} := \bigoplus_{(r,s) \in \Box_n} L_{\chi_{r,s}}$$

on $H_n$ and let $\mathcal{N}$ denote the coherent sheaf

$$\mathcal{N} := \mathcal{B}^\vee \otimes_{\mathcal{O}_{H_n}} \mathcal{V},$$

where $\mathcal{B}^\vee$ is the dual sheaf $\text{Hom}_{\mathcal{O}_{H_n}}(\pi_\ast \mathcal{O}_{F_n}, \mathcal{O}_{H_n})$.

Remark 3.1. Note that the rank of $\mathcal{N}$ as a vector bundle is $n^3$.

3.1.1 Construction of $\mathcal{N}'$

In Figure 4, let $\iota$ denote the isogeny defined by

$$\iota : T \longrightarrow T$$

$$(q,t) \longmapsto (q^{n^3}, t^{n^3}),$$

and let $\tau$ denote the composition of $\iota$ and $p_2$. We define $L_{\chi}' := p_{1,s} \circ \tau^* \mathcal{O}_{\chi}$, where $\mathcal{O}_{\chi}$ is a line bundle on $T$.

By using the same arguments as in the previous subsection we see that $L_{\chi}'$ is a $T$-equivariant line bundle on $H_n$. We define, analogous to $\mathcal{V}$ and $\mathcal{N}$, coherent sheaves

$$\mathcal{V}' := \bigoplus_{(r,s) \in \Box_n} L_{\chi_{r,s}}',$$

$$\mathcal{N}' := \mathcal{B}^\vee \otimes \mathcal{V}'$$

on $H_n$. 
3.2 Step 2.

In this section we calculate the equivariant Euler characteristic of the sheaves $N \otimes P \otimes O_{Z_n}$ (resp. $N' \otimes P \otimes O_{Z_n}$) by using an appropriate version of the Atiyah-Bott-Lefschetz identity (see (3.1) of [Hai98]);

$$
\sum_{i=0}^{n} (-1)^i \text{tr}_{H^i(H_n,F)}(\tau) = \sum_{\mu+n} (-1)^i \frac{\text{tr}_{\text{Tor}_i(C(I_{\mu}),F)}(\tau)}{\det_{C(I_{\mu})}(1-\tau)};
$$

where $F$ is a bounded complex of coherent sheaves on $H_n$, $\tau \in T$ and $C(I_{\mu})$ is the tangent space at $I_{\mu}$.

The terms in the denominator, $\det_{C(I_{\mu})}(1-\tau)$ were calculated by Ellingsrud and Strømme in [ES87]. We will calculate the numerator.

The resolution of $O_{Z_n}$ in (12) allows us to calculate $\text{Tor}_i(C(I_{\mu}), O_{Z_n} \otimes P \otimes N)$ as the homology of the complex of complex vector spaces

$$
0 \rightarrow \cdots \rightarrow C(I_{\mu}) \otimes P \otimes (B \otimes B') \otimes V \otimes (\wedge^i(B' \otimes O_t \otimes O_q)) \rightarrow \cdots \rightarrow 0.
$$

By taking the trace and using the multiplicative property of the trace with respect to tensor products we find that

$$
\sum_i (-1)^i \text{tr}_{\text{Tor}_i(C(I_{\mu}),O_{Z_n} \otimes P \otimes N)}(\tau) = n^2 \text{tr}_{C(I_{\mu})}(\tau) \cdot \text{tr}_P(\tau) \cdot \text{tr}_V(\tau) \cdot \left( \sum_i (-1)^i \text{tr}_{\wedge^i(B' \otimes O_t \otimes O_q)}(\tau) \right)
$$

(16)

Now we specialize $\tau$ to $(q,t)$. By Proposition 5.4.1 of [Hai03], we know that $\text{tr}_P(q,t)$ is the modified Macdonald polynomial. At the same time, it follows from our construction that

$$
\text{tr}_V(q,t) = [t]_n[q]_n.
$$

The description of the co-ordinate ring of the affine open sets $U_\mu$ and the $T$-action on the coordinates in (10) shows that $\text{tr}_{C(I_{\mu})}(q,t) = n$. Lastly, the final term in (16) equals $(1-t)(1-q) \prod_{\mu}$, see Theorem 2 in [Hai98]. By collecting the terms together, we see that the equivariant Euler characteristic of $O_{Z_n} \otimes P \otimes N'$ equals $n^3$ times the Frobenius characteristic of $(-1)^{n-1} \nabla p_n$.

**Remark 3.2. If we use the sheaf $N'$ instead of $N$, then we have $\text{tr}_{V'}(q,t)$ instead of $\text{tr}_V(q,t)$ in the right-hand side of equation (16). This contributes a factor $[t]_n[q]_n/n^3$ which cancels out the $n^3$ factor resulting in the desired Frobenius characteristic.**

3.3 Step 3.

In the previous subsection we calculated the equivariant Euler characteristics

$$
\sum_i (-1)^i \text{tr}_{H^i(H_n,F)}(\tau) \text{ for } \tau = (t,q) \in T
$$

10
for the sheaves $F = \mathcal{O}_Z \otimes \mathcal{P} \otimes \mathcal{N}$ (resp. $F = \mathcal{O}_Z \otimes \mathcal{P} \otimes \mathcal{N}'$). We showed that it matches the Frobenius characteristics of $(-1)^{n-1} \nabla p_n$ (upto a $n^3$ factor in the first case). In this section we will prove a cohomology vanishing result.

**Proposition 3.2.** For all $i > 0$, the cohomology groups vanish:

$$H^i(\mathcal{H}_n, \mathcal{O}_Z \otimes \mathcal{P} \otimes \mathcal{N}) = 0.$$  

(17)

The same vanishing result holds true for $\mathcal{O}_Z \otimes \mathcal{P} \otimes \mathcal{N}'$.

As a consequence of Proposition 3.2 we see that, under the derived equivalence $\Phi$, the coherent sheaf $\mathcal{O}_Z \otimes \mathcal{N} \otimes \mathcal{P}$ (resp. $\mathcal{O}_Z \otimes \mathcal{N}' \otimes \mathcal{P}$) corresponds to a representation (and not just a virtual representation) of $S_n$.

### 3.3.1 A reduction.

We first consider a useful reduction. By construction, $\mathcal{V}$ is a direct sum of finitely many line bundles of the form $L_{\chi_{r,s}}$. We have

$$\mathcal{O}_Z \otimes \mathcal{P} \otimes \mathcal{N} = \bigoplus_{(r,s) \in \square_n} \mathcal{O}_Z \otimes \mathcal{P} \otimes \mathcal{B}^\vee \otimes L_{\chi_{r,s}}$$  

(18)

and since cohomology commutes with direct sums, it suffices to show

$$H^i(\mathcal{H}_n, \mathcal{O}_Z \otimes \mathcal{P} \otimes \mathcal{B}^\vee \otimes L_{\chi_{r,s}}) = 0$$  

(19)

for $i > 0$ and for all $(r, s) \in \square_n$. The line bundles $L_{\chi_{r,s}}$ are isomorphic to the trivial bundle on $H_n$, therefore, we are further reduced to showing

$$H^i(\mathcal{H}_n, \mathcal{O}_Z \otimes \mathcal{P} \otimes \mathcal{B}^\vee) = 0$$  

(20)

for all $i > 0$. This statement is proved in the next subsection.

### 3.3.2 Proof of (20).

Let us denote the complex (12) as $V_\bullet$ for simplicity. We can calculate $H^i(\mathcal{H}_n, \mathcal{O}_Z \otimes \mathcal{P} \otimes \mathcal{B}^\vee)$ as the cohomology of the complex $\Gamma(\mathcal{H}_n, \mathcal{P} \otimes \mathcal{B}^\vee \otimes V_\bullet)$, where $\Gamma$ is the global sections functor. Each term of the complex $\mathcal{P} \otimes \mathcal{B}^\vee \otimes V_\bullet$, using the isomorphism $\mathcal{B} \otimes \mathcal{B}^\vee \cong \mathcal{O}_{H_n}^{n^2}$, is given by

$$\mathcal{P} \otimes \mathcal{N} \otimes V_i = (\mathcal{P} \otimes \wedge^i(\mathcal{B}' \oplus \mathcal{O}_t \oplus \mathcal{O}_q))^n.$$  

(21)

It suffices to show that the cohomology of the complex

$$\Gamma(\mathcal{H}_n, (\mathcal{P} \otimes \wedge^i(\mathcal{B}' \oplus \mathcal{O}_t \oplus \mathcal{O}_q)))$$  

(22)

vanishes for $i > 0$. The sheaf $\mathcal{B}'$ is a subsheaf of $\mathcal{B}$. As a result $\mathcal{P} \otimes \wedge^i(\mathcal{B}' \oplus \mathcal{O}_t \oplus \mathcal{O}_q)$ is a direct factor of the vector bundle $\mathcal{P} \otimes \mathcal{B}^{\otimes l}$ for some $l \geq 0$. The higher cohomologies of the latter sheaf vanishes by Theorem 2.1 of [Hai02]; as a result the higher cohomology of the complex (22) is also zero. This finishes our proof.
References


