

THE NEF CONE VOLUME OF GENERALIZED DEL PEZZO SURFACES

ULRICH DERENTHAL, MICHAEL JOYCE, AND ZACH TEITLER

ABSTRACT. We compute a naturally defined measure of the size of the nef cone of a Del Pezzo surface. The resulting number appears in a conjecture of Manin on the asymptotic behavior of the number of rational points of bounded height on the surface. The nef cone volume of a Del Pezzo surface Y with (-2) -curves defined over an algebraically closed field is equal to the nef cone volume of a smooth Del Pezzo surface of the same degree divided by the order of the Weyl group of a simply-laced root system associated to the configuration of (-2) -curves on Y . When Y is defined over an arbitrary perfect field, a similar result holds, except that the associated root system is no longer necessarily simply-laced.

1. INTRODUCTION

An **ordinary Del Pezzo surface** is a smooth projective rational surface X on which the anticanonical class $-K_X$ is ample. If X is defined over an algebraically closed field, then X is one of the following: \mathbb{P}^2 , $\mathbb{P}^1 \times \mathbb{P}^1$, or the blowup of \mathbb{P}^2 at up to 8 points in general position. Points are in general position if no three are collinear, no six lie on a conic, and no eight lie on a cubic with one of them a singular point of the cubic. Then X may contain (-1) -curves, but no (-2) -curves, where for $n \in \{1, 2\}$, a $(-n)$ -**curve** is a smooth rational curve on X which has self intersection number $-n$.

A **generalized Del Pezzo surface** is a smooth projective rational surface Y on which $-K_Y$ is big and nef. If Y is defined over an algebraically closed field, then Y is one of the following: \mathbb{P}^2 , $\mathbb{P}^1 \times \mathbb{P}^1$, the Hirzebruch surface \mathbb{F}_2 , or a surface obtained from \mathbb{P}^2 by a sequence of blowings-up at up to 8 points, possibly infinitely near, each not lying on any (-2) -curve. Over an algebraically closed field, a generalized Del Pezzo surface is ordinary if and only if it contains no (-2) -curves. See Section 3 for more details.

The nef cone volume of a generalized Del Pezzo surface Y is equal to the volume of the cross-section of the nef cone of Y obtained by intersecting with the hyperplane consisting of those divisor classes whose intersection with the anticanonical class $-K_Y$ is equal to 1. The resulting cross-section is a polytope. Its volume is a rational number, denoted $\alpha(Y)$. We give more details of this definition in Section 2.

In this paper we compute $\alpha(Y)$ for generalized Del Pezzo surfaces Y .

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d	8	7	6	5	4	3	2	1
N_d	1	3	6	10	16	27	56	240
$\alpha(X_d)$	1/6	1/24	1/72	1/144	1/180	1/120	1/30	1

TABLE 1. Values of $\alpha(X_d)$ for ordinary Del Pezzo surfaces X_d

The **degree** of Y is the self-intersection number $d = \langle -K_Y, -K_Y \rangle$. The degree is always in the range $1 \leq d \leq 9$, and when Y is the blowup of \mathbb{P}^2 at r points in almost general position, $d = 9 - r$.

A generalized Del Pezzo surface Y defined over a field \mathbb{K} is **split** if it is either \mathbb{P}^2 , $\mathbb{P}^1 \times \mathbb{P}^1$, \mathbb{F}_2 , or the blow-up of \mathbb{P}^2 at $1 \leq r \leq 8$ \mathbb{K} -rational points in almost general position. See Definition 3.2. Otherwise, Y is said to be non-split (e.g., the blow-up of \mathbb{P}^2 at two conjugate points). We consider only split Y until Section 6, and so the reader may assume that \mathbb{K} is algebraically closed until that point.

An investigation of $\alpha(Y)$ for split ordinary Del Pezzo surfaces was undertaken by the first author. He proved the following result:

Theorem 1.1. [Der07, Theorem 4] *Let X_d denote a split ordinary Del Pezzo surface of degree d obtained by blowing up $9 - d$ points in general position on \mathbb{P}^2 and let N_d denote the number of (-1) -curves on X_d . For $d \leq 6$,*

$$\alpha(X_d) = \frac{N_d}{d(9-d)}\alpha(X_{d+1}).$$

Combining this with simple calculations that show $\alpha(\mathbb{P}^2) = 1/3$, $\alpha(\mathbb{P}^1 \times \mathbb{P}^1) = 1/4$, $\alpha(X_8) = 1/6$, and $\alpha(X_7) = 1/24$, this theorem allows for an inductive calculation of $\alpha(X)$ for any split ordinary Del Pezzo surface X . This calculation is summarized in Table 1.

We extend this result in two directions. First, we study split generalized Del Pezzo surfaces. In Section 4, we prove the following theorem by analyzing the nef cone of such a surface Y . It allows us to compute $\alpha(Y)$ by induction on the rank of the Néron-Severi group of Y .

Theorem 1.2. *Let Y be a split generalized Del Pezzo surface of degree $d \leq 7$. For each E in the set \mathcal{C} of (-1) -curves on Y , let Y_E denote the split generalized Del Pezzo surface of degree $d + 1$ obtained by contracting E . Then*

$$\alpha(Y) = \sum_{E \in \mathcal{C}} \frac{1}{d(9-d)}\alpha(Y_E).$$

As with Theorem 1.1, using the additional calculation that $\alpha(\mathbb{F}_2) = 1/8$, this theorem allows us to compute $\alpha(Y)$ for any split generalized Del Pezzo surface Y .

The first author computed $\alpha(Y)$ for split generalized Del Pezzo surfaces Y of degree $d \geq 3$ directly, using computer programs to find a triangulation of the nef cone case by case [Der07, Section 3]. This numerical data led us to formulate the following theorem; see Section 5.3 for its proof.

Theorem 1.3. *Let Y be a split generalized Del Pezzo surface of degree $d \leq 7$ and let X be a split ordinary Del Pezzo surface of the same degree. Then*

$$\alpha(Y) = \frac{1}{\#W(R_Y)}\alpha(X),$$

where $W(R_Y)$ is the Weyl group of the root system R_Y whose simple roots are the (-2) -curves on Y .

When combined with Theorem 1.1 the computation of $\alpha(Y)$ for an arbitrary split generalized Del Pezzo surface Y of any degree is reduced to a determination of the (-2) -curves on the surface. See Section 5 for more information on the root system R_Y and its Weyl group.

We also consider the case of non-split surfaces. Suppose that Y is a generalized Del Pezzo surface and X is an ordinary Del Pezzo surface, both of the same degree and defined over the same perfect field \mathbb{K} . Then the Néron-Severi groups of X and Y coincide (Proposition 6.2) and the absolute Galois group of \mathbb{K} acts as a finite group G of automorphisms of this group (Proposition 6.1). Assume that the Galois actions associated to X and Y coincide. The Galois action on the root system R_Y allows us to associate to Y an orbit root system $\mathcal{O}(R_Y, G)$ (Definition 6.5). Our third main result is that under these assumptions

$$\alpha(Y) = \frac{\alpha(X)}{\#W(\mathcal{O}(R_Y, G))}.$$

See Corollary 7.5. The integer appearing in the denominator is the order of a Weyl group and is straightforward to compute. Thus all that is left is to compute $\alpha(X)$. There are a finite number of cases in each degree d , one for each conjugacy class of subgroups of the Weyl group of a canonically defined root system R_d (Section 5.2). We perform the computations for $d \geq 5$ in Section 7.2.

Manin's conjecture. The primary motivation for our study of the nef cone volume α is its appearance in Manin's conjecture on the number of rational points of bounded height on Fano varieties defined over number fields, as described below. Although the conjecture is now known not to hold for all Fano varieties [BT96, Theorems 3.1–3.3], it has been verified in a large number of cases, including some varieties for which the anticanonical class is big but not ample.

Let X be a smooth projective variety defined over a number field \mathbb{K} for which $-K_X$ is big and assume that the set $X(\mathbb{K})$ of rational points is Zariski dense. Equip $X(\mathbb{K})$ with an anticanonical height function H (consult [HS00, Part B] for information on height functions) and for any constructible set $U \subset X$ let

$$\mathcal{N}_U(B) := \#\{P \in U(\mathbb{K}) : H(P) \leq B\}.$$

The original formulation of the Manin conjecture [BM90, Conjecture B] posits the existence of a Zariski open set $U \subset X$ such that for any open set $V \subset U$

$$\mathcal{N}_V(B) \sim c(X)B(\log B)^{\rho-1} \quad \text{asymptotically as } B \rightarrow \infty,$$

where ρ is the Néron-Severi rank of X . The conjecture was initially made for Fano varieties, but a more ambitious version of the conjecture relaxes the condition on $-K_X$ to merely being big. The leading constant was given a conjectural interpretation by Peyre [Pey95, Definition 2.4] and Batyrev and Tschinkel [BT95, Theorem 4.4.4]. They predict that

$$c(X) = \alpha(X)\beta(X)\tau(X),$$

where $\alpha(X) \in \mathbb{Q}$ is the constant of interest in this paper, $\beta(X) \in \mathbb{N}$ is a cohomological invariant of the Galois action on the Néron-Severi group of X , and $\tau(X) \in \mathbb{R}$ is a volume of adelic points on X .

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2. DEFINITION OF THE NEF CONE VOLUME

Here we recall the definition of $\alpha(X)$, first introduced by Peyre in [Pey95, Definition 2.4].

Let X be a smooth complete variety for which $-K_X$ is big. We denote the intersection form on X by $\langle \cdot, \cdot \rangle$. Recall that a divisor class D on X is **numerically trivial** if $\langle D, C \rangle = 0$ for all curves (equivalently, all 1-cycles) C on X , and two divisor classes are **numerically equivalent** if their difference is numerically trivial. One similarly defines numerical equivalence of curves. Numerical equivalence classes of divisors on X form a finitely-generated torsion-free abelian group $N^1(X)$ whose dual group $N_1(X)$ consists of numerical equivalence classes of 1-cycles on X . Let $N^1(X)_{\mathbb{R}} = N^1(X) \otimes_{\mathbb{Z}} \mathbb{R}$ and $N_1(X)_{\mathbb{R}} = N_1(X) \otimes_{\mathbb{Z}} \mathbb{R}$ be the associated Euclidean spaces. Inside $N^1(X)_{\mathbb{R}}$ lies the **effective cone** $\text{Eff}^1(X)$, the closed convex cone spanned by the classes of effective divisors.

Recall that for a finite-dimensional real inner product space V and a convex cone $\Gamma \subset V$, the dual convex cone $\Gamma^{\vee} \subset V$ is defined by

$$\Gamma^{\vee} = \{v \in V : \langle v, c \rangle \geq 0 \text{ for all } c \in \Gamma\}.$$

The cone Γ^{\vee} is closed as a subspace of the Euclidean space V . The dual $\text{Eff}^1(X)^{\vee}$ of the effective cone of X in $N^1(X)_{\mathbb{R}}$ is the movable cone of X (see [BDPP, Theorem 2.2], [Laz04, Section 11.4.C]). Note that when X is a surface, $N_1(X) = N^1(X)$ and $\text{Eff}^1(X)^{\vee}$ is the **nef cone** of X , denoted $\text{Nef}(X)$.

Since the cone $\text{Eff}^1(X)^{\vee}$ has infinite volume in $N_1(X)_{\mathbb{R}}$, a natural means of measuring its “size” is to truncate the cone in an (anti)canonical manner. To do this, consider the hyperplane

$$\mathcal{H}_X := \{C \in N_1(X)_{\mathbb{R}} : \langle -K_X, C \rangle = 1\}.$$

Note that since $-K_X$ is big by hypothesis, \mathcal{H}_X intersects each ray of $\text{Eff}^1(X)^{\vee}$. We endow $N_1(X)_{\mathbb{R}}$ with Lebesgue measure ds normalized so that $N_1(X)$ has covolume 1, and we endow \mathcal{H}_X with the induced Leray measure $d\mu$ with respect to the linear form $\langle -K_X, \cdot \rangle$. That is, letting l be the linear form $l(v) = \langle -K_X, v \rangle$, we have $ds = d\mu \wedge dl$. We construct the polytope

$$\mathcal{P}_X := \text{Eff}^1(X)^{\vee} \cap \mathcal{H}_X$$

and define

$$\alpha(X) := \text{Vol}(\mathcal{P}_X) = \int_{\mathcal{P}_X} d\mu.$$

There are variants of this definition which differ only by a dimensional factor. Let $\rho = \dim N_1(X)_{\mathbb{R}}$ and

$$\mathcal{C}_X := \{C \in \text{Eff}^1(X)^{\vee} : \langle -K_X, C \rangle \leq 1\}$$

be the convex hull of \mathcal{P}_X and the origin. Then a simple slicing argument shows that

$$\alpha(X) = \rho \cdot \text{Vol}(\mathcal{C}_X).$$

Additionally,

$$\alpha(X) = \frac{1}{(\rho - 1)!} \int \cdots \int_{\text{Eff}^1(X)^{\vee}} \exp(-\langle -K_X, s \rangle) ds,$$

with the bigness of $-K_X$ insuring the convergence of the integral.

Example 2.1. Let us compute $\alpha(\mathbb{P}^2)$. We have $N^1(\mathbb{P}^2)_{\mathbb{R}} \cong \mathbb{R}^1$, with the real number $x \in \mathbb{R}$ corresponding to the (real) divisor class xL , where L is the class of a line in \mathbb{P}^2 . Then the nef cone $\text{Nef}(\mathbb{P}^2) = \{x \in \mathbb{R} : x \geq 0\}$ and the anticanonical class corresponds to the real number 3. The hyperplane $\mathcal{H}_{\mathbb{P}^2}$ is just $\{1/3\}$. The polytope $\mathcal{P}_{\mathbb{P}^2}$ is also $\{1/3\}$ and the convex hull $\mathcal{C}_{\mathbb{P}^2} = [0, 1/3]$. Thus $\mathcal{C}_{\mathbb{P}^2}$ has volume $1/3$ and so $\alpha(\mathbb{P}^2) = 1 \cdot \text{Vol}(\mathcal{C}_{\mathbb{P}^2}) = 1/3$.

Example 2.2. As a second example, let X_8 be the blowup of \mathbb{P}^2 at a single point. Let L be the class of the pullback of a line to X_8 and let E be the class of the exceptional divisor. Then $N^1(X_8)$ is generated by L and E . In $N^1(X_8)_{\mathbb{R}} \cong \mathbb{R}^2$, with (a, b) corresponding to $aL + bE$, the nef cone $\text{Nef}(X_8)$ is equal to $\{(a, b) : a \geq 0, a + b \geq 0\}$, that is, the cone with extremal rays spanned by L and $L - E$. The anticanonical class corresponds to the point $(3, -1)$. The hyperplane \mathcal{H}_{X_8} is the line $3a + b = 1$. One checks that \mathcal{P}_{X_8} is the segment joining the points $(1/3, 0)$ and $(1/2, -1/2)$. Then \mathcal{C}_{X_8} is the triangle with vertices the above two points together with the origin. The area of this triangle is $1/12$, and so $\alpha(X_8) = 2 \cdot \text{Vol}(\mathcal{C}_{X_8}) = 1/6$.

Terminology. Peyre [Pey95] introduced the notation $\alpha(X)$, but did not give a name to this quantity. We will refer to $\alpha(X)$ as the “nef cone volume of X ” whenever X is a surface.

3. GENERALIZED DEL PEZZO SURFACES

As stated in the introduction, a **generalized Del Pezzo surface** is a smooth projective rational surface Y on which $-K_Y$ is big and nef. If Y is defined over an algebraically closed field, Y is one of \mathbb{P}^2 , $\mathbb{P}^1 \times \mathbb{P}^1$, the Hirzebruch surface \mathbb{F}_2 , or \mathbb{P}^2 blown up at $1 \leq r \leq 8$ points in almost general position [Dem77, Definition III.2.1]. To blow up r points on \mathbb{P}^2 in almost general position is to construct a sequence of morphisms

$$Y = Y_r \rightarrow Y_{r-1} \rightarrow \cdots \rightarrow Y_1 \rightarrow Y_0 = \mathbb{P}^2,$$

where each map $Y_i \rightarrow Y_{i-1}$ is the blow-up of Y_{i-1} at a point $p_i \in Y_{i-1}$ not lying on any irreducible curves of self-intersection number -2 on Y_i .

For $n \in \{1, 2\}$, a $(-n)$ -**class** on Y is a divisor class D such that $\langle D, D \rangle = -n$ and $\langle D, -K_Y \rangle = 2 - n$. If such a class is effective, then there is necessarily a unique curve in that class. If this curve is irreducible, we use the term $(-n)$ -**curve** both for this curve and its class. It follows from the genus formula that a $(-n)$ -curve is a smooth rational curve. A simple calculation [Dem77, Tables 2, 3] shows that the sets of (-1) - and (-2) -classes on a generalized Del Pezzo surface are finite.

Let Y be a generalized Del Pezzo surface defined over a field \mathbb{K} . We denote $Y \times_{\mathbb{K}} \overline{\mathbb{K}}$ by \overline{Y} . Recall that a generalized Del Pezzo surface Y is an ordinary Del Pezzo surface if and only if the anticanonical class $-K_Y$ is ample. Equivalently, there are no (-2) -curves on \overline{Y} .

Convention 3.1. Throughout the paper, we will use X to refer to an ordinary Del Pezzo surface and Y to refer to a generalized (possibly ordinary) Del Pezzo surface.

The absolute Galois group $G_{\mathbb{K}} = \text{Gal}(\overline{\mathbb{K}}/\mathbb{K})$ acts on $N^1(\overline{Y})$.

Definition 3.2. A generalized Del Pezzo surface Y is **split** if $Y(\mathbb{K}) \neq \emptyset$ and the action of $G_{\mathbb{K}}$ on $N^1(\overline{Y})$ is trivial.

Excluding the exceptional cases where \overline{Y} is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ or \mathbb{F}_2 , the existence of a rational point assures that Y is a blow-up of \mathbb{P}^2 and the triviality of the Galois action assures that each exceptional divisor is defined over \mathbb{K} , and thus the sequence of blown-up points must themselves be defined over \mathbb{K} .

In the remainder of this section, we prove that for a split Y , the effective cone $\text{Eff}^1(Y)$ is generated by the set of (-1) - and (-2) -curves on Y , collecting a number of useful facts along the way.

By the following lemma, the group $N^1(Y)$ depends only on the degree of Y . We will make frequent use of this well-known result.

Lemma 3.3. *Let X be a split ordinary Del Pezzo surface and let Y be a split generalized Del Pezzo surface of the same degree $d \leq 7$. There is an isomorphism of $N^1(X)$ and $N^1(Y)$ which identifies the intersection forms and takes $-K_X$ to $-K_Y$.*

Proof. Say X is the blowup of \mathbb{P}^2 at points $p_1, \dots, p_r \in \mathbb{P}^2$, $r = 9 - d$, with blowdown $\pi_X : X \rightarrow \mathbb{P}^2$, and say Y is obtained by blowing up \mathbb{P}^2 at points q_1, \dots, q_r :

$$\pi_Y : Y = Y_r \rightarrow Y_{r-1} \rightarrow \dots \rightarrow Y_1 \rightarrow Y_0 = \mathbb{P}^2$$

where $Y_j = \text{Bl}_{q_j}(Y_{j-1})$, $q_j \in Y_{j-1}$. Let $E_{X,j}$ be the exceptional divisor over p_j , and let $E_{Y,j}$ be the total transform in Y of the exceptional divisor over q_j . (That is, if $f_j : Y \rightarrow Y_{j-1}$, then $E_{Y,j} = f_j^{-1}(q_j)$, scheme-theoretically.)

Then $N^1(X)$ is the free abelian group on $L_X = \pi_X^* \mathcal{O}_{\mathbb{P}^2}(1)$, $E_{X,1}, \dots, E_{X,r}$. Similarly, $N^1(Y)$ is the free abelian group on $L_Y = \pi_Y^* \mathcal{O}_{\mathbb{P}^2}(1)$, $E_{Y,1}, \dots, E_{Y,r}$. The intersection form on $N^1(X)$ is given in this basis by the diagonal matrix with entries $(1, -1, \dots, -1)$; the intersection form on $N^1(Y)$ is given in this basis by the same matrix. We have $-K_X = 3L_X - \sum E_{X,j}$ and $-K_Y = 3L_Y - \sum E_{Y,j}$. \square

Remark 3.4. Note that the identification made in the proof of Lemma 3.3 is not necessarily unique; see [Har85, Theorem 0.1].

The next lemma is a modest generalization of [HT04, Proposition 4.5].

Lemma 3.5. *Let S be a surface and let D_1, \dots, D_k be irreducible effective divisors on S . Let Γ denote the cone generated by D_1, \dots, D_k . Then the effective cone of S is equal to Γ if and only if $\Gamma^\vee \subset \Gamma$.*

Proof. If the effective cone of S is equal to Γ then it is a closed cone. The nef cone $\text{Nef}(S) = \Gamma^\vee$ is contained in the closure of the effective cone, which is just Γ .

For the converse, it is clear that Γ is contained in the effective cone of S . Let D be an effective divisor. Then we can write $D = D' + a_1 D_1 + \dots + a_k D_k$ with $a_i \geq 0$ and D' having none of the D_i as an irreducible component. It is clear that D' is contained in Γ^\vee , and by hypothesis, D' is consequently contained in Γ . Hence the same is true of D . \square

Proposition 3.6. *If Y is a split generalized Del Pezzo surface, every (-1) -class in $N^1(Y)$ is effective. Indeed, if E is any (-1) -class, then either*

- (1) E is a (-1) -curve;
- (2) E can be written as the sum of a (-1) -curve and one or more (-2) -curves;
- (3) $d = 1$ and E can be written as the sum of $-K_Y$ and one or more (-2) -curves.

Proof. See [Dem77, Theorem III.2.c]. \square

For a split generalized Del Pezzo surface Y of degree $d \geq 2$, this shows that every (-1) -class is a non-negative integral linear combination of (-1) - and (-2) -curves. By the following lemma, this holds also in degree $d = 1$ if we allow rational instead of integral coefficients.

Lemma 3.7. *For a split generalized Del Pezzo surface Y of degree 1, the anticanonical class $-K_Y$ is a linear combination of (-1) - and (-2) -curves with non-negative rational coefficients.*

Proof. Let X be an ordinary Del Pezzo surface of degree 1. It is easy to check that the sum of all (-1) -classes on X is $-240K_X$.

Using the identification of Lemma 3.3, the sum of all (-1) -classes on Y is $-240K_Y$. Using Proposition 3.6, we can write n of the (-1) -classes as the sum of a (-1) -curve and possibly some (-2) -curves, and the remaining $240 - n$ of the (-1) -classes as the sum of $-K_Y$ and some (-2) -classes. Note that $E_{Y,8}$ in the proof of Lemma 3.3 is a (-1) -curve on Y , so we have $n > 0$.

This gives us $-240K_Y$ as the sum of n (-1) -curves, $-(240 - n)K_Y$, and some (-2) -curves. We transform this equation to write $-nK_Y$ as a sum of (-1) - and (-2) -curves. \square

Lemma 3.8. *Let Y be a split generalized Del Pezzo surface and let E be a (-1) -class in $N^1(Y)$. Then E is irreducible if and only if $\langle E, C \rangle \geq 0$ for every (-2) -curve C .*

Proof. See [Dem77, Corollary on page 46]. \square

In the case of ordinary Del Pezzo surfaces, the following result is well-known.

Proposition 3.9. *Let X be a split ordinary Del Pezzo surface of degree $d \leq 7$. Then the effective cone of X is minimally generated by the (-1) -classes on X , all of which are (-1) -curves.*

Proof. This can be proved directly (see [Har77, Theorem V.4.11] for a proof when $d = 3$) or can be taken as an immediate consequence of the calculation of generators for the Cox ring given in [BP04, Theorem 3.2], making use of Lemma 3.7 in the case $d = 1$. \square

We now reach our main goal for this section.

Theorem 3.10. *If Y is a split generalized Del Pezzo surface and has degree $d \leq 7$, the effective cone of Y is finitely generated by the set of (-1) - and (-2) -curves.*

Proof. Let Γ be the cone generated by the (-1) - and (-2) -curves on Y . To prove the theorem, it suffices by Lemma 3.5 to show that $\Gamma^\vee \subset \Gamma$. Let X be a split ordinary Del Pezzo surface of the same degree as Y . Identify $N^1(X)$ and $N^1(Y)$ as in Lemma 3.3. Note that this identification takes (-1) -classes to (-1) -classes. By Proposition 3.9, $\text{Eff}^1(X)$ is generated by (-1) -classes. Each (-1) -class lies in Γ by Proposition 3.6 and Lemma 3.7. Therefore $\text{Eff}^1(X) \subset \Gamma$. It follows immediately that $\Gamma^\vee \subset \text{Eff}^1(X)^\vee$. From Lemma 3.5 we have $\text{Eff}^1(X)^\vee \subset \text{Eff}^1(X)$. Thus $\Gamma^\vee \subset \Gamma$ and hence $\Gamma = \text{Eff}^1(Y)$, again by Lemma 3.5. \square

Remark 3.11. A generalization of Theorem 3.9 has already been proved by Lahyane and Harbourne [LH05, Lemma 4.1]. We include our presentation both as a summary of results that we will use later and also because the approach here seems to have interest in its own right.

Corollary 3.12. *Let X be a split ordinary Del Pezzo surface and Y a split generalized Del Pezzo surface with $\deg(X) = \deg(Y) \leq 7$. Identifying $N^1(X)$ and $N^1(Y)$ as in Lemma 3.3, we have $\text{Eff}^1(X) \subset \text{Eff}^1(Y)$ and $\text{Nef}(X) \supset \text{Nef}(Y)$.*

Let $\Gamma \subset N^1(Y)_\mathbb{R}$ be the cone spanned by the set of (-2) -curves on Y . Then $\text{Eff}^1(Y)$ is the sum of $\text{Eff}^1(X)$ and Γ , and $\text{Nef}(Y) = \text{Nef}(X) \cap \Gamma^\vee$. \square

4. INDUCTIVE METHOD

With these preliminaries in place, we now turn to proving Theorem 1.2. For a generalized Del Pezzo surface Y and any class $D \in N^1(Y)_\mathbb{R}$, we denote by D^\perp the hyperplane

$$D^\perp := \{C \in N_1(Y)_\mathbb{R} : \langle D, C \rangle = 0\}.$$

Lemma 4.1. *Let Y be a split generalized Del Pezzo surface and E a (-1) -curve on Y . Let $\pi_E : Y \rightarrow Y_E$ be the contraction of E . Then*

$$\pi_E^* : N^1(Y_E) \longrightarrow E^\perp \cap N^1(Y)$$

is an isomorphism and induces an isomorphism of convex cones,

$$\pi_E^*(\text{Nef}(Y_E)) = \text{Nef}(Y) \cap E^\perp.$$

Proof. We have $N^1(Y) = \pi_E^*(N^1(Y_E)) \oplus \mathbb{Z}E$. We may identify $\text{Nef}(Y_E)$ with $\pi_E^*(\text{Nef}(Y_E)) \subset E^\perp$. The inclusion $\pi_E^*(\text{Nef}(Y_E)) \subset \text{Nef}(Y)$ follows immediately from the projection formula. This proves $\pi_E^*(\text{Nef}(Y_E)) \subset \text{Nef}(Y) \cap E^\perp$.

For the reverse inclusion, let $D \in \text{Nef}(Y) \cap E^\perp$. Since $E^\perp = \pi_E^*(N^1(Y_E))$, we have $D = \pi_E^* \pi_{E*} D$. Again by the projection formula, for any curve $C \subset Y_E$,

$$\langle \pi_{E*} D, C \rangle_{Y_E} = \langle D, \pi_E^* C \rangle \geq 0,$$

since $D \in \text{Nef}(Y)$. □

We now prove the first of our main theorems. We repeat it here for the convenience of the reader:

Theorem 1.2. *Let Y be a split generalized Del Pezzo surface of degree $d \leq 7$. For each E in the set \mathcal{C} of (-1) -curves on Y , let Y_E denote the split generalized Del Pezzo surface of degree $d + 1$ obtained by contracting E . Then*

$$\alpha(Y) = \sum_{E \in \mathcal{C}} \frac{1}{d(9-d)} \alpha(Y_E).$$

Proof. We follow the argument used in [Der07, Theorem 4]. Let \mathcal{E} be the set of (-1) - and (-2) -curves on Y . Then \mathcal{E} is exactly the set of generators for $\text{Eff}^1(Y)$ described in Theorem 3.10. Recall that the hyperplane \mathcal{H}_Y is defined as

$$\mathcal{H}_Y = \{C \in N_1(Y)_\mathbb{R} : \langle -K_Y, C \rangle = 1\}.$$

The intersection $\mathcal{P}_Y = \text{Nef}(Y) \cap \mathcal{H}_Y$ is a polytope with faces corresponding to $E \in \mathcal{E}$. For $E \in \mathcal{E}$, let $\mathcal{P}_E \subset \mathcal{H}_Y$ be the convex hull of the vector $\frac{1}{d}(-K_Y)$ and the face $\mathcal{P}_Y \cap E^\perp$. (Note that $-K_Y$ is nef by the definition of generalized Del Pezzo surface and $\frac{1}{d}(-K_Y)$ is in \mathcal{P}_Y since $\langle -K_Y, -K_Y \rangle = d$.) Then

$$\mathcal{P}_Y = \text{Nef}(Y) \cap \mathcal{H}_Y = \bigcup_{E \in \mathcal{E}} \mathcal{P}_E.$$

The intersection of any two of the \mathcal{P}_E has volume zero in \mathcal{H}_Y because the intersection lies in a subspace of dimension strictly less than that of \mathcal{H}_Y . Therefore,

$$\alpha(Y) = \text{Vol}(\mathcal{P}_Y) = \sum_{E \in \mathcal{E}} \text{Vol}(\mathcal{P}_E).$$

For each (-2) -curve E , $\langle K_Y, E \rangle = 0$ and hence $\frac{1}{d}(-K_Y) \in E^\perp$. Thus \mathcal{P}_E lies in the hyperplane $\mathcal{H}_Y \cap E^\perp$ of dimension $\dim(\mathcal{H}_Y) - 1$, and so \mathcal{P}_E has volume zero. We thus reduce to

$$\text{Vol}(\mathcal{P}_Y) = \sum_{E \in \mathcal{C}} \text{Vol}(\mathcal{P}_E).$$

For $E \in \mathcal{C}$, let $\pi_E : Y \rightarrow Y_E$ be the contraction. By Lemma 4.1 we have $\pi_E^* \mathcal{H}_{Y_E} = \mathcal{H}_Y \cap E^\perp$. This identifies the base of the cone \mathcal{P}_E as $\mathcal{P}_Y \cap E^\perp = \pi_E^* \mathcal{P}_{Y_E}$. Thus \mathcal{P}_E is a cone of dimension $9 - d$ with height $\frac{1}{d}$ and base volume $\text{Vol}(\pi_E^* \mathcal{P}_{Y_E})$. By Lemma 4.1, the sublattices

$N^1(Y_E) \subset N^1(Y_E)_{\mathbb{R}}$ and $\pi_E^*(N^1(Y_E)) = E^\perp \cap N^1(Y) \subset E^\perp$ are isomorphic, so π_E^* is volume-preserving and $\text{Vol}(\pi_E^*\mathcal{P}_{Y_E}) = \text{Vol}(\mathcal{P}_{Y_E}) = \alpha(Y_E)$. Consequently,

$$\text{Vol}(\mathcal{P}_E) = \frac{1}{d(9-d)} \text{Vol}(\mathcal{P}_{Y_E}) = \frac{1}{d(9-d)} \alpha(Y_E).$$

Summing over $E \in \mathcal{C}$ gives the desired result. \square

Remark 4.2. This generalization explains why Theorem 1.1 does not hold for $d = 7$. When one blows down a (-1) -curve on an ordinary Del Pezzo surface of degree d for $d \leq 7$ the result is an ordinary Del Pezzo surface of degree $d + 1$. For $d \leq 6$, the resulting ordinary Del Pezzo surfaces all have the same nef cone volume. This is no longer true when $d = 7$. Let X_d denote an ordinary Del Pezzo surface of degree d obtained by blowing up $9 - d$ points in general position on \mathbb{P}^2 . Recall that $X_7 = \text{Bl}_{p,q}(\mathbb{P}^2)$ contains three (-1) -curves: the exceptional divisors E_p and E_q , and the proper transform L_{pq} of the line through p and q . Contracting E_p or E_q results in an X_8 , while contracting L_{pq} results in $\mathbb{P}^1 \times \mathbb{P}^1$. We have

$$\alpha(X_7) = \frac{1}{14}(2\alpha(X_8) + \alpha(\mathbb{P}^1 \times \mathbb{P}^1)) = \frac{1}{24}$$

since $\alpha(X_8) = 1/6$ and $\alpha(\mathbb{P}^1 \times \mathbb{P}^1) = 1/4$.

5. ROOT SYSTEMS AND WEYL GROUPS

In this section, we recall some of the basic facts about the root system of (-2) -classes on a Del Pezzo surface and its associated Weyl group. We use this structure in our second main result which relates the nef cone volumes of split generalized and ordinary Del Pezzo surfaces of the same degree.

5.1. Root systems.

Definition 5.1. A **root system** R is a finite collection of non-zero vectors in a finite-dimensional real vector space V with a non-degenerate definite inner product $\langle \cdot, \cdot \rangle$ satisfying the following conditions.

- (1) The set R spans V , i.e. R is essential.
- (2) For each $x \in R$, let $s_x : V \rightarrow V$ be the reflection through the hyperplane orthogonal to x :

$$s_x(v) = v - 2 \frac{\langle x, v \rangle}{\langle x, x \rangle} x.$$

For each $x \in R$, it is required that s_x takes R to R .

- (3) For every $x_1, x_2 \in R$,

$$2 \frac{\langle x_1, x_2 \rangle}{\langle x_2, x_2 \rangle}$$

is an integer, i.e., R is crystallographic.

- (4) If $x \in R$ and $cx \in R$, then $c \in \{1, -1\}$, i.e., R is reduced.

Definition 5.2. A morphism of root systems from $R \subset V$ to $R' \subset V'$ is a linear map $\Phi : V \rightarrow V'$ such that

- (1) $\Phi(R) \subset R'$, and
- (2) Φ preserves inner products up to a scalar multiple, i.e., there is a $c \in \mathbb{R}$ such that $\langle \Phi(x), \Phi(y) \rangle = c \cdot \langle x, y \rangle$. Equivalently, the integers $2\langle x_1, x_2 \rangle / \langle x_2, x_2 \rangle$ are preserved for all $x_1, x_2 \in R$.

Remark 5.3. We will sometimes refer to a root system R in a vector space V even when R does not span V . Strictly speaking, R is only a root system in the subspace it spans, but this minor abuse of language should not cause any confusion.

We recall some standard notions; for details, see [Hum90, Section 1.3], [Bou02, Section VI.1.2], [Hal03, Chapter 8]. Any hyperplane in V not containing any root of R divides R into two subsets, of **positive roots** on one side (and negative roots on the other side). Those positive roots which cannot be written as a sum of other positive roots with positive coefficients form a set of **simple roots**. Each set of simple roots (for each choice of a set of positive roots) is a linearly independent set such that every root in R is either a sum of simple roots with non-negative coefficients, or a sum of simple roots with non-positive coefficients.

A decomposition of R is a disjoint union $R = R_1 \cup \dots \cup R_k$ such that the span of R is the direct sum of the spans of the R_j , each R_j is a root system in its span, and the spans of the R_j are orthogonal to each other. If R admits no non-trivial decomposition, then R is an **irreducible root system**. If R is reducible, it has a unique (up to order) decomposition into irreducible root systems, called the **irreducible components** of R .

Recall the classification of root systems by **Dynkin diagrams**. For a root system R and a choice of a set R_0 of simple roots in R , the Dynkin diagram of R is the graph with vertex set R_0 and an edge joining two vertices if and only if the corresponding roots are not perpendicular. One labels the edges of the graph according to the angle between the roots and their relative length; for details, see [Bou02]. The Dynkin diagram is independent of the choice of a set of simple roots. The irreducible root systems correspond to connected graphs. The irreducible components of a reducible root system R correspond exactly to the connected components of the Dynkin diagram of R . One has the well-known classification of irreducible root systems corresponding to Dynkin diagrams of types \mathbf{A}_n , $n \geq 1$; \mathbf{B}_n , $n \geq 2$; \mathbf{C}_n , $n \geq 3$; \mathbf{D}_n , $n \geq 4$; \mathbf{E}_n , $6 \leq n \leq 8$; \mathbf{F}_4 ; \mathbf{G}_2 .

The group of orthogonal transformations generated by all $s_x, x \in R$, is finite and is called the **Weyl group** $W(R)$. A **wall** in V is a hyperplane orthogonal to an $x \in R$. Removing the walls from V leaves a finite set of open convex cones called **chambers**. The action of $W(R)$ permutes these chambers simply transitively.

Table 2 lists all of the **simply laced** root systems (those in which all roots have the same self-intersection) and the orders of their Weyl groups. Table 3 gives the same data for the non-simply laced root systems.

root system R	\mathbf{A}_n	\mathbf{D}_n	\mathbf{E}_6	\mathbf{E}_7	\mathbf{E}_8
$\#W(R)$	$(n+1)!$	$2^{n-1} \cdot n!$	$2^7 \cdot 3^4 \cdot 5$	$2^{10} \cdot 3^4 \cdot 5 \cdot 7$	$2^{14} \cdot 3^5 \cdot 5^2 \cdot 7$

TABLE 2. The orders of simply laced Weyl groups

root system R	\mathbf{B}_n	\mathbf{C}_n	\mathbf{F}_4	\mathbf{G}_2
$\#W(R)$	$2^n \cdot n!$	$2^n \cdot n!$	$2^7 \cdot 3^2$	$2^2 \cdot 3$

TABLE 3. The orders of non-simply laced Weyl groups

5.2. Root systems on Del Pezzo surfaces. Let Y be a split generalized Del Pezzo surface of degree $d \leq 7$. By [Man86, Sections 23–25], the finite set R_d of (-2) -classes on Y is a root system in $N^1(Y)_{\mathbb{R}}$ and of course depends only on the degree d . For $d \leq 6$, the roots span the hyperplane $(-K_Y)^\perp$. The classification of this root system is shown in Table 4.

d	7	6	5	4	3	2	1
R_d	\mathbf{A}_1	$\mathbf{A}_1 \times \mathbf{A}_2$	\mathbf{A}_4	\mathbf{D}_5	\mathbf{E}_6	\mathbf{E}_7	\mathbf{E}_8

TABLE 4. Classification of root systems R_d

Not only is R_d a root system, but in fact the subset of (-2) -classes that are effective on Y gives rise to a root system, [Dem77, Theorem III.2.b]:

Theorem 5.4 (Demazure). *Let Y be a split generalized Del Pezzo surface of degree $d \leq 6$ and let R_Y^+ be the set of effective (-2) -classes on Y . Then $R_Y := R_Y^+ \cup -R_Y^+$ is a root system in $N^1(Y)$ whose simple roots are the (-2) -curves of Y and whose positive roots are R_Y^+ . It is contained in R_d . \square*

Remark 5.5. Urabe [Ura83, Main Theorem] has shown that every root system contained in R_d occurs as the root system R_Y of a generalized Del Pezzo surface Y of degree d as in Theorem 5.4, with four exceptions: the subsystem of type $7\mathbf{A}_1$ in R_2 and the subsystems of type $7\mathbf{A}_1$, $8\mathbf{A}_1$, and $\mathbf{D}_4 + 4\mathbf{A}_1$ in R_1 .

Remark 5.6. The root system R_Y can have irreducible components of the following types:

$$\mathbf{A}_1, \dots, \mathbf{A}_8, \mathbf{D}_4, \dots, \mathbf{D}_8, \mathbf{E}_6, \mathbf{E}_7, \mathbf{E}_8.$$

For Y of degree $d \geq 3$, consider the anticanonical morphism ϕ defined by the linear series $|-K_Y|$ which maps Y to a projective space of dimension d . For $d = 2$ (resp. $d = 1$), let ϕ be the morphism defined by the linear series $|-2K_Y|$ (resp. $|-3K_Y|$). Let Y' be the image of Y under ϕ . The map ϕ sends the union of (-2) -curves corresponding to any connected component of the Dynkin diagram to a singularity of Y' , while it is an isomorphism between the complement of the (-2) -curves on Y and the complement of the singularities on Y' .

Each singularity on Y' is a rational double point. Its type in the **ADE**-classification is given by the type of the corresponding irreducible Dynkin diagram. The surface Y' is a singular Del Pezzo surface, whose minimal desingularization is the generalized Del Pezzo surface Y .

5.3. Weyl groups and nef cone volume. We proceed with the proof of our second main result, which we repeat here for the convenience of the reader.

Theorem 1.3. *Let Y be a split generalized Del Pezzo surface of degree $d \leq 7$ and let X be a split ordinary Del Pezzo surface of the same degree. Then*

$$\alpha(Y) = \frac{1}{\#W(R_Y)} \alpha(X),$$

where $W(R_Y)$ is the Weyl group of the root system R_Y whose simple roots are the (-2) -curves on Y .

Proof. Identify $N^1(X)$ and $N^1(Y)$ as in Lemma 3.3.

With notation as in the statement of Theorem 5.4, let C be the open convex cone in $N_1(Y)_{\mathbb{R}}$ dual to the cone spanned by the (-2) -curves of Y . That is,

$$C = \{v \in N_1(Y)_{\mathbb{R}} : \langle v, x \rangle > 0 \text{ for all } (-2)\text{-curves } x \text{ on } Y\}.$$

Since the (-2) -curves are a system of simple roots of R_Y , C is a single chamber for the Weyl group $W(R_Y)$. Recall that by Corollary 3.12, $\text{Nef}(Y) = \text{Nef}(X) \cap \overline{C}$. Intersecting with the hyperplane \mathcal{H}_X gives $\mathcal{P}_Y = \overline{C} \cap \mathcal{P}_X$.

We have

$$N^1(X)_{\mathbb{R}} = \bigcup_{w \in W(R_Y)} \overline{wC},$$

so

$$\mathcal{P}_X = \bigcup_{w \in W(R_Y)} (\overline{wC} \cap \mathcal{P}_X).$$

The sets $\overline{wC} \cap \mathcal{P}_X$, $w \in W(R_Y)$, are pairwise disjoint except along boundaries, which have zero volume. The action of $W(R_Y)$ preserves volume and fixes $\text{Nef}(X)$ and $-K_X$. Therefore it fixes \mathcal{P}_X , and we have

$$\begin{aligned} \alpha(X) = \text{Vol}(\mathcal{P}_X) &= \sum_{w \in W(R_Y)} \text{Vol}(\overline{wC} \cap \mathcal{P}_X) = \#(W(R_Y)) \cdot \text{Vol}(\overline{C} \cap \mathcal{P}_X) \\ &= \#(W(R_Y)) \cdot \text{Vol}(\mathcal{P}_Y) = \#(W(R_Y)) \cdot \alpha(Y). \end{aligned}$$

This completes the proof. \square

Remark 5.7. As in Remark 5.6, let Y' be the singular Del Pezzo surface whose minimal desingularization is Y . The number $\#W(R_Y)$, and therefore $\alpha(Y)$, can be determined directly from the types of singularities on Y' as follows: The types R of the singularities of Y' coincide with the types of the irreducible components of R_Y . The numbers of elements of their Weyl groups $W(R)$ can be found in Table 2. Their product is $\#W(R_Y)$.

6. NON-SPLIT GENERALIZED DEL PEZZO SURFACES

We recall some facts about the geometry of generalized Del Pezzo surfaces that are not split and then introduce the notion of orbit root systems. The results collected here will be used in Section 7 to relate the nef cone volume of non-split generalized Del Pezzo surfaces to the nef cone volume of ordinary Del Pezzo surfaces.

6.1. The Galois action. Throughout this section, we let Y be a generalized Del Pezzo surface of degree $d \leq 7$ defined over a perfect field \mathbb{K} and we assume that Y contains a \mathbb{K} -rational point; we let $\bar{Y} = Y \times_{\mathbb{K}} \bar{\mathbb{K}}$. The Galois group $G_{\mathbb{K}} = \text{Gal}(\bar{\mathbb{K}}/\mathbb{K})$ acts on $N^1(\bar{Y})$, and each automorphism of $N^1(\bar{Y})$ induced by an element of $G_{\mathbb{K}}$ preserves both the intersection form and the anticanonical class.

Proposition 6.1. *The group of automorphisms of $N^1(\bar{Y})$ which preserve the intersection form $\langle \cdot, \cdot \rangle$ and the anticanonical class $-K_Y$ is canonically isomorphic to $W(R_d)$.*

Proof. The result for ordinary Del Pezzo surfaces can be found in [Man86, Theorem 23.9]. (The statement there is given only for $d \leq 6$, but the $d = 7$ case is an easy calculation.) The result holds for generalized Del Pezzo surfaces via the identification described in Lemma 3.3. \square

Thus the action of $G_{\mathbb{K}}$ factors through (a subgroup of) the finite group $W(R_d)$.

Proposition 6.2. *Let Y be a generalized Del Pezzo surface defined over the field \mathbb{K} containing a \mathbb{K} -rational point. Then $N^1(Y) = N^1(\bar{Y})^{G_{\mathbb{K}}}$.*

Recall that if S is a set on which the group G acts, the standard notation $S^G = \{s : gs = s, \forall g \in G\}$ denotes the set of fixed points of the action.

Proof. A result of Colliot-Thélène and Sansuc [CTS87, Theorem 2.1.2, Claim (iii)] assures that under the hypotheses of the proposition, $\text{Pic}(Y) = \text{Pic}(\bar{Y})^{G_{\mathbb{K}}}$. Since the intersection form on $\text{Pic}(\bar{Y})$ is non-degenerate, we have $\text{Pic}(\bar{Y}) = N^1(\bar{Y})$. Finally, to show $N^1(Y) = \text{Pic}(Y)$ it suffices to prove that a divisor is numerically trivial on $\text{Pic}(\bar{Y})$ if it is numerically trivial on $\text{Pic}(Y)$.

Suppose $D \in \text{Pic} Y$ is numerically trivial in $\text{Pic} Y$. Let E be any divisor class on \bar{Y} . Recall that the action of $G_{\mathbb{K}}$ on $N^1(\bar{Y})$ factors through the finite Weyl group $W(R_d)$, so the $G_{\mathbb{K}}$ -orbit of E is finite. Say this orbit is $\{E_1, \dots, E_s\}$. Since $G_{\mathbb{K}}$ preserves the intersection form on \bar{Y} and D is $G_{\mathbb{K}}$ -invariant,

$$\langle D, E \rangle = \frac{1}{s} \sum_i \langle D, E_i \rangle = \langle D, \frac{1}{s} \sum_i E_i \rangle = 0$$

because $(1/s) \sum E_i$ lies in $(\text{Pic} \bar{Y})^{G_{\mathbb{K}}} = \text{Pic} Y$.

Putting together the above results, $N^1(Y) = \text{Pic}(Y) = \text{Pic}(\bar{Y})^{G_{\mathbb{K}}} = N^1(\bar{Y})^{G_{\mathbb{K}}}$, proving the proposition. \square

We now explain the relation between the effective cone of Y and that of \bar{Y} .

Proposition 6.3. *The effective cone of Y is equal to the cone of $G_{\mathbb{K}}$ -invariant effective classes of \bar{Y} , i.e.,*

$$\text{Eff}^1(Y) = \text{Eff}^1(\bar{Y})^{G_{\mathbb{K}}}$$

Proof. By Proposition 6.2, we have $N^1(Y) = N^1(\bar{Y})^{G_{\mathbb{K}}}$. It is clear that $\text{Eff}^1(Y) \subseteq \text{Eff}^1(\bar{Y})^{G_{\mathbb{K}}}$. To show the reverse inclusion, first note that if D is any effective divisor on $\text{Eff}^1(\bar{Y})$, letting \mathbb{L} be a finite Galois extension of \mathbb{K} over which D is defined, then $\sum_{\sigma \in \text{Gal}(\mathbb{L}/\mathbb{K})} \sigma(D) \in \text{Eff}^1(Y)$. For any $D \in \text{Eff}^1(\bar{Y})^{G_{\mathbb{K}}}$ that is defined over a finite Galois extension \mathbb{L}/\mathbb{K} , we have

$$D = \frac{1}{\#\text{Gal}(\mathbb{L}/\mathbb{K})} \sum_{\sigma \in \text{Gal}(\mathbb{L}/\mathbb{K})} \sigma(D).$$

This completes the proof. \square

The action of $G_{\mathbb{K}}$ on $N^1(\bar{Y})$ induces an action both on the set of (-1) -curves and on the set of (-2) -curves.

Corollary 6.4. *A set of generators for the effective cone of Y consists of, for each orbit of $G_{\mathbb{K}}$ on the sets of (-1) -curves and (-2) -curves, the sum of the classes in that orbit. \square*

Note that this set of generators may fail to be minimal. (See rows 3, 6 and 9 of Table 8 for examples.)

6.2. Orbit Root Systems. We will use the following construction in Section 7.1 in the case of $G_{\mathbb{K}}$ acting on the root system $R_{\bar{Y}} \subset N^1(\bar{Y})$ (as in Theorem 5.4) of \bar{Y} , in order to obtain a root system in $N^1(Y)$.

Definition 6.5. Let $R \subset V$ be a possibly reducible root system with a chosen set Π of positive roots. Suppose a group G acts linearly on V in such a way that it permutes the elements of R , preserves the inner product between elements of R and preserves positivity. In this case, we say that G acts on R .

The set

$$\mathcal{O}(R, G) := \left\{ \sum_{x \in \mathcal{O}} x : \mathcal{O} \text{ is a } G\text{-orbit of an element of } R \right\}$$

is called the **orbit root system** of R with respect to G . (We show below that $\mathcal{O}(R, G)$ is indeed a root system.)

Proposition 6.6. *Let $R \subset V$ be an irreducible root system with a chosen positive system Π . Suppose G acts on R . Then $\mathcal{O}(R, G)$ is an irreducible root system as in Table 5. The simple (resp. positive) roots of $\mathcal{O}(R, G)$ are the sums of elements of orbits of simple (resp. positive) roots of R .*

Proof. Any group action which preserves inner products and positivity must necessarily act as an automorphism of the Dynkin diagram. Indeed, the group takes non-simple roots to non-simple roots, and thus takes simple roots to simple roots. Thus the group acts on the

R	G	$\mathcal{O}(R, G)$
\mathbf{A}_{2n}	$\mathbb{Z}/2\mathbb{Z}$	\mathbf{B}_n
\mathbf{A}_{2n+1}	$\mathbb{Z}/2\mathbb{Z}$	\mathbf{B}_{n+1}
\mathbf{D}_n	$\mathbb{Z}/2\mathbb{Z}$	\mathbf{C}_{n-1}
\mathbf{D}_4	$\mathbb{Z}/3\mathbb{Z}$ or \mathfrak{S}_3	\mathbf{G}_2
\mathbf{E}_6	$\mathbb{Z}/2\mathbb{Z}$	\mathbf{F}_4

TABLE 5. Non-trivial irreducible orbit root systems

vertices of the Dynkin diagram; since the edges (and edge labelings) are determined by the inner product, they are preserved by the group. We check case by case that all non-trivial admissible group actions on irreducible Dynkin diagrams are listed in Table 5. In each case, a direct calculation shows that $\mathcal{O}(R, G)$ is indeed a root system of the listed type. \square

We note that a list similar to Table 5 has been compiled by Kac in [Kac90, Propositions 7.9, 7.10]. The main difference between our list and Kac's is that we use the sum of roots in an orbit, while he uses the average; because of this difference Kac's approach sometimes gives the dual root system to ours.

Lemma 6.7. *Let $R \subset V$ be a possibly reducible root system with a chosen positive system Π . Suppose G acts on R . Then G acts on the irreducible components of R in the following sense. Let $R = \bigcup_{i=1}^n R_i$ be a decomposition of R into irreducible components. Then for any i and any $g \in G$, $g(R_i)$ is one of the irreducible components R_j .*

Proof. One way to see this is by considering the Dynkin diagram D of R . Each component R_i corresponds to a connected component of the graph D . As noted above, the group G acts as a graph automorphism of D . Then each element of G must take connected components of D to connected components. \square

To avoid confusion between the actions of G on R and on the set of irreducible components of R , we refer to orbits in the latter set as ‘‘component orbits’’.

Proposition 6.8. *Let $R \subset V$ be a possibly reducible root system with a chosen positive system Π . Suppose G acts on R . Let R_1, \dots, R_k be irreducible components of R which form a set of component orbit representatives, that is, each component orbit contains exactly one of the R_i . For each i , let $G_i \subset G$ be the subgroup fixing R_i . Then $\mathcal{O}(R, G)$ is a root system and*

$$\mathcal{O}(R, G) \cong \bigcup_{i=1}^k \mathcal{O}(R_i, G_i). \quad (*)$$

Proof. First, note the right-hand side is indeed a root system. For by Proposition 6.6, each $\mathcal{O}(R_i, G_i)$ is a root system contained in the subspace spanned by R_i (since each element of $\mathcal{O}(R_i, G_i)$ is a sum of one or more elements of R_i). Then if $i \neq j$, by assumption R_i and

R_j are distinct irreducible components of R , so they span perpendicular subspaces of V . Therefore $\mathcal{O}(R_i, G_i)$ and $\mathcal{O}(R_j, G_j)$ are perpendicular. Hence the union on the right-hand side of (*) is a perpendicular union of root systems.

Now, the spans of the component orbits are pairwise perpendicular, so we may treat them separately. We consider the orbit $i = 1$, the others being similar. Let the component orbit of R_1 consist of the components $R_{1,1} = R_1, R_{1,2}, \dots, R_{1,p}$. Choosing elements $g_1 = \text{id}_G, g_2, \dots, g_p \in G$ such that $g_i R_1 = R_{1,i}$ for each i , we get isomorphisms

$$(\text{span } R_1, R_1) \cong (\text{span } R_{1,2}, R_{1,2}) \cong \dots \cong (\text{span } R_{1,p}, R_{1,p}).$$

Under this identification we have an isomorphism of the diagonal

$$\Delta \subset (\text{span } R_1)^p \cong (\text{span } R_{1,1}) \oplus \dots \oplus (\text{span } R_{1,p})$$

with $\text{span}(R_1)$ by projection onto the first factor. Note that this projection preserves angles and ratios of lengths, but divides all lengths by a factor of \sqrt{p} . One can check that the projection takes

$$\mathcal{O}(R_{1,1} \cup \dots \cup R_{1,p}, G)$$

to $\mathcal{O}(R_1, G_1)$, as desired.

More precisely, if \mathcal{O} is the orbit of $r \in R_1$ under G_1 , then $\mathcal{O} \cup g_2 \mathcal{O} \cup \dots \cup g_p \mathcal{O}$ is the orbit of r under G . Then $g_i \sum_{x \in \mathcal{O}} x = \sum_{x \in g_i \mathcal{O}} x$ is an element of $\mathcal{O}(R_{1,i}, g_i G_1 g_i^{-1})$ where $g_i G_1 g_i^{-1}$ is the subgroup of G fixing $R_{1,i}$, while $\sum_{i=1}^p g_i \sum_{x \in \mathcal{O}} x$ is an element of $\mathcal{O}(R_{1,1} \cup \dots \cup R_{1,p}, G)$, which lies in Δ . It is projected to the element $\sum_{x \in \mathcal{O}} x$ of $\mathcal{O}(R_1, G_1)$. \square

Corollary 6.9. *In the setting of Proposition 6.8,*

$$W(\mathcal{O}(R, G)) \cong \prod_{i=1}^k W(\mathcal{O}(R_i, G_i)).$$

\square

7. NEF CONE VOLUME OF NON-SPLIT GENERALIZED DEL PEZZO SURFACES

Let Y be a non-split generalized Del Pezzo surface of degree at most 7, defined over a perfect field \mathbb{K} . As in Section 6 we continue to assume that Y contains a \mathbb{K} -rational point. Then $G_{\mathbb{K}} = \text{Gal}(\overline{\mathbb{K}}/\mathbb{K})$ acts on the set of (-2) -curves on \overline{Y} and on the associated root system. In this situation, we can construct an orbit root system as in Definition 6.5. As in the split case (Theorem 1.3), this allows us to relate the nef cone volume of Y to a volume associated to an ordinary Del Pezzo surface of the same degree. In Section 7.2, we compute this volume for all non-split Del Pezzo surface of degree at least 5.

7.1. Nef cone volume of pairs. Using Proposition 6.1, we can associate to a generalized Del Pezzo surface Y of degree $d \leq 7$ the pair (\overline{Y}, H_Y) , where $H_Y \subset W(R_d)$ is the image of $G_{\mathbb{K}}$ under the homomorphism

$$G_{\mathbb{K}} \rightarrow \text{Aut}(N^1(\overline{Y}), \langle \cdot, \cdot \rangle, -K_Y) \cong W(R_d).$$

Note that $G_{\mathbb{K}}$ and therefore also H_Y acts on the set of (-2) -curves on \bar{Y} and also on the set of its (-1) -curves.

Remark 7.1. To every generalized Del Pezzo surface Y over \mathbb{K} there is the associated pair (\bar{Y}, H_Y) , as described above. The “realization problem for pairs” is to describe which pairs (\bar{Y}, H) are obtained in this manner. That is, for which pairs (\bar{Y}, H) , consisting of a split generalized Del Pezzo surface \bar{Y} over $\bar{\mathbb{K}}$ of degree d and a subgroup $H \subset W(R_d)$ acting on the set of (-2) -curves, is there a Y defined over \mathbb{K} such that $\bar{Y} = Y \times_{\mathbb{K}} \bar{\mathbb{K}}$ and $H = H_Y$ is the image of $G_{\mathbb{K}}$ in $W(R_d)$?

Corn has shown that every pair (\bar{X}, H) , with \bar{X} a split *ordinary* Del Pezzo surface of degree 6 and $H \subset W(R_6)$ arbitrary, is realizable in the above sense [Cor05, Theorem 5.1].

We use pairs to circumvent this realization problem. This allows us to prove comparison theorems without having to address realization (see Corollary 7.5).

We now define the nef cone volume $\alpha(Y, H)$ of a pair (Y, H) where Y is any *split* generalized Del Pezzo surface of degree $d \leq 7$ and H is any subgroup of $W(R_d)$ that acts on the set of (-2) -curves on Y . It follows from Lemma 3.8 that H also acts on the set of (-1) -curves. Note that there is no restriction on H if Y is ordinary.

For such a pair (Y, H) , define $N^1(Y, H)$ to be $N^1(Y)^H$. Motivated by Corollary 6.4, we define $\text{Eff}^1(Y, H)$ to be the cone in $N^1(Y, H)_{\mathbb{R}}$ generated by the sum of the classes in each orbit of H acting on the sets of (-1) -curves and (-2) -curves of Y . We naturally get a dual cone

$$\text{Nef}(Y, H) := \{C \in N^1(Y, H)_{\mathbb{R}} : \langle D, C \rangle \geq 0 \forall D \in \text{Eff}^1(Y, H)\}.$$

We then have the hyperplane

$$\mathcal{H}_{Y,H} := \{C \in N^1(Y, H)_{\mathbb{R}} : \langle C, -K_Y \rangle = 1\}$$

and the polytope

$$\mathcal{P}_{Y,H} := \text{Nef}(Y, H) \cap \mathcal{H}_{Y,H}.$$

And so we define

$$\alpha(Y, H) := \text{Vol}(\mathcal{P}_{Y,H}),$$

with respect to the Leray measure $d\mu$ defined in the analogous manner to the way it was defined in Section 2.

It is immediate from Proposition 6.2 and Corollary 6.4 that if Y is any generalized Del Pezzo surface (not necessarily split), then

$$\alpha(Y) = \alpha(\bar{Y}, H_Y).$$

Lemma 7.2. *Assume that Y is split and let H_1, H_2 be two conjugate subgroups in $W(R_d)$. Then*

$$\alpha(Y, H_1) = \alpha(Y, H_2).$$

Proof. Let $w \in W(R_d)$ be such that $H_2 = wH_1w^{-1}$. Let $\mathcal{O}_i, i \in I$, denote the orbits of the (-1) - and (-2) -classes under H_1 . By definition, $\text{Eff}^1(Y, H_1)$ is generated by the sums $\sum_{D \in \mathcal{O}_i} D, i \in I$. A simple calculation shows that the orbits of these classes under H_2 are given by $w\mathcal{O}_i, i \in I$. We have

$$\alpha(Y, H_1) = \text{Vol} \left(\left\{ C \in N^1(Y, H_1)_{\mathbb{R}} : \langle -K_Y, C \rangle = 1, \langle C, \sum_{D \in \mathcal{O}_i} D \rangle \geq 0 \forall i \in I \right\} \right).$$

Making use of the fact that elements of $W(R_d)$ preserve the intersection form and anti-canonical class and noting that elements of $W(R_d)$ are orthogonal transformations and thus preserve volumes, we compute

$$\begin{aligned} \alpha(Y, H_2) &= \text{Vol} \left(\left\{ C \in N^1(Y, H_2)_{\mathbb{R}} : \langle -K_Y, C \rangle = 1, \langle C, \sum_{D \in \mathcal{O}_i} wD \rangle \geq 0 \forall i \in I \right\} \right) \\ &= \text{Vol} \left(\left\{ C \in N^1(Y, H_2)_{\mathbb{R}} : \langle -K_Y, w^{-1}C \rangle = 1, \langle w^{-1}C, \sum_{D \in \mathcal{O}_i} D \rangle \geq 0 \forall i \in I \right\} \right) \\ &= \text{Vol} \left(w \left\{ C \in N^1(Y, H_1)_{\mathbb{R}} : \langle -K_Y, C \rangle = 1, \langle C, \sum_{D \in \mathcal{O}_i} D \rangle \geq 0 \forall i \in I \right\} \right) \\ &= \alpha(Y, H_1). \end{aligned}$$

This completes the proof. \square

Corollary 7.3. *Let Y_1 and Y_2 be two generalized Del Pezzo surfaces of degree $d \leq 7$, defined over a perfect field \mathbb{K} , which are geometrically isomorphic, i.e., $\overline{Y}_1 \cong \overline{Y}_2$. Let H_1 and H_2 denote the images of $G_{\mathbb{K}}$ under the respective homomorphisms $G_{\mathbb{K}} \rightarrow W(R_d)$. If H_1 and H_2 are conjugate in $W(R_d)$, then $\alpha(Y_1) = \alpha(Y_2)$. \square*

We arrive at the following analogue of Theorem 1.3. That theorem provided a comparison between the nef cone volumes of a split generalized Del Pezzo surface and of a split ordinary Del Pezzo surface of the same degree. The following theorem generalizes this to the nef cone volumes of pairs.

Theorem 7.4. *Let Y be a split generalized Del Pezzo surface of degree $d \leq 7$, X a split ordinary Del Pezzo surface of the same degree, and H a subgroup of $W(R_d)$ acting on the set of (-2) -curves on Y . Let R_Y be the root system whose simple roots are the (-2) -curves on Y , and let $\mathcal{O}(R_Y, H)$ be the orbit root system associated to the action of H on R_Y as in Definition 6.5. Then*

$$\alpha(Y, H) = \frac{\alpha(X, H)}{\#\mathcal{O}(R_Y, H)}.$$

Proof. The proof of this theorem is a generalization of the argument that proves Theorem 1.3. Using Lemma 3.3, we identify $N^1(X)$ and $N^1(Y)$. This gives an identification of $N^1(X, H)$

and $N^1(Y, H)$. As before, $\text{Nef}(Y, H)$ is the intersection of $\text{Nef}(X, H)$ with the closure of a chamber defined by the simple roots of $\mathcal{O}(R_Y, H)$. As in the proof of Theorem 1.3, the chambers of the Weyl group $W(\mathcal{O}(R_Y, H))$ intersect only along boundaries, which have zero volume. They fill $N^1(Y)$. There are $\#W(\mathcal{O}(R_Y, H))$ of the chambers. From here, the proof is completed by the same steps as in the proof of Theorem 1.3. \square

We arrive at our third main result, the computation of the nef cone volume of a generalized Del Pezzo surface over an arbitrary perfect field:

Corollary 7.5. *Let Y be a generalized Del Pezzo surface of degree $d \leq 7$ over the perfect field \mathbb{K} and X a split ordinary Del Pezzo surface of the same degree. Let $\bar{Y} = Y \times_{\mathbb{K}} \bar{\mathbb{K}}$, and identify $N^1(\bar{Y})$ with $N^1(X)$ as in Lemma 3.3. Let $H_Y \subset W(R_d)$ be the image of $G_{\mathbb{K}}$. Let $R_{\bar{Y}} \subset R_d$ be the root system whose simple roots are (-2) -curves on \bar{Y} . Then*

$$\alpha(Y) = \alpha(\bar{Y}, H_Y) = \frac{\alpha(X, H_Y)}{\#W(\mathcal{O}(R_{\bar{Y}}, H_Y))}.$$

\square

Using Proposition 6.6 and Corollary 6.9, the integer appearing in the denominator is straightforward to compute. This reduces the computation of the nef cone volume of an arbitrary generalized Del Pezzo surface over a non-closed field to the computation of the nef cone volume of a pair involving a split ordinary Del Pezzo surface.

7.2. Pairs involving ordinary Del Pezzo surfaces of high degree. As examples, let us compute $\alpha(X)$ for the various possible non-split ordinary Del Pezzo surfaces X of degree $d \geq 5$.

For $d \geq 7$ there are very few possible non-trivial Galois actions on \bar{X} , and we list these cases briefly:

- (1) In fact there are no non-trivial possibilities with $d = 9$: we must have $\bar{X} \cong \mathbb{P}^2$, the Galois action is trivial, and $\alpha(X) = 1/3$.
- (2) For $d = 8$, the only possible non-trivial form occurs when X is a twist of $\mathbb{P}^1 \times \mathbb{P}^1$ in which the Galois action permutes the two generating rulings. In this case $\alpha(X) = 1/2$.
- (3) For $d = 7$, the only possible non-trivial form occurs when X is the blow-up of two conjugate rational points on \mathbb{P}^2 , so that the Galois action interchanges the points. In this case $\alpha(X) = 1/6$.

For $d = 5, 6$ there are many more cases. For the remainder of this section, let X be a possibly non-split ordinary Del Pezzo surface of degree 5 or 6 defined over a non-closed perfect field \mathbb{K} . Let $\bar{X} = X \times_{\mathbb{K}} \bar{\mathbb{K}}$. As above, we have

$$\alpha(X) = \alpha(\bar{X}, H_X)$$

where H_X is the image of the Galois group in $W(R_d)$. We compute $\alpha(X)$ by finding the values of $\alpha(\bar{X}, H)$ for all subgroups H of $W(R_d)$. (As noted in Remark 7.1, it is not obvious

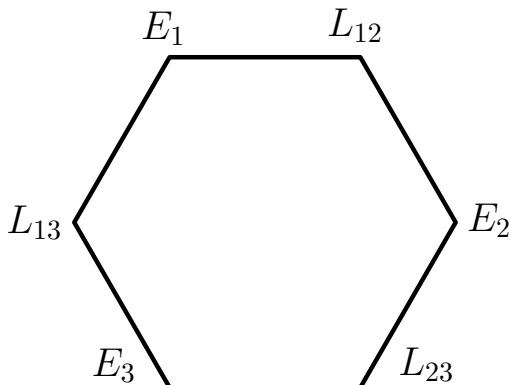


FIGURE 1. Configuration of (-1) -curves on an ordinary Del Pezzo surface of degree 6

which $H \subset W(R_d)$ arise as images of Galois groups, so a priori some values $\alpha(\overline{X}, H)$ might not correspond to any $\alpha(X)$.)

For the case $d = 6$, recall that \overline{X} is obtained by blowing up three non-collinear points in \mathbb{P}^2 and the cone $\text{Eff}^1(\overline{X})$ is minimally generated by the (-1) -curves on \overline{X} . Let E_1, E_2, E_3 denote the exceptional curves and L denote the pullback of a line. The set of (-1) -curves is shown schematically in Figure 1. In this graph, the vertices correspond to the generating classes for $\text{Eff}^1(\overline{X})$, with the convenient shorthand $L_{ij} = L - E_i - E_j$. Two classes intersect if and only if the corresponding vertices in the graph are connected by an edge.

In Table 6, we consider the various subgroups of

$$W(R_6) = W(\mathbf{A}_1) \times W(\mathbf{A}_2) \cong \mathbb{Z}/2\mathbb{Z} \times \mathfrak{S}_3 \cong D_6.$$

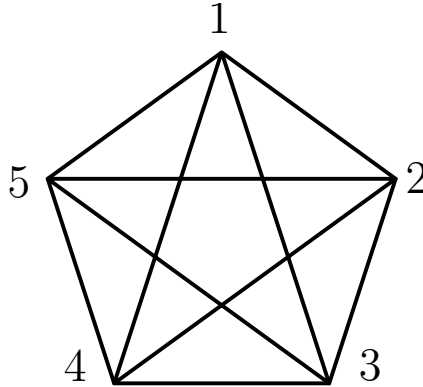
By Lemma 7.2, it suffices to consider subgroups up to conjugacy. For each conjugacy class, we choose a representative subgroup H and give the order $\#H$ of H , the orbit structure of H on the generators of $\text{Eff}^1(\overline{X})$, the rank ρ of $N^1(\overline{X}, H)$, the number m of generators in the minimal generating set of $\text{Eff}^1(\overline{X}, H)$, and finally the nef cone volume $\alpha(\overline{X}, H)$. We describe H in terms of generators, using the generator $s_{123} := s_{L-E_1-E_2-E_3}$ (180° rotation) of $W(\mathbf{A}_1)$ and the generators $s_{12} := s_{E_1-E_2}$ (the flip swapping E_1 and E_2) and $s_{23} := s_{E_2-E_3}$ (the flip swapping E_2 and E_3) of $W(\mathbf{A}_2)$.

Given H , we may compute $\alpha(\overline{X}, H)$ as follows. We explicitly compute the sums of elements in each orbit of the action of H on the generators of $\text{Eff}^1(\overline{X})$, obtaining a set of generators of the cone $\text{Eff}^1(\overline{X}, H)$. We compute the dual cone in $N^1(\overline{X}, H)$, obtaining $\text{Nef}(\overline{X}, H)$. Intersecting with the hyperplane $\mathcal{H}_{\overline{X}, H}$ gives the polytope $\mathcal{P}_{\overline{X}, H}$, whose volume is $\alpha(\overline{X}, H)$.

For the case when $d = 5$, recall that \overline{X} is the blowup of \mathbb{P}^2 at 4 points in general position. Similarly to the case $d = 6$, the cone $\text{Eff}^1(\overline{X})$ is generated by the (-1) -curves E_i for $1 \leq i \leq 4$ and $L_{ij} = L - E_i - E_j$ for $1 \leq i < j \leq 4$.

In Figure 2, we use a different diagram to exhibit the full symmetry of the configuration of these 10 curves with respect to $W(R_5) = \mathfrak{S}_5$. (It seems impossible to make visible all of the

H	$\#H$	Orbit structure	ρ	m	$\alpha(\overline{X}, H)$
$\langle s_{123}, s_{12}, s_{23} \rangle$	12				
$\langle s_{23}s_{123}, s_{23}s_{12} \rangle$	6				
$\langle s_{123}s_{12}s_{23} \rangle$	6	$\{E_1, E_2, E_3, L_{12}, L_{13}, L_{23}\}$	1	1	1
$\langle s_{12}, s_{23} \rangle$	6				
$\langle s_{12}s_{23} \rangle$	3	$\{E_1, E_2, E_3\}, \{L_{12}, L_{13}, L_{23}\}$	2	2	1/3
$\langle s_{123}, s_{23} \rangle$	4	$\{E_1, L_{23}\}, \{E_2, E_3, L_{12}, L_{13}\}$	2	2	1/2
$\langle s_{123}s_{12} \rangle$	2	$\{E_1, L_{12}\}, \{E_2, L_{13}\}, \{E_3, L_{23}\}$	2	2	1/2
$\langle s_{123} \rangle$	2	$\{E_1, L_{23}\}, \{E_2, L_{13}\}, \{E_3, L_{12}\}$	3	3	1/8
$\langle s_{12} \rangle$	2	$\{E_1, E_2\}, \{E_3\}, \{L_{12}\}, \{L_{13}, L_{23}\}$	3	4	1/12
$\langle e \rangle$	1	$\{E_1\}, \{E_2\}, \{E_3\}, \{L_{12}\}, \{L_{13}\}, \{L_{23}\}$	4	6	1/72

TABLE 6. Values of $\alpha(\overline{X}, H)$ for \overline{X} a split ordinary Del Pezzo surface of degree 6FIGURE 2. Configuration of (-1) -curves on an ordinary Del Pezzo surface of degree 5

symmetries when drawing a diagram analogous to Figure 7.2). Here the minimal generators of $\text{Eff}^1(\overline{X})$ correspond to *edges* of the graph, and two generating classes intersect if and only if the corresponding edges do *not* share a common vertex. The action of $W(R_5) = \mathfrak{S}_5$ corresponds to permuting the 5 vertices. Table 7 shows the correspondence between the edges of the diagram and the generating classes, where we use the notation $[ij]$ to indicate the edge connecting vertex i with vertex j .

We note that the enumeration of the conjugacy classes of subgroups of \mathfrak{S}_5 has been made by Götz Pfeiffer and is available online [Pfe07]. Table 8 contains the values of $\alpha(\overline{X}, H)$ for the various possible conjugacy classes of subgroups of \mathfrak{S}_5 .

[12]	[13]	[14]	[15]	[23]	[24]	[25]	[34]	[35]	[45]
E_1	E_2	E_3	E_4	L_{34}	L_{24}	L_{23}	L_{14}	L_{13}	L_{12}

TABLE 7. Correspondence of edges in Figure 2 to generators of the effective cone of an ordinary Del Pezzo surface of degree 5

H	$\#H$	Orbit structure	ρ	m	$\alpha(\overline{X}, H)$
$\langle(12), (12345)\rangle$	120				
$\langle(12)(34), (253)\rangle$	60				
$\langle(1234), (13)(24), (12543)\rangle$	20	$\{E_1, E_2, E_3, E_4, L_{12},$			
$\langle(12)(34), (13542)\rangle$	10	$L_{13}, L_{14}, L_{23}, L_{24}, L_{34}\}$	1	1	1
$\langle(12), (1234)\rangle$	24	$\{E_1, E_2, E_3, L_{14}, L_{24}, L_{34}\},$			
$\langle(123), (12)(34), (14)(23)\rangle$	12	$\{E_4, L_{12}, L_{13}, L_{23}\}$	2	2	2/3
$\langle(12), (34), (345)\rangle$	12				
$\langle(12)(34), (345)\rangle$	6	$\{E_1\}, \{L_{12}, L_{13}, L_{14}\}$			
$\langle(12), (345)\rangle$	6	$\{E_2, E_3, E_4, L_{23}, L_{24}, L_{34}\},$	2	2	1/2
$\langle(12), (123)\rangle$	6	$\{E_1, E_2, L_{34}\}, \{E_3, L_{14}, L_{24}\},$			
$\langle(123)\rangle$	3	$\{E_4, L_{13}, L_{23}\}, \{L_{12}\}$	3	4	5/24
$\langle(12345)\rangle$	5	$\{E_1, E_4, L_{12}, L_{14}, L_{34}\},$			
		$\{E_2, E_3, L_{13}, L_{23}, L_{24}\}$	1	1	1
$\langle(12)(34), (13)(24)\rangle$	4	$\{E_1\}, \{E_2, E_3, L_{24}, L_{34}\},$			
		$\{E_4, L_{23}\}, \{L_{12}, L_{13}\}, \{L_{14}\}$	3	4	1/6
$\langle(12), (34)\rangle$	4	$\{E_1, E_3, L_{14}, L_{34}\}, \{E_2, L_{24}\},$			
		$\{E_4, L_{12}, L_{13}, L_{23}\}$	2	2	2/3
$\langle(12), (34), (13)(24)\rangle$	8	$\{E_1, L_{14}\}, \{E_2, L_{24}\}, \{E_3, L_{34}\},$			
$\langle(1234)\rangle$	4	$\{E_4, L_{12}, L_{13}, L_{23}\}$	2	2	2/3
$\langle(12)(34)\rangle$	2	$\{E_1\}, \{E_2, L_{24}\}, \{E_3, L_{34}\},$			
		$\{E_4, L_{23}\}, \{L_{12}, L_{13}\}, \{L_{14}\}$	3	4	1/6
$\langle(12)\rangle$	2	$\{E_1\}, \{E_2, L_{34}\}, \{E_3, L_{24}\},$			
		$\{E_4, L_{23}\}, \{L_{12}\}, \{L_{13}\}, \{L_{14}\}$	4	7	1/24
$\langle e \rangle$	1	$\{E_1\}, \{E_2\}, \{E_3\}, \{E_4\}, \{L_{12}\},$			
		$\{L_{13}\}, \{L_{14}\}, \{L_{23}\}, \{L_{24}\}, \{L_{34}\}$	6	10	1/144

TABLE 8. Values of $\alpha(\overline{X}, H)$ for \overline{X} a split ordinary Del Pezzo surface of degree 5

REFERENCES

[BDPP] Sébastien Boucksom, Jean-Pierre Demailly, Mihai Paun, and Thomas Peternell, *The pseudo-effective cone of a compact Kähler manifold and varieties of negative Kodaira dimension*, arXiv: math.AG/0405285.

- [BM90] Victor V. Batyrev and Yuri I. Manin, *Sur le nombre des points rationnels de hauteur borné des variétés algébriques*, Math. Ann. **286** (1990), no. 1-3, 27–43.
- [Bou02] Nicolas Bourbaki, *Lie groups and Lie algebras. Chapters 4–6*, Elements of Mathematics (Berlin), Springer-Verlag, Berlin, 2002, Translated from the 1968 French original by Andrew Pressley. MR MR1890629 (2003a:17001)
- [BP04] Victor V. Batyrev and Oleg N. Popov, *The Cox ring of a Del Pezzo surface*, Arithmetic of higher-dimensional algebraic varieties (Palo Alto, CA, 2002) (Boston, MA), Progr. Math., vol. 226, Birkhäuser Boston, 2004, pp. 85–103. MR MR2029863 (2005h:14091)
- [BT95] Victor V. Batyrev and Yuri Tschinkel, *Rational points of bounded height on compactifications of anisotropic tori*, Int. Math. Res. Not. **12** (1995), 591–635.
- [BT96] ———, *Rational points on some Fano cubic bundles*, C. R. Acad. Sci. Paris **323** (1996), no. 1, 41–46.
- [Cor05] Patrick Corn, *Del Pezzo surfaces of degree 6*, Math. Res. Lett. **12** (2005), no. 1, 75–84. MR MR2122732 (2006d:14035)
- [CTS87] Jean-Louis Colliot-Thélène and Jean-Jacques Sansuc, *La descente sur les variétés rationnelles. II*, Duke Math. J. **54** (1987), no. 2, 375–492. MR MR899402 (89f:11082)
- [Dem77] Michel Demazure, *Surfaces de Del Pezzo, $i - v$* , Séminaire sur les Singularités des Surfaces (B. Teissier, ed.), Centre de Mathématiques de l’Ecole Polytechnique, Palaiseau, 1976-1977, pp. 21–69 (French).
- [Der07] Ulrich Derenthal, *On a constant arising in Manin’s conjecture for Del Pezzo surfaces*, Math. Res. Lett. **14** (2007), no. 3, 481–489.
- [Hal03] Brian C. Hall, *Lie groups, Lie algebras, and representations*, Graduate Texts in Mathematics, vol. 222, Springer-Verlag, New York, 2003, An elementary introduction. MR MR1997306 (2004i:22001)
- [Har77] Robin Hartshorne, *Algebraic geometry*, Springer-Verlag, New York, 1977, Graduate Texts in Mathematics, No. 52. MR MR0463157 (57 #3116)
- [Har85] Brian Harbourne, *Blowings-up of \mathbf{P}^2 and their blowings-down*, Duke Math. J. **52** (1985), no. 1, 129–148. MR MR791295 (86m:14026)
- [HS00] Marc Hindry and Joseph Silverman, *Diophantine Geometry: An Introduction*, Graduate Texts in Mathematics, vol. 201, Springer-Verlag, 2000.
- [HT04] Brendan Hassett and Yuri Tschinkel, *Universal torsors and Cox rings*, Arithmetic of higher-dimensional algebraic varieties (Palo Alto, CA, 2002) (Boston, MA), Progr. Math., vol. 226, Birkhäuser Boston, 2004, pp. 149–173. MR MR2029868 (2005a:14049)
- [Hum90] James E. Humphreys, *Reflection groups and Coxeter groups*, Cambridge studies in advanced mathematics, Cambridge University Press, 1990.
- [Kac90] Victor Kac, *Infinite dimensional lie algebras*, third ed., Cambridge University Press, 1990.
- [Laz04] Robert Lazarsfeld, *Positivity in algebraic geometry. II*, Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 49, Springer-Verlag, Berlin, 2004, Positivity for vector bundles, and multiplier ideals. MR MR2095472
- [LH05] Mustapha Lahyane and Brian Harbourne, *Irreducibility of -1 -classes on anticanonical rational surfaces and finite generation of the effective monoid*, Pacific J. Math. **218** (2005), no. 1, 101–114.
- [Man86] Yuri I. Manin, *Cubic forms*, second ed., North-Holland Mathematical Library, vol. 4, North-Holland Publishing Co., Amsterdam, 1986, Algebra, geometry, arithmetic, Translated from the Russian by M. Hazewinkel. MR MR833513 (87d:11037)
- [Pey95] Emmanuel Peyre, *Hauteurs et mesures de Tamagawa sur les variétés de Fano*, Duke Math. J. **79** (1995), no. 1, 101–218. MR MR1340296 (96h:11062)
- [Pfe07] Götz Pfeiffer, *The subgroups of S_5* , <http://schmidt.nuigalway.ie/subgroups/s5.pdf>, 2007.
- [Ura83] Tohsuke Urabe, *On singularities on degenerate del Pezzo surfaces of degree 1, 2*, Singularities, Part 2 (Arcata, Calif. 1981), Proc. Sympos. Pure Math., vol. 40, Amer. Math. Soc., 1983, pp. 587–591.

INSTITUT FÜR MATHEMATIK, UNIVERSITÄT ZÜRICH, WINTERTHURERSTRASSE 190, 8057 ZÜRICH,
SWITZERLAND

E-mail address: `ulrich.derenthal@math.unizh.ch`

DEPARTMENT OF MATHEMATICS, TULANE UNIVERSITY, NEW ORLEANS LA 70118

E-mail address: `mjoyce@math.tulane.edu`

DEPARTMENT OF MATHEMATICS, SLU 10687, HAMMOND, LA 70402

E-mail address: `zzeitler@selu.edu`