PROBLEM SET #1

DUE: THURSDAY, MAY 31

(1) Knapp pages 20-21, problems #2, 5, 7, 8, 9.

(2) Let A be a real or complex algebra, not necessarily associative. A derivation is a linear map $D : A \to A$ satisfying the Leibniz rule:

$$D(ab) = aD(b) + D(a)b.$$

(a) Show that $\mathrm{Der}(A)$, the derivations of A, form a Lie algebra under the bracket $[D, E] = D \circ E - E \circ D$.

(b) The automorphisms of the algebra A are automorphisms of A as a vector space that also preserve the multiplication structure, and so $\mathrm{Aut}(A)$ forms a subgroup of $\mathrm{GL}(A)$. Show that the Lie algebra of $\mathrm{Aut}(A)$ is equal to $\mathrm{Der}(A)$. Hint: Use ideas from Section 0.1 to show that $\mathrm{Lie}(\mathrm{Aut}(A)) \subset \mathrm{Der}(A)$. Then prove inductively that a derivation D satisfies

$$D^n(ab) = \sum_{k=0}^{n} \binom{n}{k} D^k(a) D^{n-k}(b)$$

for $n > 0$. Use this to show that $e^D(ab) = (e^D a)(e^D b)$.

(c) If the algebra A is a Lie algebra, show that the map $A \to \mathrm{Der}(A)$ given by $X \mapsto D_X$ where $D_X(Y) = [X, Y]$ is a Lie algebra homomorphism (i.e. $D_{[X,Y]} = [D_X, D_Y]$).

(3) The group $PGL_n(\mathbb{C})$ is the quotient of the group $GL_n(\mathbb{C})$ by its center, which consists of scalar multiples of the identity. Show that $PGL_n(\mathbb{C})$ is a matrix group by constructing an injective homomorphism

$$PGL_n(\mathbb{C}) \hookrightarrow GL_N(\mathbb{C})$$

for some N (which will be larger than n). Hint: You're looking for an action of $PGL_n(\mathbb{C})$ on a vector space such that only the identity element in $PGL_n(\mathbb{C})$ fixes every vector. This is equivalent to an action of $GL_n(\mathbb{C})$ on a vector space such that only scalar multiples of the identity matrix fix every vector.

(4) The point of this problem is to construct a Lie group that is not a matrix group.

(a) Let N be the group of matrices of the form

$$\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}$$

with $a, b, c \in \mathbb{R}$. Let Z be the subgroup of matrices of the form

$$\begin{pmatrix} 1 & 0 & n \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

with $n \in \mathbb{Z}$. Show that Z is contained in the center of N. (In particular, this implies that Z is normal in N, so N/Z is a group.)

(b) The group N/Z is called the Heisenberg group and comes up in the study of quantum mechanics. Show that N/Z has a unique structure of Lie group so that the quotient map $N \to N/Z$ is morphism of Lie groups.

(c) Consider the subgroup $\mathbb{T}$ of N/Z consisting of matrices of the form

$$\begin{pmatrix} 1 & 0 & t \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

with $t \in \mathbb{R}$. As a group, $\mathbb{T} \cong \mathbb{R}/\mathbb{Z}$. We will identify the matrix given above with the element $t \in \mathbb{R}/\mathbb{Z}$. Show that

(i) $\mathbb{T}$ is contained in the center of N/Z.

(ii) Every element of $\mathbb{T}$ is a commutator of N/Z, i.e. can be written in the form $uvu^{-1}v^{-1}$, with $u, v \in N/Z$.

(d) Suppose V is a finite-dimensional vector space and

$$\rho : N/Z \to \mathrm{GL}(V)$$

is a homomorphism (this is equivalent to a N/Z action on V). Argue that ρ cannot be injective as follows:

Consider the restriction of ρ to $\mathbb{T}$. A consequence of Fourier theory is that V can be decomposed as

$$V = \bigoplus_{n \in \mathbb{Z}} V_n,$$

where all but finitely many of the $V_n = 0$, and for each $t \in \mathbb{T}$, $\rho(t)$ restricted to V_n is equal to the linear transformation given by multiplication by $e^{2\pi i n t}$. Take this important fact for granted. (We'll prove this next week.) Show that because of 4(c)i, V_n is an invariant subspace for all of N/Z, i.e. if $g \in N/Z$ and $v \in V_n$, then $\rho(g)(v) \in V_n$. Then use 4(c)ii to show that t must act on V_n by a transformation of determinant 1. Conclude that $V = V_0$ and that $\rho(t)$ is the identity transformation for every $t \in \mathbb{T}$.

Since ρ cannot be injective, this means that N/Z cannot be embedded as a subgroup of matrices.