PROBLEM SET #2

DUE: THURSDAY, JUNE 14

(1) Fulton-Harris exercises #1.11, 1.12, 1.13.
(2) Fulton-Harris exercises #11.10, 11.11, 11.13, 11.14
(3) Prove the claim at the end of Section 1 of Lecture 11 that begins with "Observe that when we exponentiate..." on the bottom of page 150.
(4) (a) The *center* of a Lie algebra $\mathfrak{g}$ is by definition the set of all $X \in \mathfrak{g}$ such that $[X, Y] = 0$ for all $Y \in \mathfrak{g}$. Suppose that $H \subset G \subset GL_n$ are connected closed linear groups, and $\mathfrak{h} \subset \mathfrak{g} \subset \mathfrak{gl}_n$ are the corresponding Lie algebras. Show that H is contained in the center of G if and only if $\mathfrak{h}$ is contained in the center of $\mathfrak{g}$.
   (b) A Lie algebra $\mathfrak{g}$ is *abelian* if $[X, Y] = 0$ for all $X, Y \in \mathfrak{g}$. If $G \subset GL_n$ is a connected closed linear group and $\mathfrak{g} \subset \mathfrak{gl}_n$ its corresponding Lie algebra, show that G is abelian if and only if $\mathfrak{g}$ is abelian.
   (c) A Lie subalgebra $\mathfrak{h}$ of a Lie algebra $\mathfrak{g}$ is an *ideal* if $[\mathfrak{g}, \mathfrak{h}] \subset \mathfrak{h}$. Suppose $H \subset G \subset GL_n$ are connected closed linear groups, and $\mathfrak{h} \subset \mathfrak{g} \subset \mathfrak{gl}_n$ are the corresponding Lie algebras. Show that H is normal in G if and only if $\mathfrak{h}$ is an ideal of $\mathfrak{g}$.
   (d) Remind yourself of the definition of a solvable group. (There are several equivalent definitions, so choose whichever one you prefer.) Using the dictionary we have constructed in the previous parts of this exercise as motivation, propose a definition of a solvable Lie algebra. Can you show that if $G \subset GL_n$ is a connected closed linear group with corresponding Lie algbera $\mathfrak{g} \subset \mathfrak{gl}_n$, then G is solvable if and only if $\mathfrak{g}$ is solvable? (Note: All results in this exercise hold for arbitrary connected Lie groups, not just connected closed linear groups.)
(5) Let $\rho : G \to GL(V)$ be a representation of a Lie group G on a finite dimensional vector space V, and let $d\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ be the corresponding representation of $\mathfrak{g}$ on V. Prove that a subspace $W \subset V$ is invariant under G if and only if it is invariant under $\mathfrak{g}$, i.e. $\rho(G)(W) \subset W$ if and only if $d\rho(\mathfrak{g})(W) \subset W$. In particular, conclude that ρ is irreducible if and only if $d\rho$ is irreducible.
(6) Are the two three-dimensional real Lie algebras $\mathfrak{sl}_2(\mathbb{R})$ and $\mathfrak{su}_2$ isomorphic? What about the two three-dimensional complex Lie algebras $\mathfrak{sl}_2(\mathbb{R}) \otimes \mathbb{C} = \mathfrak{sl}_2(\mathbb{C})$ and $\mathfrak{su}_2 \otimes \mathbb{C}$?