# An Exact Solution Layer Potential Test Case for an Interior Dirichlet Problem on an Ellipse

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## 1 Elliptical Coordinates

We use elliptical coordinates:

$$\begin{aligned} x &= \mu \cosh(\rho)\cos(\theta) \\ y &= \mu \sinh(\rho)\sin(\theta) \end{aligned}$$

For $\rho = R$, a constant, these coordinates define the ellipse $\Omega$:

$$\frac{x^2}{\mu^2\cosh^2(R)} + \frac{y^2}{\mu^2\sinh^2(R)} = 1$$

## 2 The Problem

Our Dirichlet problem is given by

$$\begin{aligned} \Delta u &= 0 \text{ , in } \Omega \\ u &= f \text{ , on } \partial\Omega \end{aligned} \tag{1}$$

Now, the function $(x+iy)^3$ is analytic. Therefore $\operatorname{Im}\left((x+iy)^3\right) = 3x^2y - y^3$ is harmonic. Let $u_{in}(x,y) = x^2y - y^3$ inside $\Omega$. In elliptical coordinates,

$$u_{in}(\rho,\theta) = (\mu\cosh(\rho)\cos(\theta))^2(\mu\sinh(\rho)\sin(\theta)) - (\mu\sinh(\rho)\sin(\theta))^3$$

Expanding this, and using the triple angle identity

$$\sin^3(\theta) = \frac{3}{4}\sin(\theta) - \frac{1}{4}\sin(3\theta),$$

we can write $u_{in}$

$$u_{in} = \mu^3\left[\left(\frac{3}{4}\cosh^2(\rho)\sinh(\rho) - \frac{3}{4}\sinh^3(\rho)\right)\sin(\theta) + \left(\frac{1}{4}\sinh^3(\rho) + \frac{3}{4}\cosh^2(\rho)\sinh(\rho)\right)\sin(3\theta)\right]$$

We can now see that $u_{in}$ satisfies (1) with

$$f(\theta) = \mu^3\left[\left(\frac{3}{4}\cosh^2(R)\sinh(R) - \frac{3}{4}\sinh^3(R)\right)\sin(\theta) + \left(\frac{1}{4}\sinh^3(R) + \frac{3}{4}\cosh^2(R)\sinh(R)\right)\sin(3\theta)\right]$$

## 3 The Double Layer Potential

Our goal is to find a $\psi(\theta)$ so that the double layer potential

$$u(x_0, y_0) = u(\mathbf{x}_0) = \frac{1}{2\pi} \int_{\partial\Omega} \psi(\mathbf{x}(\theta)) \frac{\partial}{\partial n_{\mathbf{x}}} \log \frac{1}{|\mathbf{x}_0(\theta) - \mathbf{x}(\theta)|} ds(\mathbf{x}(\theta))$$

is equal to $u_{in}$ in $\Omega$ and therefore satisfies (1). To do this, we define the function $u_{out}$ for $\rho > R$ of the form

$$u_{out}(\rho, \theta) = Ae^{-3\rho} \sin(3\theta) + Be^{-\rho} \sin(\theta)$$

with the hope that together $u_{in}$ and $u_{out}$ can be a double layer potential with the right choices of $A$ and $B$. We can find such an $A$ and $B$ using the fact that the normal derivative of a double layer potential is continuous across $\partial\Omega$:

$$\frac{\partial u^+}{\partial n} - \frac{\partial u^-}{\partial n} = \left[\frac{\partial u}{\partial \mathbf{n}}\right] = 0 \tag{2}$$

With the surface of the ellipse defined by the curve $(\mu \cosh(R) \cos(\theta), \mu \sinh(R) \sin(\theta))$, the outward unit normal is

$$\begin{aligned}\mathbf{n} &= \frac{(\mu \sinh(R) \cos(\theta), \mu \cosh(R) \sin(\theta))}{\sqrt{\mu^2 \sinh^2(R) \cos^2(\theta) + \mu^2 \cosh^2(R) \sin^2(\theta)}} \\ &= \frac{(\sinh(R) \cos(\theta), \cosh(R) \sin(\theta))}{\sqrt{\sinh^2(R) + \sin^2(\theta)}}\end{aligned}$$

where we have used the identity $\cosh^2(R) = \sinh^2(R) + 1$ in the denominator. We therefore can write

$$\begin{aligned}\frac{\partial u}{\partial \mathbf{n}} &= \mathbf{n} \cdot \nabla u \\ &= n_1 \frac{\partial u}{\partial x} + n_2 \frac{\partial u}{\partial y} \\ &= \frac{\sinh(R) \cos(\theta)}{\sqrt{\sinh^2(R) + \sin^2(\theta)}} \frac{\partial u}{\partial x} + \frac{\cosh(R) \sin(\theta)}{\sqrt{\sinh^2(R) + \sin^2(\theta)}} \frac{\partial u}{\partial y}\end{aligned}$$

Compare this to the derivative with respect to $\rho$

$$\begin{aligned}\frac{\partial u}{\partial \rho}|_{\rho=R} &= \frac{\partial u}{\partial x}\frac{\partial x}{\partial \rho}|_{\rho=R} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \rho}|_{\rho=R} \\ &= \mu \sinh(R) \cos(\theta) \frac{\partial u}{\partial x} + \mu \cosh(R) \sin(\theta) \frac{\partial u}{\partial y}\end{aligned}$$

By comparing these two derivatives, we see that

$$\frac{\partial u}{\partial \mathbf{n}} = \frac{1}{\mu \sqrt{\sinh^2(R) + \sin^2(\theta)}} \frac{\partial u}{\partial \rho}|_{\rho=R}$$

Using this and (2), we have

$$\left[\frac{\partial u}{\partial \mathbf{n}}\right] = \frac{1}{\mu\sqrt{\sinh^2(R)+\sin^2(\theta)}}\left[\frac{\partial u}{\partial \rho}\right] = 0$$

We can easily compute $\frac{\partial u_{in}}{\partial \rho}$ and $\frac{\partial u_{out}}{\partial \rho}$, and this condition then allows us to find $A$ and $B$ in $u_{out}$. We spare you the details and just report the results, but you may want to check this your yourself.

$$\begin{aligned} A &= -\mu^3 e^{3R}\left(\frac{3}{4}\cosh(R)\sinh^2(R) + \frac{1}{4}\cosh^3(R)\right) \\ B &= \mu^3 e^{R}\left(\frac{3}{4}\cosh(R)\sinh^2(R) - \frac{3}{4}\cosh^3(R)\right) \end{aligned}$$

Knowing $A$ and $B$, we can now use the fact that for a double layer potential

$$u^+ - u^- = -\psi$$

to find the density $\psi$ and have a complete test problem. After some algebra,

$$\begin{aligned} \psi(\theta) = u^- - u^+ &= -\frac{\mu^3}{4}\left(\cosh(R)+\sinh(R)\right)^3\sin(3\theta) \\ &+ \frac{3}{4}\mu^3\left(-\cosh^3(R) - \cosh^2(R)\sinh(R) + \cosh(R)\sinh^2(R) + \sinh^3(R)\right)\sin(\theta) \end{aligned}$$

# 4 A Couple Numerical Items

To evaluate the layer potential, we need the arclength in elliptical coordinates:

$$\begin{aligned} ds(\mathbf{x}) &= \sqrt{\left(\frac{\partial x}{\partial \theta}\right)^2 + \left(\frac{\partial y}{\partial \theta}\right)^2}\,d\theta \\ &= \sqrt{\mu^2\cosh^2(R)\sin^2(\theta) + \mu^2\sinh^2(R)\cos^2(\theta)}d\theta \\ &= \mu\sqrt{\sinh^2(R)+\sin^2(\theta)}d\theta \end{aligned}$$

We also need the curvature of an ellipse to evaluate the integrand when $\mathbf{x_0} = \mathbf{x}$

$$\begin{aligned} \kappa(\theta) &= -\frac{\mu^2\cosh(R)\sinh(R)}{\left(\mu^2\cosh^2(R)\sin^2(\theta) + \mu^2\sinh^2(R)\cos^2(\theta)\right)^{3/2}} \\ &= -\frac{\cosh(R)\sinh(R)}{\mu\left(\sinh^2(R)+\sin^2(\theta)\right)^{3/2}} \end{aligned}$$