Spaces of ordered commuting elements in Lie groups

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Outline

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Joint with Fred Cohen.

Spaces of Homomorphisms

Let G be a Lie group and π be a finitely generated discrete group.
Consider the set of homomorphisms $Hom(\pi, G)$.

Topology: If π has n generators $x_1, \ldots, x_n$, then $Hom(\pi, G)$ can be identified as a subset of G^n as follows:

$$f \in Hom(\pi, G) \hookrightarrow G^n$$

$$f \sim f(x_1, \ldots, x_n) = (g_1, \ldots, g_n).$$

This can also be seen from the inclusion induced by a quotient $F_n \to \pi$.

Therefore, we can endow $Hom(\pi, G)$ with the subspace topology.

Examples

1. Let $\pi = F_n$ be the free group on n letters. Then $Hom(F_n, G) = G^n$. ($\pi = F_n$ has no relations)
2. Let $\pi = \mathbb{Z}^n$. $Hom(\mathbb{Z}^n, G)$ can be identified with the set of elements in G^n whose coordinates pairwise commute:

   $$f \in Hom(\mathbb{Z}^n, G) \subseteq G^n$$

   $$f \sim f(x_1, ..., x_n) = (g_1, ..., g_n), \text{ such that } g_i g_j = g_j g_i, \text{ all } i, j.$$

   We call $Hom(\mathbb{Z}^n, G)$ **the space of ordered pairwise commuting n–tuples in G.**
3. If $\pi = \mathbb{Z}^n$ and G is abelian, then $Hom(\mathbb{Z}^n, G) = G^n$.

Remark: G acts by conjugation on $Hom(\pi, G)$. The space $Hom(\pi, G)/G = R(\pi, G)$ is also widely studied (not considered here). $R(\pi, G)$ is called the *representation space*.

Remarks

- The spaces $Hom(\mathbb{Z}^n, G)$ "appear in physics":
  - $Hom(\mathbb{Z}^n, G)/G$ is the moduli space of flat $G-$bundles over the compact $n-$torus $(S^1)^n$.
  - These moduli spaces form critical level sets of Lagrangians for important quantum-field theories such as the Chern-Simons and Yang-Mills theories.
  - Connections to work of E. Witten in quantum-field theory.
- The study of the spaces $Hom(\mathbb{Z}^n, G)$ for finite groups G, leads to problems as hard as the Feit-Thompson theorem.

Some known results

- (Goldman) If π is finitely generated, then $Hom(\pi, G)$ is a real algebraic variety.
- (Adem & Cohen) If G is a closed subgroup of $GL_n(\mathbb{C})$, then there is a homotopy equivalence

$$\Sigma Hom(\mathbb{Z}^n, G) \to \bigvee_{1 \leq k \leq n} \Sigma \bigvee^{\binom{n}{k}} Hom(\mathbb{Z}^k, G)/S_k(G).$$

- (G. H. Rojo) Computes the number of connected components of $Hom(\mathbb{Z}^k, O(n))$ and $Hom(\mathbb{Z}^k, SO(n))$ (*formulas upon request*).

Some known results (cnt'd)

- (Sjerve & Torres-Giese) The homotopy type of $Hom(\mathbb{Z}^k, SO(3))$ is given by

$$Hom(\mathbb{Z}^k, SO(3)) \to Hom(\mathbb{Z}^k, SO(3))_1 \bigsqcup (\bigsqcup_{\# < \infty} S^3/Q_8).$$

- (Pettet & Souto) If G is a reductive algebraic group and $K \subset G$ is a maximal compact subgroup, then $Hom(\mathbb{Z}^k, K)$ is a strong deformation retract of $Hom(\mathbb{Z}^k, G)$.
  - There is a homotopy equivalence $Hom(\mathbb{Z}^k, O(n)) \simeq Hom(\mathbb{Z}^k, GL_n(\mathbb{R}))$.
  - There is an isomorphism $\pi_1(Hom(\mathbb{Z}^k, G)) = \pi_1(G)^k$.

- (Bergeron) If Γ is a finitely generated nilpotent group, then there is a strong deformation retract of $Hom(\Gamma, G)$ onto $Hom(\Gamma, K)$.
- (Adem & Gomez) The space $B(2, G)$ is an infinite loop space.

Connectedness

Recall the following classical definitions:

The group $T \subset G$ is a **maximal torus** of G if it is a compact, connected torus of maximal rank.

For some special Lie groups G, T has the additional property that "*every abelian subgroup of G is conjugate to a subgroup of T*". Such groups include $U(n), SU(n), Sp(n)$.
Groups that do not have this property include $SO(2n+1), G_2$ etc.

Theorem (Adem & Cohen)

If G has a maximal torus T with the property that every abelian subgroup of G is conjugate to a subgroup of T, then $Hom(\mathbb{Z}^n, G)$ is path connected for all n.

For any $f \in Hom(\mathbb{Z}^n, G)$ we can apply the classifying space functor to obtain $Bf \in Map_*(B\mathbb{Z}^n, BG)$. If we pass to the path components we obtain

$$B_0 : \pi_0(Hom(\mathbb{Z}^n, G)) \to [(S^1)^n, BG],$$

where $[(S^1)^n, BG]$ classifies principal G-bundles over the n-torus $(S^1)^n$.

A classical method for studying $Hom(\mathbb{Z}^n, G)$

For A an abelian subgroup of G, there is a map

$$\Theta : G \times A^n \longrightarrow Hom(\mathbb{Z}^n, G)$$

$$(g, t_1, ..., t_n) \mapsto (t_1^g, ..., t_n^g)$$

Θ is $A-$invariant, so it factors through $G \times_A A^n$, where A acts by conjugation, i.e. trivially, on A^n and acts by left multiplication on G.
We get a map $\widehat{\Theta} : G \times_A A^n \longrightarrow Hom(\mathbb{Z}^n, G)$.

Definition

Let T be a maximal torus of G. Then the **Weyl group** *of G is the group $W = NT/T$, where NT is the normalizer of T in G.*

Then W acts on $G \times_T T^n = G/T \times T^n$ by conjugation on T^n and by left multiplication on the cosets G/T. $\widehat{\Theta}$ is W-invariant, so it factors through $G/T \times_W T^n = G \times_{NT} T^n$ and we get

$$\widetilde{\Theta} : G \times_{NT} T^n \longrightarrow Hom(\mathbb{Z}^n, G).$$

The space $Comm(G)$

- Problems involving $Hom(\mathbb{Z}^k, G)$ are delicate, in general.
- Instead we construct a space called $Comm(G)$ that assembles all the spaces $Hom(\mathbb{Z}^k, G)$ into a single one.
- $Comm(G)$ is more *tractable*.
- There is a decomposition of $Comm(G)$ which *tells* that this space is *the smallest* space containing all spaces $Hom(\mathbb{Z}^n, G)$.

The space $Comm(G)$ (cnt'd)

From now on G is a compact and connected Lie group.

Recall the *James reduced product* on a pointed CW-complex, denoted by $J(X)$. The space $J(X)$ is given by

$$J(X) := \bigsqcup_{n \geq 1} X^n / \sim,$$

where $\sim$ is generated by $(x_1, ..., *, ..., x_n) \sim (x_1, ..., \widehat{*}, ..., x_n)$.

Definition

Let G be a Lie group. Then $Comm(G)$ is defined by

$$Comm(G) := \bigsqcup_{n \geq 1} Hom(\mathbb{Z}^n, G) / \sim,$$

where $\sim$ is generated by the same relation.

We now obtain a map

$$G \times_{NT} J(T) \longrightarrow Comm(G).$$

Let

$$Comm(G)_1 = \bigsqcup_{n \geq 1} Hom(\mathbb{Z}^n, G)_1 / \sim,$$

where $Hom(\mathbb{Z}^n, G)_1$ is the path component of $(1, ..., 1)$.
Then the following is a surjection

$$\widetilde{\Theta} : G \times_{NT} T^n \longrightarrow Hom(\mathbb{Z}^n, G)_1,$$

which gives a surjection

$$G \times_{NT} J(T) \longrightarrow\!\!\!\!\!\rightarrow Comm(G)_1.$$

In the special case where G has the property that every abelian subgroup can be conjugated to T, it follows that $Comm(G) = Comm(G)_1$ and we have a surjection

$$G \times_{NT} J(T) \longrightarrow\!\!\!\!\!\rightarrow Comm(G).$$

Stable decompositions

Theorem

Let G be a compact Lie group with maximal torus T and NT acting on T by conjugation. There is a homotopy equivalence

$$\Sigma(G \times_{NT} J(T)) \simeq \Sigma(G/NT \vee (\bigvee_{n\geq 1} G \times_{NT} \widehat{T}^n / G \times_{NT} \{1\})).$$

Theorem

Let G be a compact and connected Lie group. There is a stable homotopy equivalence

$$\Sigma Comm(G) \simeq \Sigma \bigvee_{n\geq 1} \widehat{Hom}(\mathbb{Z}^n, G),$$

where $\widehat{Hom}(\mathbb{Z}^n, G) = Hom(\mathbb{Z}^n, G)/S(Hom(\mathbb{Z}^n, G))$.

The homology of $Comm(G)$

Recall the map $\Theta : G \times T^n \longrightarrow Hom(\mathbb{Z}^n, G)$.
Now, let $R = \mathbb{Z}[|W|^{-1}]$. (For simplicity one can also work over $\mathbb{Q}$).

Lemma

The reduced homology of $\Theta^{-1}(g_1, ..., g_n)$ with coefficients in R is trivial, i.e.

$$\widetilde{H}_k(\Theta^{-1}(g_1, ..., g_n); R) = 0.$$

Theorem

Let G be compact and connected with maximal torus T and Weyl group W. Then there is an isomorphism

$$H_*(G \times_{NT} J(T); R) \cong H_*(Comm(G)_1; R).$$

Homological computations

There is a short exact sequence of groups

$$1 \longrightarrow T \longrightarrow NT \longrightarrow W \longrightarrow 1$$

and thus a fibration

$$(G \times J(T))/T \longrightarrow (G \times J(T))/NT = G \times_{NT} J(T) \longrightarrow BW.$$

There is a spectral sequence

$$E^2_{p,q} = H_p(BW; H_q(G/T \times J(T); R))$$

which converges to $H_*(G \times_{NT} J(T); R)$.

If $p > 0$ then $E^2_{p>0,q} = 0$ and

$$E^2_{0,q} = E^\infty_{0,q} = H_q(G/T \times J(T); R)_W$$

Therefore,

$$H_*(G \times_{NT} J(T); R)) \cong H_*(G/T \times J(T); R)_W$$

and

$$\begin{aligned} &H_*(G/T \times J(T); R)_W \\ &\cong \big(H_*(G/T; R) \otimes H_*(J(T); R)\big)_W \\ &\cong \big(H_*(G/T; R) \otimes T[V]\big)_W, \end{aligned}$$

where V is the reduced homology of T as a $W-$module, and $T[V]$ is the tensor algebra of V.

Therefore the we have the following:

Theorem

Let G be a compact and connected Lie group with maximal torus T and Weyl group W. Then the homology of $Comm(G)_1$ with coefficients in R is given by

$$H_*(Comm(G)_1; R) \cong \left(H_*(G/T; R) \otimes T[V]\right)_W$$

As a corollary we have:

Theorem

If G has a maximal torus T with the property that every abelian subgroup of G is conjugate to a subgroup of T, then the homology of $Comm(G)$ with coefficients in R is given by

$$H_*(Comm(G); R) \cong \left(H_*(G/T; R) \otimes T[V]\right)_W.$$

Remarks

- This theorem works for any compact and connected Lie group G, including the exceptional groups G_2, F_4, E_6, E_7, E_8.
- The representation theory of W gives the homology for these spaces (Classical representation theory).
- The same construction does not inform on finite groups G.

Ungraded homology

Let H_*^U denote ungraded homology and $T_U[V]$ denote the ungraded tensor algebra of V. Then the following theorem holds:

Theorem

Let G be a compact and connected Lie group with maximal torus T and Weyl group W. Then the ungraded homology of $Comm(G)_1$ with coefficients in R is given by

$$H_*^U(Comm(G); R) \cong T_U[V].$$

Example 1: $SO(3)$

E. Torres Giese and D. Sjerve prove that

$$Hom(\mathbb{Z}^n; SO(3)) = Hom(\mathbb{Z}^n; SO(3))_1 \bigsqcup \Big(\bigsqcup_{m<\infty} S^3/Q_8 \Big).$$

Therefore,

Proposition

There is a homeomorphism

$$Comm(SO(3)) = Comm(SO(3))_1 \bigsqcup \Big(\bigsqcup_{\infty} S^3/Q_8 \Big),$$

where $Comm(SO(3))_1 = \bigsqcup_{n \geq 1} Hom(\mathbb{Z}^n; SO(3))_1/\sim$.

The homology of $Comm(SO(3))_1$ is given by

$$H_*(Comm(SO(3)); \mathbb{Z}[2^{-1}]) \cong \big(H_*(SO(3)/S^1; \mathbb{Z}[2^{-1}]) \otimes T[V] \big)_{\Sigma_2}.$$

Example 2: $U(2)$

If $G = U(2)$, then $T = S^1 \times S^1$ and every abelian subgroup of $U(2)$ is conjugate to a subgroup of T.
Therefore,

$$H_*(Comm(U(2)); \mathbb{Z}[2^{-1}]) \cong \big(H_*(U(2)/S^1 \times S^1; \mathbb{Z}[2^{-1}]) \otimes T[V]\big)_{\Sigma_2},$$

where $H_*(U(2)/S^1 \times S^1; \mathbb{Z}[2^{-1}]) = \mathbb{Z}[2^{-1}]\Sigma_2$, the group ring (ungraded).

Hilbert-Poincaré series

Assume there is a tri-graded series $\sum_{i,j,k} A(i,j,k) q^i s^j t^k$, where $A(i,j,k)$ is the rank of the sub-module in $H_*(Comm(G)_1; R)$ equal to the tensor product of the i-th homology in G/T, and k-th homology in $J(T)$ which is given by j-tensors. If we consider cohomology, $A(i,j,k)$ is the rank of

$$\sum_{\substack{k=r_1+\cdots+r_j \\ r_q>0}} (H^i(G/T) \otimes \wedge^{r_1}\mathbb{Q}^n \otimes \cdots \otimes \wedge^{r_j}\mathbb{Q}^n).$$

Using methods in algebra V. Reiner proved the following in an appendix:

Theorem

If G is a compact, connected Lie group with maximal torus T, and Weyl group W, then

$$Hilb\left(\left(H^*(G/T;R) \otimes \mathcal{T}^*[\widetilde{E}]\right)^W, q, s, t\right)$$
$$= \frac{\prod_{i=1}^n (1-q^{2d_i})}{|W|} \sum_{w\in W} \frac{1}{\det(1-q^2 w)\,(1-t(\det(1+sw)-1))}.$$

Example

Let $G = U(2)$:

1. The Weyl group W is Σ_2 with elements 1, and $w \neq 1$.
2. The homology of the space $G/T = U(2)/T = S^2$ is $\mathbb{R}$ in degrees zero and two, and is $\{0\}$ otherwise.
3. The degrees (d_1, d_2) in the theorem are given by $(d_1, d_2) = (1, 2)$.

Then using the formula

$$Hilb\left(\left(C^* \otimes T^*[\widetilde{E}]\right)^W, q, s, t\right) = \frac{(1-q^2)(1-q^4)}{2}(A_1 + A_w),$$

where

$$A_1 = \frac{1}{(1-q^2)^2(1-t[(1+s)^2-1])}$$

and

$$A_w = \frac{1}{(1-q^2)(1+q^2)(1-t[(1+s)(1-s)-1])}.$$

Example (cnt'd)

Thus

$$\frac{(1-q^2)(1-q^4)}{2}(A_1) = \frac{1+q^2}{2(1-t(s^2+2s))}, \text{ and}$$
$$\frac{(1-q^2)(1-q^4)}{2}(A_w) = \frac{1-q^2}{2(1+s^2t)}.$$

The Hilbert-Poincaré series is then given by

$$Hilb\left(\left(C^* \otimes T^*[\widetilde{E}]\right)^W, q, s, t\right) = \frac{1+q^2}{2(1-t(s^2+2s))} + \frac{1-q^2}{2(1+s^2t)}.$$

From this information, it follows that the coefficient of $t^m, m > 0$, is

$$\frac{1}{2}\left[(1+q^2)(s^2+2s)^m + (1-q^2)(-s^2)^m\right]$$
$$= \sum_{1\le j\le m} 2^{j-1}\binom{m}{j} s^{2m-j} + \begin{cases} s^{2m} & \text{if } m \text{ is even, and} \\ q^2 s^{2m} & \text{if } m \text{ is odd.} \end{cases}$$

Corollary

Then there are additive isomorphisms

$$\widetilde{H}^d(Hom(\mathbb{Z}^m, G); \mathbb{R}) \to \sum_{1 \leq s \leq m} \sum_{i+j=d} \left(\sum_{\substack{j=k_1+\cdots+k_s \\ i \geq 0}} \oplus_{\binom{m}{s}} (M_{(i,j,s)})^W \right).$$

Right-angled Artin groups

A right-angle Artin group can be described as the fundamental group of a polyhedral product

$$\pi(K) := \pi_1\left(Z_K(S^1, *)\right),$$

where K is a simplicial complex with n vertices. We can study the space of homomorphisms

$$Hom(\pi(K), G).$$

Theorem

There is a homotopy equivalence

$$\Sigma Hom(\pi(K), G) \to \Sigma X \vee \bigvee_{\sigma \in M} \Sigma(Hom(\mathbb{Z}^{m(\sigma)}, G))$$

for the space $X = Hom(\pi(K), G)/\left(\bigvee_{\sigma \in M}(Hom(\mathbb{Z}^{m(\sigma)}, G))\right)$.

Right-angled Artin groups (cnt'd)

Define the space

$$G[\sigma] = \{(g_1, \dots, g_n) \in G^n : [g_i, g_i] = 1 \text{ if } (ij) \in \sigma\}.$$

Then

$$Hom(\pi(K), G) = \bigcap_{\sigma \in K} G[\sigma],$$

and

$$G^m - Hom(\pi(K), G) \approx \bigcup_{\sigma \in K} (G^m - G[\sigma]).$$

gives a Mayer-Vietoris spectral sequence abutting to the homology of $G^m - Hom(\pi(K), G)$.

Problems

- Analogous approaches for finitely generated discrete groups π, other than $\mathbb{Z}^n$.
- Further decompositions of $Hom(\mathbb{Z}^n, G)$.
- The integer homology of the spaces $Hom(\mathbb{Z}^n, G)$.
- The number of path components $\pi_0(Hom(\mathbb{Z}^n, G))$ for compact and connected G.
- Prove the equivalent form of the Feit-Thompson theorem.

THANK YOU