Polyhedral products and monodromy

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Combinatorial Constructions in Topology
Regina, SK - August 17-21, 2015

Outline

Polyhedral Products

Ingredients:

- a simplicial complex K
- a sequence of pairs of spaces $(X_1, A_1), \ldots, (X_n, A_n)$ denoted by $(\underline{X}, \underline{A})$
- a functor $D : K \to Top$, where $D(\sigma) = Y_1 \times \cdots \times Y_n$, with $Y_i = A_i$ if $i \notin \sigma$ and $Y_i = X_i$ if $i \in \sigma$.

Definition

The *polyhedral product* denoted $Z_K(\underline{X}, \underline{A})$ is the topological space defined by the colimit

$$Z_K(\underline{X}, \underline{A}) = \operatorname*{colim}_{\sigma \in K} D(\sigma) = \bigcup_{\sigma \in K} D(\sigma) \subseteq \prod_{i=1}^{n} X_i,$$

where the maps are the inclusions and the topology is the subspace topology.

Hence, *polyhedral products* are a generalized version of *moment-angle complexes*given by $Z_K(D^2, S^1)$.

Example

Some classical examples of polyhedral products are the following:

1. *Let* $K = \{\{1\}, \ldots, \{n\}\}$, $X_i = X$ *and* A_i *be the basepoint* $* \in X$. *Then*

$$Z_K(\underline{X}, \underline{A}) = X \vee \cdots \vee X,$$

   the n*-fold wedge sum of the space* X.

2. *Let* $K = 2^{[n]}$, *then* $Z_K(\underline{X}, \underline{A}) = X_1 \times \cdots \times X_n$.
3. *Let* $K = \{\{1\}, \{2\}\}$ *and* $(\underline{X}, \underline{A}) = (D^n, S^{n-1})$. *Then*

$$Z_K(\underline{X}, \underline{A}) = D^n \times S^{n-1} \cup S^{n-1} \times D^n = \partial D^{2n} = S^{2n-1}.$$

4. *If* $|K|$ *is the boundary of the simplex* $\Delta[n-1]$, *then* $Z_K(\underline{X})$ *is the fat wedge of the spaces* $X_1, \ldots, X_n$.

Some facts about polyhedral products:

1. (Denham-Suciu) The polyhedral product $Z_K(X, A)$ depends only on the relative homotopy type of the pair (X, A).
   E.g. if G is finite, then $(EG, G) \simeq_{rel} ([0,1], F)$, where F is a finite subset of the unit interval with $|G|$ elements. Hence,
   $$Z_K(EG, G) \simeq Z_K([0,1], F)$$
   for any K.
2. (Bahri-Bendersky-Cohen-Gitler) If $(\underline{X}, \underline{A})$ is a sequence of pointed and connected CW-pairs, then there is a homotopy equivalence
   $$\Sigma(Z_K(\underline{X}, \underline{A})) \to \Sigma\left(\bigvee_{I \leq [n]} \widehat{Z}_{K_I}(\underline{X_I}, \underline{A_I})\right).$$
3. (Wedge Lemma) When applied to polyhedral products, follows from the decomposition above; i.e. if $A_i \subset X_i$ is null-homotopic, then
   $$\Sigma(Z_K(\underline{X}, \underline{A})) \to \Sigma\left(\bigvee_{I}\Big(\bigvee_{\sigma \in K_I} |\Delta(K_I)_{<\sigma}| * \widehat{D}(\sigma)\Big)\right).$$

A fibration

Consider the circle S^1 and its classifying space $BS^1 \simeq \mathbb{C}P^\infty$.

There is an inclusion map

$$BS^1 \vee \cdots \vee BS^1 \hookrightarrow BS^1 \times \cdots \times BS^1,$$

with homotopy fibre proved by V. Buchstaber and T. Panov to be the polyhedral product

$$Z_K(ES^1, S^1),$$

where K is a set of n vertices and ES^1 is the universal cover of BS^1, hence there is a fibration

$$Z_K(ES^1, S^1) \to BS^1 \vee \cdots \vee BS^1 \hookrightarrow BS^1 \times \cdots \times BS^1.$$

Denham-Suciu fibrations

G. Denham and A. Suciu gave a generalization to this fibration:

Let G be a topological group with BG its classifying space and EG the universal cover of BG. Then the inclusion map

$$Z_K(BG,*) \hookrightarrow \prod_n BG$$

has homotopy fibre $Z_K(EG,G)$, where K is any simplicial complex with n vertices.

Definition

The fibrations

$$Z_K(EG,G) \to Z_K(BG,*) \hookrightarrow \prod_n BG$$

will be called **Denham-Suciu fibrations**.

The fundamental group

Theorem

Let G be a finite group. The fundamental group of $Z_K(BG,*)$ is is given by

$$\pi_1(Z_K(BG,*)) \cong \prod_{K_1} G,$$

where K_1 is the 1-skeleton of K and the product is the graph product of groups.

This theorem holds also if we change $(BG,*)$ by $(\underline{BG},\underline{*})$ and $G_1,\ldots,G_n$ are countable discrete groups:

$$\pi_1(Z_K(\underline{BG},\underline{*})) \cong \prod_{K_1} G_i.$$

Eilenberg-MacLane spaces

Recall that an **Eilenberg-Maclane space** X has the property that all but one homotopy groups are zero. If $\pi_n(X) = A \neq 0$, then the space is denoted by $K(A, n)$ since it is unique up to homotopy.

The simplicial complex K is called a *flag complex* if any finite set of vertices, which are pairwise connected by edges, spans a simplex in K.

M. Davis showed the following:

Theorem (M. Davis)

If K is a flag complex and G is a finite group then $Z_K(BG, *)$ is a $K(A, 1)$ with $\prod_{K_1} G$.

The converse is also true:

Theorem

If G is finite and $Z_K(BG, *)$ is a $K(A, 1)$, then K is a flag complex

Short Exact Sequence

Let K be a flag complex and $G_1, \ldots, G_n$ be finite groups.
Then the fibration

$$Z_K(\underline{EG}, \underline{G}) \to Z_K(\underline{BG}, \underline{*}) \hookrightarrow \prod_{i=1}^{n} BG_i$$

gives a short exact sequence of groups

$$\pi_1(Z_K(EG, G)) \to \prod_{K_1} G_i \to \prod_{i=1}^{n} G_i.$$

Note that if K is only the set of n vertices, then the short exact sequence becomes

$$1 \to F_{N_n} \to G_1 * \cdots G_n \to \prod G_i \to 1.$$

The followings are true:

- If K is a set of vertices, then $Z_K(EG, G)$ is homotopy equivalent to a graph
- The fundamental group is a free group F_{N_n} with

$$N_n = (n-1)\prod_{i=1}^{n} m_i - \sum_{i=1}^{n}(\prod_{j\neq i} m_j) + 1.$$

- Finding the algebraic monodromy amounts to finding the action of the fundamental group on a graph.

Monodromy action

For any locally trivial fibration $F \to E \to B$ the fundamental group of the base acts on the fibre.

For the free group F_n and $n \geq 2$, there is a short exact sequence of groups

$$1 \longrightarrow \mathrm{Inn}(F_n) \longrightarrow \mathrm{Aut}(F_n) \longrightarrow \mathrm{Out}(F_n) \longrightarrow 1$$

and hence, a commutative diagram

$$\begin{tikzcd}
1 \arrow[r] \arrow[d] & F_{N_n} \arrow[r] \arrow[d, "\Psi"] & G_1 * \cdots * G_n \arrow[r] \arrow[d, "\Theta"] & G_1 \times \cdots \times G_n \arrow[r] \arrow[d, "\widetilde{\Theta}"] & 1 \\
1 \arrow[r] & \mathrm{Inn}(F_{N_n}) \arrow[r] & \mathrm{Aut}(F_{N_n}) \arrow[r] & \mathrm{Out}(F_{N_n}) \arrow[r] & 1.
\end{tikzcd}$$

where $G_1, \ldots, G_n$ are finite discrete groups. So the map

$$\Theta : G_1 * \cdots * G_n \to \mathrm{Aut}(F_{N_n})$$

induces a map

$$\widetilde{\Theta} : G_1 \times \cdots \times G_n \to \mathrm{Out}(F_{N_n}),$$

which is the representation we are interested in.

Monodromy

Consider the fibration

$$Z_K(\underline{EG}, \underline{G}) \to Z_K(\underline{BG}, \underline{*}) \hookrightarrow \prod_{i=1}^{n} BG_i$$

Then the fundamental group $\pi_1(\prod_{i=1}^n BG_i) = \prod_{i=1}^n G_i$ acts on the fibre $Z_K(\underline{EG}, \underline{G})$; hence acts on the fundamental group $\pi_1(Z_K(\underline{EG}, \underline{G}))$ and on the homology $H_1 = \pi_1/[\pi_1, \pi_1]$. (note $n \geq 2$)

If K is a flag complex, then $Z_K(\underline{EG}, \underline{G})$ is a $K(\pi, 1)$.

Initially we consider K to be the zero skeleton of a finite simplicial complex.

Monodromy, $n = 2$

Let G_1 and G_2 be finite cyclic groups with order m and n, respectively, such that $G_1 = \{1, x_1, x_1^2, \dots, x_1^{m-1}\}$ and $G_2 = \{1, x_2, x_2^2, \dots, x_2^{n-1}\}$.

The zero simplicial complex with two vertices is $K_0 = \{\{1\}, \{2\}\}$.

Assume there are bijections of finite sets $G_1 \approx F_1$ and $G_2 \approx F_2$ given by

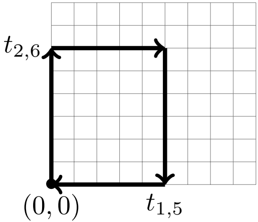
$$G_1 = \{1, x_1, x_1^2, \dots, x_1^{m-1}\} \approx F_1 = \{0 = t_{1,0} < t_{1,1} < \cdots < t_{1,m-1} = 1\} \subset I,$$
$$G_2 = \{1, x_2, x_2^2, \dots, x_2^{n-1}\} \approx F_2 = \{0 = t_{2,0} < t_{2,1} < \cdots < t_{2,n-1} = 1\} \subset I,$$

identifying x_i^k with $t_{i,k}$.

Monodromy, $n = 2$

Then we have

$$Z_{K_0}(\underline{EG}, \underline{G}) \simeq Z_{K_0}(\underline{I}, \underline{F}) = D(\{1\}) \cup D(\{2\}) = I \times F_2 \cup F_1 \times I,$$



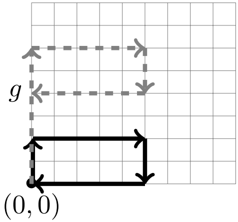
where $1 \leq i \leq m-1$ and $1 \leq j \leq n-1$.
Consider the cycles γ_ω starting at the basepoint $* = (0,0)$, given by the words

$$\omega = [x_1^i, x_2^j] = x_1^i x_2^j x_1^{-i} x_2^{-j}.$$

The set of words $\mathcal{W} = \{\omega_{ij} = [x_1^i, x_2^j] | 1 \le i \le m-1, 1 \le j \le n-1\}$ is a minimal generating set for all the cycles $\gamma_\omega \in Z_{K_0}(\underline{I}, \underline{F})$; i.e.

- Every cycle can be represented by a product of elements in $\mathcal{W}$.
- $|\mathcal{W}|$ equals the rank of F_{N_2}.

In general, for any $g \in G_1 * \cdots * G_n$, denote $\Theta(g) = \varphi_g \in \mathrm{Aut}(F_{N_n})$.



$G_1 \times \cdots \times G_n$ acts on the loops in the fiber by conjugation, that is,

$$g \cdot \gamma_\omega = \gamma_{gwg^{-1}},$$

hence

$$x_1 \omega_{ij} x_1^{-1} = x_1 [x_1^i, x_2^j] x_1^{-1} = [x_1^{i+1}, x_2^j][x_2^j, x_1] = \omega_{i+1,j} {\omega_{1j}}^{-1}.$$

Looking at the induced map of φ_{x_1} onto the abelianization

$$\bigoplus_{(r-1)(m-1)} \mathbb{Z} \cong F_{(r-1)(m-1)}/[F_{(r-1)(m-1)}, F_{(r-1)(m-1)}]$$

then

$$\widetilde{\varphi}_{x_1}(\omega_{11}, \ldots, \omega_{(r-1)(m-1)}) = (\omega_{2,1} - \omega_{1,1}, \omega_{2,2} - \omega_{1,2}, \ldots, -\omega_{(r-1),(m-1)})$$

which is given by the matrix

$$[\widetilde{\varphi}_{x_1}] = \begin{pmatrix} -I_{m-1} & I_{m-1} & 0 & 0 & \cdots & 0 \\ 0 & -I_{m-1} & I_{m-1} & 0 & \cdots & 0 \\ 0 & 0 & -I_{m-1} & I_{m-1} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & -I_{m-1} & I_{m-1} \\ 0 & \cdots & 0 & 0 & 0 \cdots & -I_{m-1} \end{pmatrix} \tag{1}$$

with respect to the basis $\mathcal{W}_2$, where I_{m-1} is the $(m-1) \times (m-1)$ identity matrix. Hence, clearly φ_{x_1} is not an element of IA_k.

For $\varphi_{x_2} \in \mathrm{Aut}(F_k)$:

$$x_2 \omega_{ij} {x_2}^{-1} = x_2[x_1^i, x_2^j]{x_2}^{-1} = [x_2, x_1^i][x_1^i, x_2^{j+1}] = {\omega_{i,1}}^{-1}\omega_{i,j+1}.$$

Similarly, looking at the induced map of φ_{x_2} onto the abelianization of F_k we get

$$\widetilde{\varphi}_{x_2}(\omega_{11}, \ldots, \omega_{(r-1)(m-1)}) = (-\omega_{1,1} + \omega_{1,2} - \omega_{1,1} + \omega_{1,3}, \ldots, -\omega_{(r-1),(m-1)}),$$

which is given by the matrix

$$\widetilde{\varphi}_{x_2}] = \begin{pmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_{r-1} \end{pmatrix}, \ A_i = \begin{pmatrix} -1 & 1 & 0 & 0 & \cdots & 0 \\ -1 & 0 & 1 & 0 & \cdots & 0 \\ -1 & 0 & 0 & 1 & \cdots & 0 \\ -1 & 0 & 0 & 0 & \ddots & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & 1 \\ -1 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \text{ for all } i.$$

with respect to the basis $\mathcal{W}$. Hence, φ_{x_2} is not an element of IA_k.

Monodromy for any K_0

Recall that for a group G, there is a sequence of subgroups called the **descending central series** of G given by

$$G = \Gamma^1(G) \trianglerighteq \Gamma^2(G) \trianglerighteq \cdots \trianglerighteq \Gamma^n(G) \trianglerighteq \cdots ,$$

where the second stage is $\Gamma^2(G) = [G, G]$ and the $(n+1)$–st stage is given inductively by $\Gamma^{n+1}(G) = [\Gamma^n(G), G]$. The Lie algebra of G associated to the descending central series is given by

$$\mathrm{gr}_*(G) = \bigoplus_{i \geq 1} \Gamma^i(G)/\Gamma^{i+1}(G)$$

with $\mathrm{gr}_p(G) = \Gamma^p(G)/\Gamma^{p+1}(G)$.

Monodromy for any K_0

Lemma

Let $\{G_i\}_{i=1}^n$ *be a collection of finite discrete groups and* K_0 *be the* 0*–simplicial complex on* n *vertices. Let* $\rho : \prod_{i=1}^n G_i \to \mathrm{Out}(F_N)$ *be the monodromy representation where* F_N *is isomorphic to the kernel of the projection* $p : G_1 * \cdots * G_n \to \prod_{i=1}^n G_i$*. Then the following hold:*

1. *There is a choice of a generating set for* F_{N_n} *that consists of elements of the form*

$$f = [g_{i_1}, [g_{i_2}, [\ldots, [g_{i_{k-1}}, g_{i_k}] \ldots]]] \in \Gamma^k(G_1 * \cdots * G_n)$$

   such that $g_{i_j} \in G_{i_j}$*, for all* i_j*.*

2. *For any* $g \in G_1 * \cdots * G_n$*, the map* $\rho(g) \in \mathrm{Aut}(F_N)$ *satisfies* $\rho(g)(f) = \Delta \cdot f$*, where* $\Delta \in \Gamma^{k+1}(G_1 * \cdots * G_n)$*. That is,* Δ *is trivial in* $\mathrm{gr}_p(G_1 * \cdots * G_n)$ *for* $p \leq k$*.*

Remark:

The part of the problem that makes it algebraically inaccessible is that, even though we know the rank of the free group, we do not have an explicit list of elements that forms a set of generators for this free group.

Hence, using the polyhedral product model of the fibre, we get a geometric description of the monodromy action.

General K

It would be interesting to determine whether the representations for various choices of K are related.

If $\pi = \pi_1(Z_K(\underline{EG}, \underline{G}))$, then we are interested in the following diagram:

$$\begin{tikzcd}
 & \mathrm{Out}(F_N) \arrow[dd, dotted, "?"] \\
G_1 \times \cdots \times G_n \arrow[ur, "\rho_{K_0}"] \arrow[dr, "\rho_K"'] & \\
 & \mathrm{Out}(\pi)
\end{tikzcd}$$

Open questions:

- determine the algebraic interpretation of monodromy for a finite number of cyclic groups.
- determine whether the monodromy for K_0 informs about the monodromy for other K.

An extension problem

Assume $\{G_1, \ldots, G_n\}$ is a family of subgroups of a finite discrete group G. Then there is a natural map

$$G_1 * \cdots * G_n \xrightarrow{\varphi} G.$$

It is natural to ask the following: for what abstract simplicial complexes K on $[n]$ does the map $BG_1 \vee \cdots \vee BG_n \xrightarrow{B\varphi} BG$ extend to $Z_K(\underline{BG})$, i.e following diagram commutes:

$$\begin{tikzcd} BG_1 \vee \cdots \vee BG_n \arrow[r, "B\varphi"] \arrow[d, "i"] & BG \\ Z_K(\underline{BG}) \arrow[ur] & \end{tikzcd}$$

If the map in question extends, then we detect commuting elements in G. Algebraically, determine K s.t. the following diagram commutes:

$$\begin{tikzcd} G_1 * \cdots * G_n \arrow[r, "B\varphi"] \arrow[d, "i_\#"] & G \\ \prod_{SK_1} G_i \arrow[ur] & \end{tikzcd}$$

Thank you for your attention...

References:
arXiv:1402.3270
arXiv:1310.3504