Polyhedral products and toric topology

The theme of toric topology/polyhedral products is to study spaces that interpolate between a point and a cartesian product of spaces. Reference: "Toric Topology" by Buchstaber-Panov AMS 2015

Ex:

- $(X,*)$ a CW pair.
  - $\{*\}\times\{*\} \subset X\times\{*\} \cup \{*\}\cup X \subset X\times X$
  - $*\times\cdots\times * \subset X\vee\cdots\vee X \subset X\times\cdots\times X$.
- G. Porter '66: $T_i(X_1,\dots,X_n) \subset X_1\times\cdots\times X_n$
  consisting of n-tuples with at least i coordinates $= *$.
  T_1 is called a fat wedge
  T_{n-1} is called a wedge.
- K = simplicial complex, R = comm. ring w/ 1
  The Stanley-Reisner ring: $R[K] := R[v_1,\dots,v_n]/I_K$
  I_K = Stanley-Reisner ideal $= \{v_{i_1}\cdots v_{i_t} \mid \{i_1,\dots,i_t\}\notin K\}$
- Davis-Januszkiewicz '91
  $DJ(K)$ with $H^*(DJ(K);\mathbb{Z}) \cong \mathbb{Z}[K]$.

$*^n \subseteq DJ(K) \subseteq (\mathbb{C}P^\infty)^n$

Defn <u>moment-angle complex</u>: (D^2, S^1) = pair of spaces

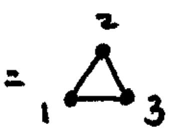
and K = simplicial complex with n vertices.

$$Z_K(D^2, S^1) = (D^2, S^1)^K = \bigcup_{\sigma \in K} D_\sigma = \operatorname{colim}_{\sigma \in K} D_\sigma \subseteq (D^2)^n.$$

where $D_\sigma = \{ (x_1, \dots, x_n) \in (D^2)^n \mid x_i \in S^1 \text{ if } i \notin \sigma \}$

ex:
- $K = \overset{1}{\bullet}\ \overset{2}{\bullet} = \{\{1\}, \{2\}\}$; $D_{\{1\}} = D^2 \times S^1$, $D_{\{2\}} = S^1 \times D^2$

  $\Rightarrow (D^2, S^1)^K = D^2 \times S^1 \cup S^1 \times D^2 = \partial (D^2 \times D^2) = S^3.$

- $K =$  , $(D^2, S^1)^K = D^2 \times D^2 \times S^1 \cup D^2 \times S^1 \times D^2 \cup S^1 \times D^2 \times D^2$
  $= \partial (D^2 \times D^2 \times D^2) = S^7$

- $K = \partial \Delta^{n-1}$, $(D^2, S^1)^K = S^{2n-1}$.

Defn: <u>real moment-angle complex</u>: (D^1, S^0) = pair of spaces,

K = s.c. on $[n] = \{1, \dots, n\}$. Then

$$Z_K(D^1, S^0) = (D^1, S^0)^K = \operatorname{colim}_{\sigma \in K} D_\sigma \subseteq (D^1)^n$$

where $D_\sigma = \{ (x_1, \dots, x_n) \in (D^1)^n \mid x_i \in S^0 \text{ if } i \notin \sigma \}$.

ex:
- $K = \overset{1}{\bullet}\ \overset{2}{\bullet} \Rightarrow (D^1, S^0)^K = S^1$
- $K = \partial \Delta^{n-1} \Rightarrow (D^1, S^0)^K = S^{n-1}$.
- K = n-gon $\Rightarrow (D^1, S^0)^K = M_g$, $g = 1 + (n-4) 2^{n-3}$.

  Coxeter '38

Defn Polyhedral Product:

$(\underline{X},\underline{A}) = \{(X_i,A_i)\}_{i=1}^{n}$ sequence of pairs of spaces, K s.c. on $[n]$.

$Z_K(\underline{X},\underline{A}) = (\underline{X},\underline{A})^K = \operatorname{colim}_{\sigma\in K} D_\sigma$, $D_\sigma = \{(x_1,\dots,x_n)\in \prod X_i \mid x_i \in A_i \text{ if } i\notin\sigma\}$

(Can be thought of as a colim of a diagram of spaces $D: K \to Top$.)

Ex: • Momen-angle complexes & real moment-angle cx

& Applications are polyhedral products

- $X\vee\cdots\vee X = (X,*)^{K^0}$, where K^0 = 0-skeleton of Δ^{n-1}.
- $T_i(X_1,\dots,X_n) = (\underline{X},\underline{*})^{K^{n-i}}$, K^{n-i} = $(n-i)$-skeleton of Δ^{n-1}.
- $DJ(K) = ET^n \times_{T^n} (D^2,S^1)^n \simeq (BS^1,*)^K = (\mathbb{C}P^\infty,*)^K \subseteq (\mathbb{C}P^\infty)^n$.

Proved by Buchstaber-Panov.

- Davis-Januszkiewicz '91 Every quasi-toric manifold can be realized as the quotient of a moment-angle cx. by the free action of a real torus.

  $(M_P(\chi) = T^n \times P/\sim$, $(t,p)\sim(u,q)$ if $p=q$ & $tu^{-1}\in \operatorname{im}(\chi: T^m\to T^n)$

  (K = dual of ∂P, free action of $T^{m-n} = \ker(\chi)$.)

- Goresky-MacPherson '88 Complement of of subspace arrangements:

  $\mathcal{U}(K) = \mathbb{C}^n \setminus \bigcup_{I\notin K} \{z\in\mathbb{C}^n \mid z_i = 0 \text{ if } i\in I\}$

G-M studied the cohomology of $\mathcal{U}(K)$.

<u>Buchstaber-Panov</u> showed $\mathcal{U}(K) \simeq (\mathbb{C}, \mathbb{C}^*)^K \simeq (D^2, S^1)^K$ and

$$H^*(Z_K(D^2, S^1); R) = \mathrm{Tor}_{\underbrace{R[v_1, \dots, v_n]}_{M}}(R[K], R)$$

H_* of free M-resolution of R, $\otimes_M$ with $R[K]$.

- <u>Toral Rank Conjecture</u>: $T^k \curvearrowright X$ almost freely, X fin. dim'l.
  (S. Halperin '85)
  Then $\dim H^*(X; \mathbb{Q}) \geq 2^k$.

  <u>Y. Ustinovsky</u> $\Rightarrow$ TRC holds for moment-angle complexes.

- <u>Bahri-Bendersky-Cohen-Gitler 09</u>:

$$\Sigma (X, A)^K \xrightarrow{\simeq} \Sigma \bigvee_{I \subseteq [n]} \widehat{(\underline{X_I}, \underline{A_I})^{K_I}}$$

  Far-reaching generalization of $\Sigma X * X = \Sigma X \wedge X \vee \Sigma X \vee \Sigma X$.

- <u>Grbic-Panov-Theriault-Wu</u>: K = flag complex. TFAE:

  1. $\Gamma = K^1$ is a chordal graph
  2. $(D^2, S^1)^K$ is a wedge of spheres
  3. $\mathbb{k}[K]$ is a Golod ring
  4. Multiplication in $H^*((D^2, S^1)^K; \mathbb{Q})$ is trivial.

A variety of contexts

$(\underline{X}, \underline{A})$	$(\underline{X}, \underline{A})^K$
(D^2, S^1)	toric geometry & topology
(D^1, S^0)	surfaces, number theory
(S^1, S^1_+)	robotics
$(\mathbb{C}, \mathbb{C}^*)$	complements of hyperplane arrangements
(EG, G)	free groups
$(BG, *)$	monodromy, combinatorics
$(PX, \Omega X)$	homotopy theory.

Monodromy representations:

$G =$ top group. Then

$$\begin{array}{ccc} Z_K(EG, G) & \longrightarrow & Z_K(BG, *) \\ & & \downarrow \\ & & (BG)^n \end{array}$$

Study the case when G is finite. (abelian / non-abelian).

{Group Actions}

Denham - Suciu: $G_1, \dots, G_n$ are topological groups, $\underbrace{BG_i}_{\text{class. sp.}}$, $\underbrace{EG_i}_{\simeq * \text{ space w/ free } G_i\text{-action}}$.

$$Z_K(EG_i, G_i) \longrightarrow \underbrace{\prod_{i=1}^{n} EG_i \times_{\prod_{i=1}^{n} G_i} Z_K(EG_i, G_i)}_{\simeq Z_K(BG_i, *)} \longrightarrow \prod_{i=1}^{n} BG_i \qquad \underline{\text{bundle}}$$

$\left((D^2, S^1) \text{ is a special case} \right)$.

$$Z_K(EG_i, G_i) \longrightarrow Z_K(BG_i) \hookrightarrow \prod_{i=1}^{n} BG_i$$

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What happens when G_i = finite discrete:

$$Z_K(EG_i, G_i) \longrightarrow Z_K(BG_i) \hookrightarrow \prod BG_i$$

doesn't contain the boundary of 2-simplex or higher.

<u>Thm</u>: $Z_K(BG_i, *) \simeq K(\pi, 1)$ iff K is a <u>flag complex</u>. (M. Davis $\Leftarrow$)

In general: $\pi_1 Z_K(BG_i) = \prod_{K_1} G_i$ = graph product of G_i.

<u>K = flag</u>: $1 \to \pi_1(Z_K(EG_i, G_i)) \longrightarrow \prod_{K_1} G_i \longrightarrow \prod G_i \to 1$.

$\prod G_i \circlearrowright H_1(Z_K(EG_i, G_i))$

<u>K = flag = K_0</u>: $1 \to \underbrace{\pi_1(Z_K(EG_i, G_i))}_{\text{free gp.}} \longrightarrow * G_i \longrightarrow \prod G_i \to 1$.

<u>Prop</u>: Rank of free gp is $(n-1) \prod_{i=1}^{n} |G_i| - \sum_{i=1}^{n} \left(\prod_{j \neq i} |G_j| \right) + 1$

an early result of Nielsen using algebra. Here we use topology.

<u>when is π_1(fibre) free?</u>: <u>Answ</u>. Iff K_1 is a chordal graph (i.e. triangulated graph.)

<u>What is the monodromy rep</u>?

K_0 w/ 2 vertices, $G_1 = \mathbb{Z}/2 = \langle x \rangle = \{1, x\}$, $G_3 = \mathbb{Z}/3 = \langle y \rangle = \{1, y, y^2\}$

$$1 \to F_2 \longrightarrow \mathbb{Z}_2 * \mathbb{Z}_3 \longrightarrow \mathbb{Z}_2 \times \mathbb{Z}_3 \to 1$$

w/ $F_2 = \langle w_1, w_2 \rangle$, $w_1 = [x,y]$, $w_2 = [x,y^2]$.

$x w_1 x^{-1} = w_1^{-1}$
$x w_2 x^{-1} = w_2^{-1}$

$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ in homology

$y w_1 y^{-1} = w_1^{-1} w_2$
$y w_2 y^{-1} = w_1^{-1}$

$\begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}$.

$\mathbb{Z}_2 * \mathbb{Z}_3 \hookrightarrow Aut(F_2)$

$\mathbb{Z}_2 \times \mathbb{Z}_3 \hookrightarrow SL_2(\mathbb{Z})$.

For any finite cyclic $\mathbb{Z}/n$, $\mathbb{Z}/m$

$\mathbb{Z}_n * \mathbb{Z}_m \hookrightarrow Aut(F_N)$

$\mathbb{Z}_n \times \mathbb{Z}_m \longrightarrow \cancel{Aut}_N(\mathbb{Z})$

(*) Note that none of these aut. are in IA_N .

<u>If we</u> choose non-abelian groups, then the reps. don't necessarily land in $SL_N(\mathbb{Z})$. e.g. $G_1 = \mathbb{Z}/2$, $G_2 = \Sigma_3$.

<u>For a finite # of $G_1, \dots, G_n$:</u> The fibre has fundamental group w/ a generating set the iterated commutators

$$[g_j, g_i], [g_{k_1}, [g_j, g_i]], \dots, [g_{k_1}, \dots, [g_{k_\ell}, [g_j, g_i]] \dots],$$

where $g_t \in G_t$, w/ $k_1 < \dots < k_\ell < j$ & $j > i \neq k_r$ $\forall r$.

e.g. $G_1 = \{1, x\}$, $G_2 = \{1, y\}$, $G_3 = \{1, z, z^2\}$

$S = \{ [z,x], [z^2,x], [z,y], [z^2,y], [y,x], [x,[z,y]], [x,[z^2,y]],$
$[y,[z,x]], [y,[z^2,x]] \}$, $|S| = 9$

$\boxed{\text{Connections w/ Feit-Thompson theorem.}}$

For the monodromy rep., see Magma codes.

<u>Theorem</u>: Let $G_1, \dots, G_n$ be finite abelian gps. Then the faithful monodromy rep induces a faithful representation $G_1 \times \dots \times G_n \longrightarrow SL(\rho_n, \mathbb{Z})$.

<u>Theorem</u>: $\prod_{K^1} G_i \hookrightarrow Aut(F_{\rho_n})$ K^1 = chordal

Right-angled Artin / Coxeter groups.