DISCRETE AND CONTINUOUS
DYNAMICAL SYSTEMS SERIES B
Volume **10**, Number **4**, November **2008**

Website: http://aimSciences.org

pp. **801–822**

THE STOCHASTIC PRIMITIVE EQUATIONS IN TWO SPACE DIMENSIONS WITH MULTIPLICATIVE NOISE

Nathan Glatt-Holtz and Mohammed Ziane

Department of Mathematics
University of Southern California
Los Angeles, CA 90089, USA

(Communicated by Thomas Witelski)

Abstract. We study the two dimensional primitive equations in the presence of multiplicative stochastic forcing. We prove the existence and uniqueness of solutions in a fixed probability space. The proof is based on finite dimensional approximations, anisotropic Sobolev estimates, and weak convergence methods.

1. **Introduction.** The Primitive equations are a ubiquitous model in the study of geophysical fluid dynamics. They can be derived from the compressible Navier-Stokes equations by taking advantage of various properties common to geophysical flows. In particular, one uses the Boussinesq approximation, that fluctuations in the density of the fluid are much smaller than the mean density throughout. A scale analysis relying on the relative shallowness of the ocean at geophysical scales is employed to show that the pressure and the gravitational forces are the only relevant terms in the vertical momentum equation. This is referred to as the hydrostatic approximation. For further background and detailed physical derivations see [9] or [27], for example.

The mathematical study of the Primitive equations was initiated in the early 1990's with the work of Lions, Temam and Wang [21], [20], [19]. In these initial works the existence of global weak solutions and weak attractors were established and numerical schemes were developed. The case of local strong solutions was addressed by Guillén-González, Masmoudi, and Rodríguez-Bellido [14] and by Hu, Temam and Ziane [15]. Also see the survey article of Temam and Ziane[30]. Recent breakthroughs have yielded global existence of strong solutions in three space dimensions. The case of Neumann boundary conditions was addressed by Cao and Titi [6] and later independently by Kobelkov [17]. In subsequent work of Kukavica and Ziane [18] a different proof was discovered which covers the case of physically relevant boundary conditions.

For the two dimensional deterministic setting we mention the work of Petcu, Temam, and Wirosoetisno in [28] and Bresch, Kazhikhov and Lemoine [4] where both the cases of weak and strong solutions are considered. The 2-D primitive equations seem to be more difficult mathematically than the 2-D Navier-Stokes

2000 *Mathematics Subject Classification.* Primary: 76D03, 60H15; Secondary: 37L65.
Key words and phrases. Primitive Equations, Geophysical Fluid Dynamics, Stochastic Partial Differential Equations, Galerkin Approximation, Existence and Uniqueness.
The second author is supported by NSF grant DMS-0505974.

equations. For instance, it is still an open problem whether weak solutions of the Primitive equations in the deterministic setting are unique. This is an easy exercise for the 2-D Navier Stokes equations.

The addition of white noise driven terms to the basic governing equations for a physical system is natural for both practical and theoretical applications. For example, these stochastically forced terms can be used to account for numerical and empirical uncertainties and thus provide a means to study the robustness of a basic model. Specifically in the context of fluids, complex phenomena related to turbulence may also be produced by stochastic perturbations. For instance, in the recent work of Mikulevicius and Rozovsky [25] such terms are shown to arise from basic physical principals.

A wide body of mathematical literature exists for the stochastic Navier-Stokes equations. This analytic program dates back to the early 1970's with the work of Bensoussan and Temam [2]. For the study of well-posedness new difficulties related to compactness often arise due to the addition of a probabilistic parameter. For situations where continuous dependence on initial data remains open (for example in $d = 3$ when the initial data merely takes values in L^2) it has proven fruitful to consider Martingale solutions. Here one constructs a probabilistic basis as part of the solution. For this context we refer the reader to the works of Cruzeiro [8], Capinski and Gatarek [7], Flandoli and Gatarek [13] and of Mikulevicius and Rozovskii [23].

On the other hand, when working in spaces where continuous dependence on the initial data can be expected, existence of solutions can sometimes be established on a preordained probability space. Such solutions are often referred to as "strong" or "pathwise" solutions. In the two dimensional setting, Da Prato and Zabczyk [11] and later Breckner [3] as well as Menaldi and Sritharan [22] established the existence of pathwise solutions where $u \in L^\infty([0,T], L^2), \mathbb{P} - a.s.$ On the other hand, Bensoussan and Frehse [1] have established local solutions in 3-d for the class $C^\beta([0,T]; H^{2s})$ where $3/4 < s < 1$ and $\beta < 1 - s$. In the works of Mikulevicius and Rozovsky [26] and of Brzezniak and Peszat [5] the case of arbitrary space dimensions for local solutions evolving in Sobolev spaces of type $W^{1,p}$ for $p > d$ is addressed.

It is with this background in mind that we present the following examination of the two dimensional Primitive equations in the presence of multiplicative white noise terms on a preordained probability space. In the first section we introduce the model, providing an overview of the relevant function spaces and establish some anisotropic Sobolev type estimates on the nonlinear terms of the equation. A variational definition for solutions is then presented. We next turn to the Galerkin scheme. Since the best estimates for the nonlinear terms are closer to those currently available for the three dimensional Navier Stokes equations we need to make use of special cancellation properties available for the z direction. In this way we are able to infer uniform bounds for $\partial_z u^{(n)}$ in $L^p(\Omega; L^2(0,T;V) \cap L^\infty(0,T;H))$ along with those typical for $u^{(n)}$. To establish existence we apply weak convergence methods to identify a limit system. A comparison technique is then employed taking advantage of the additional regularity established for $u^{(n)}$. This allows us to identify certain point-wise limits from which we conclude that the system is indeed the desired equation. Uniqueness is established to conclude the final section.

2. **A model for the stochastic primitive equations in two space dimensions.** The two dimensional Primitive equations can be formally derived from the

full three dimensional system under the assumption of invariance with respect to the second horizontal variable y. As such we will assume that the initial data and the external forcing are independent of y. By adding a second external forcing term driven by a white noise, we arrive at the following non-linear stochastic evolution system:

$$\partial_t u - \nu\Delta u + u\partial_x u + w\partial_z u + \partial_x p = f + g(u,t)\dot{W}(t) \tag{1a}$$

$$\partial_x u + \partial_z w = 0. \tag{1b}$$

In this formulation we have ignored the coupling with the temperature and salinity equations in order to focus our attention towards difficulties arising from the non-linear term (see [4]). This omission will be remedied in future work. The unknowns (u, w), p represent the field of the flow and the pressure respectively. The fluid fills a domain $\mathcal{M} = [0, L] \times [0, -h] \ni (x, z)$. Note that p does not depend on the vertical variable z.

We partition the boundary into the top $\Gamma_i = \{z = 0\}$, the bottom $\Gamma_b = \{z = -h\}$ and the sides $\Gamma_s = \{x = 0\} \cup \{x = L\}$. Regarding the boundary conditions, we assume the Dirichlet condition $u = 0$ on Γ_s, while on $\Gamma_i \cup \Gamma_b$ we posit the free boundary condition $\partial_z u = 0, w = 0$. We further suppose with no loss of generality that:

$$\int_{-h}^{0} f\,dz = 0, \quad \int_{-h}^{0} g\,dz = 0, \quad \int_{-h}^{0} u\,dz = 0. \tag{2}$$

Due to (1b) we have that[1]:

$$w(x,z) = -\int_{-h}^{z} \partial_x u(x,\tilde{z})d\tilde{z}. \tag{3}$$

The stochastic term can be written in the expansion:

$$g(u,t)\dot{W}(t) = \sum_k g_k(u,t)\dot{\beta}_k(t). \tag{4}$$

The $\dot{\beta}_k$ are the formal time derivatives of a collection of independent standard Brownian motions β_k relative to some ambient, filtered, right continuous, probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$. The series converges in the appropriate function spaces and is subject to uniform Lipschitz conditions in u. All of this will be formulated rigorously below.

3. **Definitions.**

We will be working on the Hilbert spaces:

$$H = \left\{ v \in L^2(\mathcal{M}) : \int_{-h}^{0} v\,dz = 0 \right\} \tag{5}$$

and

$$V = \left\{ v \in H^1(\mathcal{M}) : \int_{-h}^{0} v\,dz = 0, v = 0 \text{ on } \Gamma_s \right\}. \tag{6}$$

These spaces are endowed with the L^2 and H^1 norms which we respectively denote by $|\cdot|$ and $\|\cdot\|$. We shall also need the intermediate space:

$$X = \{v \in H : \partial_z v \in H\} \tag{7}$$

[1] In the geophysical literature w is often referred as a diagnostic variable as its value is completely determined by u

with the norm $|u|_X = (|u|^2 + |\partial_z u|^2)^{1/2}$. Take V' to be the dual of V with the pairing notated by $\langle \cdot, \cdot \rangle$. The Leray operator P_H is the orthogonal projection of $L^2(\mathcal{M})$ onto H. The action of this operator is given explicitly by:

$$P_H v = v - \frac{1}{h}\int_{-h}^{0} v\, dz. \tag{8}$$

It will also be useful to consider:

$$\mathcal{V} = \left\{ v \in C^\infty(\bar{\mathcal{M}}) : \int_{-h}^{0} v\, dz = 0, v = 0 \text{ on } \Gamma_s, \partial_z v = 0 \text{ on } \Gamma_b \cup \Gamma_i \right\} \tag{9}$$

which is a dense subset of H, X and V.

We would now like to make (1) precise as an equation in V'. To this end, we define a Stokes-type operator A as a bounded map from V to V' via:

$$\langle v, Au \rangle = ((v, u)).$$

A can be extended to an unbounded operator from H to H according to $Au = -P_H \Delta u$ with the domain:

$$D(A) = \left\{ v \in H^2(\mathcal{M}) : \int_{-h}^{0} v\, dz = 0, v = 0 \text{ on } \Gamma_s, \partial_z v = 0 \text{ on } \Gamma_b \cup \Gamma_i \right\}. \tag{10}$$

By applying the theory of symmetric compact operators for A^{-1}, one can prove the existence of an orthonormal basis $\{e_k\}$ for H of eigenfunctions of A. Here the associated eigenvalues $\{\lambda_k\}$ form an unbounded, increasing sequence. Define:

$$H_n = \mathrm{span}\{e_1, \ldots, e_n\}$$

and take P_n to be the projection from H onto this space. Let $Q_n = I - P_n$.

Remark 1. For $u \in D(A)$:

$$\int_{-h}^{0} \partial_{xx} u\, dz = \partial_{xx} \int_{-h}^{0} u\, dz = 0, \qquad \int_{-h}^{0} \partial_{zz} u\, dz = \partial_z u]_{z=-h}^{z=0} = 0.$$

Thus, as in the case of periodic boundary conditions, we have $-P_H \partial_{xx} = -\partial_{xx}$ and $-P_H \partial_{zz} = -\partial_{zz}$ and therefore that $-P_H \Delta = -\Delta$ on $D(A)$. The eigenvalue problem for A reduces to:

$$\Delta u = \lambda u$$
$$\int_{-h}^{0} u\, dz = 0$$
$$u(0, z) = 0 = u(L, z), \quad \partial_z u(x, -h) = 0 = \partial_z u(x, 0).$$

As such, the eigenfunctions and associated eigenvalues can be explicitly identified:

$$\left\{ \frac{2}{\sqrt{hL}} \sin(\frac{k_1 \pi x}{L}) \cos(\frac{k_2 \pi z}{h}) \right\}_{k_1, k_2 \geq 1}, \qquad \left\{ \pi^2 \left(\frac{k_1^2}{L^2} + \frac{k_2^2}{h^2} \right) \right\}_{k_1, k_2 \geq 1}.$$

This has the useful consequence that when $j \neq l$:

$$\langle \partial_{zz} e_j, e_l \rangle = 0 \tag{11}$$

and hence:

$$P_n(-\partial_{zz} v) = -\partial_{zz} v \qquad \text{whenever } v \in H_n. \tag{12}$$

Next we address the nonlinear term. In accordance with (3) we take:

$$\mathcal{W}(v)(x,z) := -\int_{-h}^{z} \partial_x v(x,\tilde{z})d\tilde{z} \qquad v \in \mathcal{V}$$

and let:

$$B(u,v) := P_H(u\partial_x v + \mathcal{W}(u)\partial_z v).$$

Below it will sometimes be convenient to denote $B(u) := B(u,u)$. One would like to establish that B is a well defined and continuous mapping from $V \times V \to V'$ according to:

$$\langle B(u,v),\phi\rangle = b(u,v,\phi)$$

where the associated trilinear form is given by:

$$b(u,v,\phi) := \int_{\mathcal{M}} (u\partial_x v\phi - \mathcal{W}(u)\partial_z v\phi)\, d\mathcal{M} := b_1(u,v,\phi) - b_2(u,v,\phi).$$

This and more is contained in the following lemma:

Lemma 3.1.

(i) b is a continuous linear form on $V \times V \times V$ and:

$$|b(u,v,\phi)| \le C\left(|u|^{1/2}\|u\|^{1/2}\|v\||\phi|^{1/2}\|\phi\|^{1/2} + |\partial_x u||\partial_z v||\phi|^{1/2}\|\phi\|^{1/2}\right) \tag{13}$$

for any $u,v,\phi \in V$

(ii) b satisfies the cancellation property $b(u,v,v) = 0$

(iii) b is also a continuous form on $D(A) \times D(A) \times H$

(iv) For $u \in D(A)$ we have the additional cancellation property:

$$\langle B(u), \partial_{zz} u\rangle = 0$$

(v) Moreover, for any $\epsilon > 0$:

$$\begin{aligned}|\langle B(u),\partial_{xx}u\rangle| &\le C(|\partial_x u|^2\|\partial_x u\| + |\partial_x u|^{1/2}\|\partial_x u\|^{3/2}|\partial_z u|^{1/2}\|\partial_z u\|^{1/2})\\ &\le \epsilon\|\partial_x u\|^2 + C(\epsilon)(|\partial_x u|^4 + |\partial_x u|^2|\partial_z u|^2\|\partial_z u\|^2)\end{aligned} \tag{14}$$

(vi) If $u,v \in V$ and $\partial_z v \in V$ then:

$$|B(u,v)|_{V'} \le C((|u| + |\partial_z u|)|\partial_x v| + |u|\|\partial_z v\| + |u||\partial_z v|^{1/2}\|\partial_z v\|^{1/2}) \tag{15}$$

Proof. Fix $u,v,\phi \in \mathcal{V}$. The first term b_1 admits the classical 2-D estimate:

$$|b_1(u,v,\phi)| \le C|u|^{1/2}\|u\|^{1/2}\|v\||\phi|^{1/2}\|\phi\|^{1/2}.$$

The second term is estimated anisotropically:

$$\begin{aligned}&|b_2(u,v,\phi)|\\ &\quad\le \int_0^L \left(\sup_{z\in[-h,0]}\left\{\int_{-h}^{z}\partial_x u\, d\bar{z}\right\}\int_0^L |\partial_z v\phi|\, dz\right) dx\\ &\quad\le C\int_0^L \left(\int_{-h}^0 |\partial_x u|^2\, dz \cdot \int_{-h}^0 |\partial_z v|^2\, dz \cdot \int_{-h}^0 |\phi|^2\, dz\right)^{1/2} dx\\ &\quad\le C\sup_{x\in[0,L]}\left(\int_{-h}^0 |\phi|^2\, dz\right)^{1/2}\int_0^L\left(\int_{-h}^0 |\partial_x u|^2\, dz \cdot \int_{-h}^0 |\partial_z v|^2\, dz\right)^{1/2} dx\\ &\quad\le C|\partial_x u||\partial_z v||\phi|^{1/2}\|\phi\|^{1/2}.\end{aligned}$$

For the final inequality we make use of the boundary conditions:

$$\sup_{x\in[0,L]}\left\{\int_{-h}^{0}|\phi|^2\,dz\right\}=\sup_{x\in[0,L]}\left\{\int_0^x\partial_x\int_{-h}^0\phi^2\,dz\,dx\right\}\leq 2|\phi|\|\phi\|.$$

To establish the cancellation property in (ii):

$$\begin{aligned} b(u,v,v)&=-\frac{1}{2}\int_{\mathcal{M}}\partial_x uv^2\,d\mathcal{M}-\frac{1}{2}\int_{\mathcal{M}}\left(\int_{-h}^{z}\partial_x ud\tilde{z}\right)\partial_z(v^2)\,d\mathcal{M}\\ &=-\frac{1}{2}\int_{\mathcal{M}}\partial_x uv^2\,d\mathcal{M}+\frac{1}{2}\int_{\mathcal{M}}\partial_x uv^2\,d\mathcal{M}=0.\end{aligned}$$

Property (iii) is a direct application of Hölder's Inequality and Sobolev embedding inequalities and is omitted.

For (iv), noting that $-P_H\partial_{zz}=-\partial_{zz}$ on $D(A)$:

$$\langle B(u),\partial_{zz}u\rangle=\int_{\mathcal{M}}(-\partial_x u(\partial_z u)^2-u\partial_{xz}u\partial_z u+\frac{1}{2}\partial_x u(\partial_z u)^2)\,d\mathcal{M}=0.$$

The inequality given in (v) is addressed by estimating:

$$\begin{aligned}|\langle B(u),\partial_{xx}u\rangle|&\leq\frac{1}{2}|\partial_x u|^3_{L^3(\mathcal{M})}+\left|\int_{\mathcal{M}}(\mathcal{W}(u)\partial_{zx}u\partial_x u+\partial_x\mathcal{W}(u)\partial_z u\partial_x u)\,d\mathcal{M}\right|\\ &\leq|\partial_x u|^3_{L^3(\mathcal{M})}+\int_{\mathcal{M}}|\partial_x\mathcal{W}(u)\partial_z u\partial_x u|\,d\mathcal{M}.\end{aligned}$$

For the first term above we use the Sobolev embedding $H^{1/3}\subset L^3$. For the second term we have $H^{1/2}\subset L^4$ which justifies the estimate:

$$\begin{aligned}\int_{\mathcal{M}}|\partial_x\mathcal{W}(u)\partial_z u\partial_x u|\,d\mathcal{M}&\leq C|\partial_{xx}u||\partial_z u|_{L^4}|\partial_x u|_{L^4}\\ &\leq C|\partial_x u|^{1/2}\|\partial_x u\|^{3/2}|\partial_z u|^{1/2}\|\partial_z u\|^{1/2}.\end{aligned}$$

The second inequality is just an application of ϵ-Young.

For the final item (vi) fix $\phi\in V$. In this case we estimate b_1 anisotropically:

$$\begin{aligned}b_1(u,v,\phi)=&\int_0^L\int_{-h}^0 u\partial_x v\phi\,d\mathcal{M}\\ \leq& C\int_0^L\left(\int_{-h}^0(|\partial_z u|+|u|)^2dz\cdot\int_{-h}^0|\partial_x v|^2dz\cdot\int_{-h}^0|\phi|^2dz\right)^{1/2}dx\\ \leq& C(|\partial_z u|+|u|)|\partial_x v||\phi|^{1/2}\|\phi\|^{1/2}.\end{aligned}\tag{16}$$

For b_2, by integrating by parts in, x we find:

$$\begin{aligned}b_2(u,v,\phi)=&\int_0^L\int_{-h}^0\left(\int_{-h}^z\partial_x u\,d\bar{z}\right)\partial_z v\phi\,d\mathcal{M}\\ =&-\int_0^L\int_{-h}^0\left(\int_{-h}^z u\,d\bar{z}\right)\partial_{zx}v\phi\,d\mathcal{M}\\ &-\int_0^L\int_{-h}^0\left(\int_{-h}^z u\,d\bar{z}\right)\partial_z v\partial_x\phi\,d\mathcal{M}\\ :=&T_1(u,v,\phi)+T_2(u,v,\phi).\end{aligned}\tag{17}$$

For T_1 the anisotropic estimates yield:

$$\begin{aligned} |T_1(u,v,\phi)| \leq& C\int_0^L \left(\int_{-h}^0 |u|^2 dz \cdot \int_{-h}^0 |\partial_{zx} v|^2 dz \cdot \int_{-h}^0 |\phi|^2 dz\right)^{1/2} dx \\ \leq& |u|\|\partial_z v\||\phi|^{1/2}\|\phi\|^{1/2}. \end{aligned} \tag{18}$$

The estimate for T_2 is similar except that we make the L^∞_x estimate on the middle term:

$$\begin{aligned} |T_2(u,v,\phi)| \leq& C\int_0^L \left(\int_{-h}^0 |u|^2 dz \cdot \int_{-h}^0 |\partial_z v|^2 dz \cdot \int_{-h}^0 |\partial_x\phi|^2 dz\right)^{1/2} dx \\ \leq& C|u||\partial_z v|^{1/2}\|\partial_z v\|^{1/2}\|\phi\|. \end{aligned} \tag{19}$$

□

It remains to examine the stochastically forced term $g = \{g_k\}_{k\geq 1}$ in order to make precise the Lipschitz condition alluded to above. For this purpose we introduce some notation. Suppose U is any (separable) Hilbert space. One defines $\ell^2(U)$ via the inner product:

$$(h,g)_{\ell^2(U)} = \sum_k (h_k, g_k)_U.$$

For any normed space Y, we say that $g : Y \times [0,T] \times \Omega \to \ell^2(U)$ is uniformly Lipschitz with constant K_Y if:

$$|g(x,t,\omega) - g(y,t,\omega)|_{\ell^2(U)} \leq K_Y |x-y|_Y \quad \text{for } x,y \in Y \tag{20}$$

and

$$|g(x,t,\omega)|_{\ell^2(U)} \leq K_Y(1 + |x|_Y) \tag{21}$$

where K_Y is independent of t and ω. We denote the collection of all such mappings $\mathrm{Lip}_u(Y, \ell^2(U))$. For the analysis below we will frequently assume that $g \in \mathrm{Lip}_u(H, \ell^2(H)) \cap \mathrm{Lip}_u(X, \ell^2(X))$. It is worth noting at this juncture that the condition imposed on g is not overly restrictive by considering some examples where the above conditions are satisfied:

Example 3.2.

- (*Independently Forced Modes*) Suppose $(\kappa_k(t,\omega))$ is any sequence uniformly bounded in $L^\infty([0,T]\times\Omega)$. We force the modes independently defining:

  $$g_k(v,t,\omega) = \kappa_k(t,\omega)(v, e_k)e_k.$$

  In this case the Lipschitz constants can be taken to be $K_H = K_X = K_V = \sup_{\omega,k,t} |\kappa_k(t,\omega)|$.
- (*Uniform Forcing*) Given a uniformly square summable sequence $a_k(t,\omega)$ we can take:

  $$g_k(v,t,\omega) = a_k(t,\omega)v$$

  with $K_H = K_X = K_V = (\sup_{t,\omega}\sum_k a_k(t,\omega)^2)^{1/2}$ as the Lipschitz constants.
- (*Additive Noise*) We can also include the case when the noise term does not depend on the solution:

  $$g_k(v,t,\omega) = g_k(t,\omega)$$

  Here the uniform constants can be taken to be $K_U := \sup_{t,\omega}\left(\sum_k |g_k(t,\omega)|_U^2\right)^{1/2}$ for $U = H, X, V$ as desired.

With the above framework in place we now give a variational definition for solutions of the system (1). Note that *weak* refers to the spatial-temporal regularity of the solutions. Strong refers to the fact that the probabilistic basis is given in advance (see Remark 2 below).

Definition 3.3 (Weak-Strong Solutions)**.** Suppose that $(\Omega, \mathbb{P}, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, (\beta_k))$ is a fixed stochastic basis, $T > 0$ and $p \in [2, \infty]$. For the data assume that $u_0 \in L^p(\Omega; H)$ and is $\mathcal{F}_0$-measurable. We suppose that f and g are respectively H and $\ell^2(H)$ valued, predictable processes with:

$$f \in L^p(\Omega, L^2(0, T; V')), \quad g \in \mathrm{Lip}_u(H, \ell^2(H)). \tag{22}$$

We say that an $\mathcal{F}_t$ adapted process u is a *weak-strong solution* to the stochastically forced primitive equation if:

$$u \in C([0, T]; H) \text{ a.s.} \quad u \in L^p(\Omega; L^\infty([0, T]; H) \cap L^2([0, T]; V)) \tag{23}$$

and satisfies:

$$\begin{gathered} d\langle u(t), v\rangle + \langle \nu A u + B(u), v\rangle dt = \langle f, v\rangle dt + \sum_{k=1}^{\infty} \langle g_k(u, t), v\rangle d\beta_k \\ \langle u(0), v\rangle = \langle u_0, v\rangle \end{gathered} \tag{24}$$

for any $v \in V$.

Several remarks are in order regarding this definition:

Remark 2.

- Note that, as in the theory of the Navier-Stokes equations, the pressure disappears in the variational formulation. Suppose that $\partial_x p$ in (1) is integrable and does not depend on the vertical variable z. Then:
$$\int_0^L \int_{-h}^0 \partial_x p v \, dz \, dx = \int_0^L \partial_x p \left(\int_{-h}^0 v \, dz \right) dx = 0 \tag{25}$$
for every $v \in V$.
- For the probabilistically 'strong' solutions we consider, the stochastic basis is given in advance. Such solutions can be understood pathwise. This is in contrast to the theory of *Martingale solutions* considered for many non linear systems where the underlying probability space is constructed as part of the solution. See [10] chapter 8 or [24].
- One has to check that each of the terms given in (24) are well defined. In particular, the stochastic terms deserve special attention. Recall that the collection $\mathcal{M}^2(0, T)$ of continuous square integrable martingales is a Banach space under the norm:
$$\|X\|_{\mathcal{M}^2} = \left(\mathbb{E} \sup_{0\leq t\leq T} |X(t)|^2 \right)^{1/2}.$$
Applying standard Martingale inequalities and making use of the uniform Lipschitz assumption one establishes that:
$$\sum_k \langle g_k(u, t), v\rangle d\beta_k \quad \text{for all } v \in V$$
converges in this space. See [16] or [10] for further details on the general construction of stochastic integrals.

4. **The Galerkin systems and a priori estimates.** We now introduce the Galerkin systems associated to the original equation and establish some uniform a priori estimates.

Definition 4.1 (The Galerkin System)**.** An adapted process $u^{(n)}$ in $C(0,T;H_n)$ is a solution to the *Galerkin System of Order* n if for any $v \in H_n$:

$$\begin{aligned} d\langle u^{(n)}, v\rangle + \langle \nu A u^{(n)} + B(u^{(n)}), v\rangle dt &= \langle f, v\rangle dt + \sum_{k=1}^{\infty} \langle g_k(u^{(n)}, t), v\rangle d\beta_k \\ \langle u^{(n)}(0), v\rangle &= \langle u_0, v\rangle. \end{aligned} \tag{26}$$

These systems also be written as equations in H_n:

$$\begin{aligned} du^{(n)} + (\nu A u^{(n)} + P_n B(u^{(n)}))dt &= P_n f dt + \sum_{k=1}^{\infty} P_n g_k(u^{(n)}, t) d\beta_k \\ u^{(n)}(0) &= P_n u_0. \end{aligned} \tag{27}$$

We note that the second formulation (27) allows one to treat $u^{(n)}$ as a process in $\mathbb{R}^n$. As such one can apply the finite dimensional Itô calculus to the Galerkin systems above.

We next establish some uniform estimates on $u^{(n)}$ (independent of n). To simplify notation we drop the (n) superscript for the remainder of the section.

Lemma 4.2 (A Priori Estimates)**.**

(i) Assume that u is the solution of the Galerkin System of Order n. Suppose that $p \geq 2$ and:

$$g \in \mathrm{Lip}_u(H, \ell^2(H)), \quad f \in L^p(\Omega; L^2(0,T;V')), \quad u_0 \in L^p(\Omega, H) \tag{28}$$

then:

$$\mathbb{E}\left(\sup_{t\in[0,T]} |u|^p + \int_0^T \|u\|^2 |u|^{p-2} dt \right) \leq C_W \tag{29}$$

and:

$$\mathbb{E}\left(\int_0^T \|u\|^2 dt \right)^{p/2} \leq C_W \tag{30}$$

for an appropriate constant

$$C_W = C_W(p, \nu, \lambda_1, \mathbb{E}|u_0|^p, T, |f|_{L^p(\Omega;L^2(0,T;V'))}, K_H),$$

that does not depend on n.

(ii) Given the additional assumptions on the data:

$$g \in \mathrm{Lip}_u(X, \ell^2(X)), \ \ \partial_z f \in L^p(\Omega; L^2(0,T;V')), \ \ \partial_z u_0 \in L^p(\Omega; H) \tag{31}$$

then:

$$\mathbb{E}\left(\sup_{t\in[0,T]} |\partial_z u|^p + \int_0^T \|\partial_z u\|^2 |\partial_z u|^{p-2} dt \right) \leq C_I \tag{32}$$

where $C_I = C_I(p, \nu, \lambda_1, \mathbb{E}|\partial_z u_0|^p, T, |\partial_z f|_{L^p(\Omega;L^2(0,T;V'))}, K_X)$, independent of n.

(iii) Finally assume that in addition to (28):

$$g \in \mathrm{Lip}_u(V, \ell^2(V)), \quad f \in L^p(\Omega; L^2(0,T;H)), \quad u_0 \in L^p(\Omega, V). \tag{33}$$

If τ is a stopping time taking values in $[0,T]$, and M a positive constant so that:

$$\int_0^\tau (\|u\|^2 + |\partial_z u|^2 \|\partial_z u\|^2) dt \leq M \tag{34}$$

then:

$$\mathbb{E}\left(\sup_{t\in[0,\tau]} \|u\|^p + \int_0^\tau |Au|^2 \|u\|^{p-2} dt \right) \leq C_S e^{C_S M} \tag{35}$$

where $C_S = C_S(p, \nu, \mathbb{E}\|u_0\|^p, T, |f|_{L^p(\Omega;L^2(0,T;H))}, K_V)$.

Proof. By applying Itô's formula one finds a differential for $|u|^2$ and then for $e^\phi |u|^p$:

$$\begin{aligned}
&de^\phi |u|^p + p\nu e^\phi \|u\|^2 |u|^{p-2} dt \\
=&pe^\phi \langle f, u\rangle |u|^{p-2} dt + \frac{p}{2} e^\phi \sum_{k=1}^\infty |P_n g_k(u,t)|^2 |u|^{p-2} dt \\
&+ \frac{p(p-2)}{2} e^\phi \sum_{k=1}^\infty \langle g_k(u,t), u\rangle^2 |u|^{p-4} dt \\
&+ pe^\phi \sum_{k=1}^\infty \langle g_k(u,t), u\rangle |u|^{p-2} d\beta_k + \phi' |u|^p e^\phi dt.
\end{aligned} \tag{36}$$

Here, ϕ is a non-positive element in $C^1(0,T)$ to be determined below. This function will be used to cancel off terms involving $\int_0^T e^\phi |u|^p dt$. For the deterministic external forcing term we estimate:

$$\begin{aligned}
&\int_0^T pe^\phi \langle f, u\rangle |u|^{p-2} dt \\
\leq& C(\nu, p, \lambda_1) \int_0^T e^\phi |f|^2_{V'} |u|^{p-2}\, dt + \frac{\nu p}{2} \int_0^T e^\phi \|u\|^2 |u|^{p-2}\, dt \\
\leq& C(\nu, p, \lambda_1) \left(\sup_{t\in[0,T]} (e^{(p-2)\phi/p} |u|^{p-2}) \right) \int_0^T |f|^2_{V'} dt \\
&+ \frac{\nu p}{2} \int_0^T e^\phi \|u\|^2 |u|^{p-2}\, dt \\
\leq& \frac{1}{4} \sup_{t\in[0,T]} (e^\phi |u|^p) + \frac{p\nu}{2} \int_0^T e^\phi \|u\|^2 |u|^{p-2}\, dt \\
&+ C(\nu, p, \lambda_1) |f|^p_{L^2(0,T;V')}.
\end{aligned} \tag{37}$$

Taking advantage of the Lipschitz condition assumed for g:

$$\begin{aligned}
&\sum_{k=1}^\infty \int_0^T e^\phi |g_k(u,t)|^2 |u|^{p-2} dt \\
\leq& K_H \int_0^T e^\phi (1 + |u|^2) |u|^{p-2} dt \\
\leq& \frac{1}{4} \sup_{t\in[0,T]} (e^\phi |u|^p) + C(p, K_H, T) \left(1 + \int_0^T e^\phi |u|^p\, dt \right).
\end{aligned} \tag{38}$$

Applying the above estimates, absorbing terms and rearranging one deduces:

$$
\begin{aligned}
&\sup_{t\in[0,T]}(e^{\phi}|u|^p) + p\nu\int_0^T e^{\phi}\|u\|^2|u|^{p-2}dt \\
\leq& C(p,\nu,\lambda_1)\left(|u_0|^p + |f|^p_{L^2(0,T;V')} + 1\right) \\
&+ C_1(p,K_H,T)\int_0^T e^{\phi}|u|^p dt + 2\int_0^T \phi' e^{\phi}|u|^p dt \\
&+ 2p\sup_{t\in[0,T]}\left|\int_0^t e^{\phi}\sum_{k=1}^{\infty}\langle g_k(u,t),u\rangle|u|^{p-2}d\beta_k\right|.
\end{aligned}
\tag{39}
$$

For the final term involving the Itô integral we apply the Burkholder-Davis-Gundy (BDG) inequality (see [16]). This yields the following:

$$
\begin{aligned}
&2p\mathbb{E}\sup_{t\in[0,T]}\left|\sum_{k=1}^{\infty}\int_0^t e^{\phi}\langle g_k(u,s),u\rangle|u|^{p-2}d\beta_k\right| \\
\leq& C(p)\mathbb{E}\left(\int_0^T e^{2\phi}\sum_{k=1}^{\infty}\langle g_k(u,t),u\rangle^2|u|^{2(p-2)}dt\right)^{1/2} \\
\leq& C(K_H,p)\mathbb{E}\left(\int_0^T e^{2\phi}(1+|u|^2)|u|^{2(p-1)}dt\right)^{1/2} \\
\leq& \frac{1}{4}\mathbb{E}\left(\sup_{t\in[0,T]}(e^{\phi}|u|^p)\right) + C_2(p,T,K_H)\mathbb{E}\left(1+\int_0^T e^{\phi}|u|^p dt\right)
\end{aligned}
\tag{40}
$$

Note that the constants in estimates above are independent of ϕ. Set $\phi(t) = -(C_1 + C_2)t$, where C_1, C_2 are the constants arising in (39) and (40) respectively. This choice, used in conjunction with the preceding estimates implies (29).

From (36) with $p = 2$ and $\phi = 0$, one deduces that for any $r > 2$:

$$
\begin{aligned}
\mathbb{E}\left(\int_0^T\|u\|^2dt\right)^{r/2} \leq& C\mathbb{E}\left(1+|u_0|^r + |f|^r_{L^2(0,T;V')} + \sup_{t\in[0,T]}|u|^r\right) \\
&+ C\mathbb{E}\sup_{t\in[0,T]}\left|\int_0^t\sum_{k=1}^{\infty}\langle g_k(u,t),u\rangle d\beta_k\right|^{r/2}.
\end{aligned}
\tag{41}
$$

For the second term we again employ BDG and infer:

$$
\begin{aligned}
&C\mathbb{E}\sup_{t\in[0,T]}\left|\sum_{k=1}^{\infty}\int_0^t\langle g_k(u,s),u\rangle d\beta_k\right|^{r/2} \\
\leq& C\mathbb{E}\left(\int_0^T\sum_{k=1}^{\infty}\langle g_k(u,t),u\rangle^2dt\right)^{r/4} \\
\leq& C\mathbb{E}\left(\sup_{t\in[0,T]}|u|^r+1\right) + \frac{1}{2}\mathbb{E}\left(\int_0^T\|u\|^2\right)^{r/2}.
\end{aligned}
\tag{42}
$$

Absorbing terms and applying estimates already obtained in (29) yields (30).

To obtain the desired estimates in (ii) we make use of the commutativity of P_n and ∂_{zz} on $D(A)$ (see Remark 1). In particular, note that vertical cancellation property established in Lemma 3.1 along with this commutativity means that:

$$\langle P_n B(u), \partial_{zz} u\rangle = \langle B(u), \partial_{zz} u\rangle = 0.$$

Thus, when we apply Itô's formula for $|\partial_z v|^2$, the nonlinear term disappears as above. Bootstrapping to $p \geq 2$ with a second application of Itô we arrive at the differential:

$$\begin{aligned} &d|\partial_z u|^p + p\nu\|\partial_z u\|^2|\partial_z u|^{p-2}dt \\ =&p\langle f, \partial_{zz} u\rangle|\partial_z u|^{p-2}dt \\ &+ \frac{p}{2}\sum_{k=1}^{\infty}|\partial_z P_n g_k(u,t)|^2|\partial_z u|^{p-2}dt \\ &+ \frac{p(p-2)}{2}\sum_{k=1}^{\infty}\langle g_k(u,t), -\partial_{zz} u\rangle^2|\partial_z u|^{p-4}dt \\ &+ p\sum_{k=1}^{\infty}\langle g_k(u,t), -\partial_{zz} u\rangle|\partial_z u|^{p-2}d\beta_k. \end{aligned} \tag{43}$$

We bound the first term on the right hand side of (43) as in (37). For the second term we utilize the Lipschitz condition imposed in (31) and the uniform bound (30) established in the previous case:

$$\begin{aligned} &\frac{p}{2}\mathbb{E}\int_0^T\sum_{k=1}^{\infty}|\partial_z P_n g_k(u,t)|^2|\partial_z u|^{p-2}dt \\ \leq& C(p)\mathbb{E}\int_0^T\sum_{k=1}^{\infty}|\partial_z g_k(u,t)|^2|\partial_z u|^{p-2}dt \\ \leq& C(K_X,p)\mathbb{E}\int_0^T(\|u\|^2+1)|\partial_z u|^{p-2}dt \\ \leq& C(K_X,p)\mathbb{E}\left(\int_0^T(\|u\|^2+1)\,dt\right)^{p/2} + \frac{1}{8}\mathbb{E}\left(\sup_{t\in[0,T]}|\partial_z u|^p\right). \end{aligned} \tag{44}$$

The third term in (43) is estimated in the same manner. The final term is handled using BDG:

$$\begin{aligned} &p\mathbb{E}\sup_{t\in[0,T]}\left|\sum_{k=1}^{\infty}\int_0^t\langle g_k(u,s), -\partial_{zz} u\rangle|\partial_z u|^{p-2}d\beta_k\right| \\ \leq& C(p)\mathbb{E}\left(\int_0^T\sum_{k=1}^{\infty}\langle\partial_z g_k(u,t), \partial_z u\rangle^2|\partial_z u|^{2(p-2)}dt\right)^{1/2} \\ \leq& C(p,K_X)\mathbb{E}\left(\int_0^T(1+|u|_X^2)|\partial_z u|^{2(p-1)}dt\right)^{1/2} \\ \leq& \frac{1}{8}\mathbb{E}\left(\sup_{t\in[0,T]}|\partial_z u|^p\right) + C(p,K_X)\mathbb{E}\left(\int_0^T(1+\|u\|^2)dt\right)^{p/2}. \end{aligned} \tag{45}$$

For the (iii), we do not have cancellation in the nonlinear term. Here the differential reads:

$$\begin{aligned}
&de^{\phi}\|u\|^p + p\nu e^{\phi}|Au|^2\|u\|^{p-2}dt \\
=&pe^{\phi}\langle f - B(u), Au\rangle\|u\|^{p-2}dt \\
&+ \frac{p}{2}e^{\phi}\sum_{k=1}^{\infty}|\nabla P_n g_k(u,t)|^2\|u\|^{p-2}dt \\
&+ \frac{p(p-2)}{2}e^{\phi}\sum_{k=1}^{\infty}(\nabla g_k(u,t), \nabla u)^2\|u\|^{p-4}dt \\
&+ pe^{\phi}\sum_{k=1}^{\infty}(\nabla g_k(u,t), \nabla u)\|u\|^{p-2}d\beta_k + \phi'\|u\|^p e^{\phi}dt.
\end{aligned} \tag{46}$$

Once again ϕ will be a non-positive function chosen further on to cancel off terms. Integrating up to $t \wedge \tau$, then taking a supremum in t and estimating terms as previously reveals:

$$\begin{aligned}
&\mathbb{E}\left(\sup_{t\in[0,\tau]}(\|u\|^p e^{\phi}) + p\nu\int_0^{\tau} e^{\phi}|Au|^2\|u\|^{p-2}dt\right) \\
\leq& C\mathbb{E}(\|u_0\|^p + |f|^p_{L^2(0,T;H)}) \\
&+ 2\mathbb{E}\int_0^{\tau} e^{\phi}|(B(u), \partial_{xx}u)|\|u\|^{p-2}dt \\
&+ C_3(p, K_V)\mathbb{E}\int_0^{\tau} e^{\phi}\|u\|^p dt + 2\mathbb{E}\int_0^{\tau}\phi' e^{\phi}\|u\|^p dt.
\end{aligned} \tag{47}$$

We apply Lemma 3.1, (v) with $\epsilon = p\nu/4$ inferring:

$$\begin{aligned}
&\int_0^{\tau} e^{\phi}|(B(u), \partial_{xx}u)|\|u\|^{p-2}dt \\
\leq& \frac{p\nu}{4}\int_0^{\tau} e^{\phi}|Au|^2\|u\|^{p-2}dt \\
&+ C_4(\nu, p)\int_0^{\tau} e^{\phi}\|u\|^p(\|u\|^2 + |\partial_z u|^2\|\partial_z u\|^2)dt.
\end{aligned} \tag{48}$$

Taking the previous estimates into account we set:

$$\phi(t) = -C_4\int_0^t (\|u\|^2 + |\partial_z u|^2\|\partial_z u\|^2) - tC_3$$

where C_3 and C_4 are the constants appearing in (47) and (48) respectively. Given the assumption (34), $e^{\phi(\tau)} \geq C(\nu, p, T)e^{-CM}$. With this we can apply (48) to (47) and conclude the final bound (35). $\square$

Remark 3. If one could find a subsequence n_k, a stopping time τ with $\mathbb{P}(\tau > 0) = 1$ and a positive constant M such that:

$$\sup_{n_k}\int_0^{\tau}(\|u^{(n_k)}\|^2 + |\partial_z u^{(n_k)}|^2\|\partial_z u^{(n_k)}\|^2)ds \leq M \quad \text{a.s} \tag{49}$$

then the existence of solutions taking values in $L^p(\Omega; L^2([0,T]; D(A)) \cap L^\infty([0,T]; V))$ would follow.

We conclude this section with some comments concerning the existence and uniqueness of solutions to the Galerkin systems. Regarding existence one uses that B is locally Lipschitz in conjunction with the a priori bounds established above. These properties taken together allows one to establish global existence on any compact time interval via Picard iteration methods. See [12] for detailed proofs. Uniqueness is established as below for the full infinite dimensional system.

5. **Existence and uniqueness of solutions.** With the uniform estimates on the solutions of the Galerkin systems in hand, we proceed to identify a (weak) limit u. This element is shown to satisfy a stochastic differential (54) with unknown terms corresponding to the nonlinear portions of the equation. Next we prove a comparison lemma that establishes a sufficient condition (67) for the identification of the unknown portions of the differential. This lemma, in conjunction with some further estimates, provides the final step in the main theorem concerning existence below.

We will assume the following conditions on the data throughout this section:

$$\begin{gathered} f, \partial_z f \in L^p(\Omega; L^2(0,T;V')), \quad g \in \mathrm{Lip}_u(H, \ell^2(H)) \cap \mathrm{Lip}_u(X, \ell^2(X)) \\ u_0, \partial_z u_0 \in L^p(\Omega; H). \end{gathered} \tag{50}$$

Here $p \geq 4$ so that the sequence $P_n B(u^{(n)})$ will have a weakly convergent subsequence. These assumptions may be weakened slightly for several of the lemmas leading up to the main result. In particular the limit system (54) can be obtained by merely assuming:

$$u_0 \in L^p(\Omega; H), \quad f \in L^p(\Omega; L^2(0,T;V')), \quad g \in \mathrm{Lip}_u(H, \ell^2(H)). \tag{51}$$

Lemma 5.1 (Limit System)**.** *There exists adapted processes u, B^* and g^* with the regularity:*

$$\begin{gathered} u \in L^p(\Omega, L^2(0,T;V) \cap L^\infty(0,T;H)) \\ \partial_z u \in L^p(\Omega, L^2(0,T;V) \cap L^\infty(0,T;H)) \\ u \in C(0,T;H) \text{ a.s.} \end{gathered} \tag{52}$$

and:

$$B^* \in L^2(\Omega; L^2(0,T;V')), \quad g^* \in L^2(\Omega; L^2(0,T;\ell^2(H))) \tag{53}$$

such that u, B^* and g^* satisfy:

$$\begin{gathered} d\langle u, v\rangle + \langle \nu A u + B^*, v\rangle dt = \langle f, v\rangle dt + \sum_{k=1}^{\infty} \langle g_k^*(t), v\rangle d\beta_k \\ \langle u(0), v\rangle = \langle u_0, v\rangle \end{gathered} \tag{54}$$

for any test function $v \in V$.

Remark 4. We use the following elementary facts regarding weakly convergent sequences in the proof below.

(i) Suppose B_1, B_2 are Banach spaces and that $L : B_1 \to B_2$ is a bounded linear mapping. If $x_n \rightharpoonup x$ in B_1 then $Lx_n \rightharpoonup Lx$ in B_2.

(ii) For $p \in [1, \infty)$ take:

$$L_1(w)(t) = \int_0^t w ds \qquad w \in L^p(\Omega \times [0,T]).$$

If $x_n \rightharpoonup x$ in $L^p(\Omega \times (0,T))$ and then $L_1(x_n) \rightharpoonup L_1(x)$ in the same space.

(iii) Take:

$$L_2(v)(t) = \sum_k \int_0^t v^k d\beta_k$$

for $v = \{v^k\} \in L^2(\Omega, L^2(0,T;\ell^2))$. Given that $v_n \rightharpoonup v$ in this space then $L_2(v_n) \rightharpoonup L_2(v)$ in $L^2(\Omega; L^2(0,T))$

(Proof- Lemma 5.1). Applying the estimate (29) with Alaoglu's theorem we deduce the existence of a subsequence of Galerkin elements $u^{(n)}$ and an element $u \in L^p(\Omega, L^2(0,T;V) \cap L^\infty(0,T;H))$ so that:

$$u^{(n)} \rightharpoonup u \quad \text{ in } L^p(\Omega; L^2(0,T;V)) \tag{55}$$

and:

$$u^{(n)} \rightharpoonup^* u \quad \text{ in } L^p(\Omega; L^\infty(0,T;H)). \tag{56}$$

To deduce the desired regularity for $\partial_z u$ we apply the uniform estimates given in (32) and thin our subsequence again so that $\partial_z u \in L^p(\Omega, L^2(0,T;V) \cap L^\infty(0,T;H))$ with:

$$\partial_z u^{(n)} \rightharpoonup \partial_z u \quad \text{ in } L^p(\Omega, L^2(0,T;V)) \tag{57}$$

as well as:

$$\partial_z u^{(n)} \rightharpoonup^* \partial_z u \quad \text{ in } L^p(\Omega, L^\infty(0,T;H)). \tag{58}$$

By an application of (13):

$$\begin{aligned} &\mathbb{E}\int_0^T |P_n B(u^{(n)})|_{V'}^2 dt \\ \leq &C\mathbb{E}\left[\sup_{t\in[0,T]} (|u^{(n)}|^4 + |\partial_z u^{(n)}|^4) + \left(\int_0^T \|u^{(n)}\|^2 dt\right)^2\right]. \end{aligned} \tag{59}$$

The later quantity is uniformly bounded as a consequence of (30) and (32). Thinning $u^{(n)}$ as necessary we find an element $B^* \in L^2(\Omega; L^2(0,T;V'))$ so that:[2]

$$P_n B(u^{(n)}) \rightharpoonup B^* \quad \text{ in } L^2(\Omega; L^2(0,T;V')). \tag{60}$$

Finally the Lipschitz assumption along with the Poincaré inequality imply:

$$\mathbb{E}\int_0^T \sum_{k=1}^\infty |P_n g_k(t,u^{(n)})|^2 dt \leq C\mathbb{E}\left(\int_0^T (\|u^{(n)}\|^2 + 1) dt\right). \tag{61}$$

This gives the uniform bounds needed to infer $g^* \in L^2(\Omega; L^2(0,T;\ell^2(H)))$ and implies that:

$$P_n g(u^{(n)}) \rightharpoonup g^* \quad \text{ in } L^2(\Omega; L^2(0,T;\ell^2(H))). \tag{62}$$

We next establish that u satisfies (54). To this end, fix any measurable set $E \subset \Omega \times [0,T]$ and $v \in V$. By applying (55) and using that each $u^{(n)}$ is a solution

[2]Given only the weaker assumptions on the initial data (51) one can still show that $B^* \in L^{4/3}(\Omega; L^2(0,T;V'))$ with the estimate:

$$\mathbb{E}\int_0^T |P_n B(u^{(n)}))|_{V'}^{4/3} dt \leq C\mathbb{E}\left[\sup_{t\in[0,T]} |u^{(n)}|^2 + \left(\int_0^T \|u^{(n)}\|^2 dt\right)^{3/2}\right].$$

to an associated Galerkin system we infer:

$$
\begin{aligned}
\mathbb{E}\int_0^T \chi_E(u,v)dt = &\lim_{n\to\infty} \mathbb{E}\int_0^T \chi_E(u^{(n)},v)dt \\
= &\lim_{n\to\infty}\Bigg(\mathbb{E}\int_0^T \chi_E(P_n u_0, v)dt \\
&- \mathbb{E}\int_0^T \chi_E\left[\int_0^t \langle \nu A u^{(n)} + P_n B(u^{(n)}) - P_n f, v\rangle ds\right] dt \\
&+ \mathbb{E}\int_0^T \chi_E\left[\sum_{k=1}^{\infty}\int_0^t \langle P_n g_k(u^{(n)},s), v\rangle d\beta_k(s)\right] dt\Bigg) \\
=&\mathbb{E}\int_0^T \chi_E\left[(u_0,v) - \int_0^t \langle \nu A u + B^* - f, v\rangle ds\right] dt \\
&+ \mathbb{E}\int_0^T \chi_E\left[\sum_{k=1}^{\infty}\int_0^t \langle g_k^*(s), v\rangle d\beta_k(s)\right] dt
\end{aligned}
\tag{63}
$$

For the final equality above we use Remark 4. Observe that:

$$\langle P_n B(u^{(n)}), v\rangle \rightharpoonup \langle B^*, v\rangle \quad \text{in } L^2(\Omega\times[0,T]) \tag{64}$$

$$\langle A u^{(u)}, v\rangle \rightharpoonup \langle Au, v\rangle \quad \text{in } L^2(\Omega\times[0,T]) \tag{65}$$

along with:

$$[(P_n g_k(u^{(n)}), v)]_{k\geq 1} \rightharpoonup [(g_k^*, v)]_{k\geq 1} \quad \text{in } L^2(\Omega; L^2([0,T],\ell^2)). \tag{66}$$

Since E is arbitrary in (63), the equality (54) follows up to a set of measure zero. Referring then to results in [29], chapter 2 we find that u has modification so that $u \in C([0,T]; H)$ a.s. □

With a candidate solution in hand, it remains to show that $B(u) = B^*$ and $g^* = g(u)$. A sufficient condition for these equalities, at least up to a stopping time, is captured in the following:

Lemma 5.2. *If $0 \leq \tau \leq T$ is any stopping time such that:*

$$\mathbb{E}\int_0^\tau \|u - u^{(n)}\|^2 dt \to 0 \tag{67}$$

then, $\lambda\times\mathbb{P}$-a.e.[3]*:*

$$B(u(t))\mathbb{1}_{t\leq\tau} = B^*(t)\mathbb{1}_{t\leq\tau} \tag{68}$$

and for every k:

$$g_k(u,t)\mathbb{1}_{t\leq\tau} = g_k^*(t)\mathbb{1}_{t\leq\tau}. \tag{69}$$

Proof. Let E, a measurable subset of $\Omega\times[0,T]$ and $v\in V$ be given. It is sufficient to show that:

$$\mathbb{E}\int_0^T \chi_E\langle (B(u(t)) - B^*(t))\mathbb{1}_{t\leq\tau}, v\rangle dt = 0 \tag{70}$$

and that for any k:

$$\mathbb{E}\int_0^T \chi_E\langle (g_k(u(t),t) - g_k^*(t))\mathbb{1}_{t\leq\tau}, v\rangle dt = 0 \tag{71}$$

[3] On $[0,T]$ we use the Lebesgue measure λ.

where χ_E is the characteristic function of E. Due to the weak convergence (60) established above:

$$\mathbb{E}\int_0^T \chi_E\langle (P_n B(u^{(n)}(t)) - B^*(t))1\!\!1_{t\leq\tau}, v\rangle dt \to 0 \quad \text{as } n\to\infty. \tag{72}$$

Using that:

$$\begin{aligned}&|\langle B(u) - P_n B(u^{(n)}), v\rangle| \\ \leq & C\|u\|^2\|Q_n v\| + C(\|u\|\|u-u^{(n)}\| + \|u^{(n)}\|\|u-u^{(n)}\|)\|v\|\end{aligned} \tag{73}$$

we estimate:

$$\begin{aligned}&\left|\mathbb{E}\int_0^T \chi_E\langle (P_n B(u^{(n)},t) - B(u,t))1\!\!1_{t\leq\tau}, v\rangle ds\right| \\ \leq & C\|(I-P_n)v\|\mathbb{E}\left[\int_0^T \|u\|^2 ds\right] \\ &+ C\|v\|\left[\mathbb{E}\int_0^T (\|u\| + \|u^{(n)}\|)^2 ds\right]^{1/2}\left[\mathbb{E}\int_0^\tau \|u-u^{(n)}\|^2 ds\right]^{1/2}\end{aligned} \tag{74}$$

which, also vanishes as $n\to\infty$. These estimates imply the first item.

For the second item (71) we again exploit the weak convergence in (62) to infer that for every k:

$$\mathbb{E}\int_0^T \chi_E\langle P_n(g_k(u^{(n)},t) - g_k^*(t))1\!\!1_{t\leq\tau}, v\rangle dt \to 0 \text{ as } n\to\infty. \tag{75}$$

Also:

$$\left|\mathbb{E}\int_0^T \chi_E\langle Q_n g(u,t)1\!\!1_{t\leq\tau}, v\rangle dt\right| \leq K_H|Q_n v|\mathbb{E}\int_0^T (1+|u|)dt \to 0 \text{ as } n\to\infty. \tag{76}$$

Finally by the assumption (67), the Lipschitz continuity of g and the Poincaré Inequality:

$$\begin{aligned}&\mathbb{E}\int_0^T \chi_E\langle P_n(g_k(u^{(n)},t) - g_k(u,t))1\!\!1_{t\leq\tau}, v\rangle\, dt \\ \leq & C(K_H,T)|v|\left(\mathbb{E}\int_0^\tau |u^{(n)}-u|^2\, dt\right)^{1/2} \to 0\end{aligned} \tag{77}$$

as $n\to\infty$. Combining (75), (76) and (77) provides the second equality (69). $\square$

With this lemma in mind we compare u to the sequence $u^{(n)}$ of Galerkin estimates to show that (67) is satisfied for a sequence of stopping times τ_n. Since we are able to choose τ_n so that $\tau_n \uparrow T$ a.s., this is sufficient to deduce the existence result. Here we are adapting techniques used in [3].

Theorem 5.3 (Existence of Weak-Strong Solutions)**.** *Suppose that $p\geq 4$ and $f,\partial_z f \in L^p(\Omega; L^2(0,T;V'))$, $g\in \mathrm{Lip}_u(H,\ell^2(H))\cap \mathrm{Lip}_u(X,\ell^2(X))$ and $u_0,\partial_z u_0 \in L^p(\Omega;H)$. Then there exists an H continuous* weak-strong *solution u with the additional regularity:*

$$\partial_z u \in L^p(\Omega, L^2(0,T;V)\cap L^\infty(0,T;H)). \tag{78}$$

Proof. The needed regularity conditions for u follow from Lemma 5.1. Moreover we found that u satisfies the differential (54). Thus it remains only to show that for almost every t, ω:

$$B(u) = B^* \quad g_k(u) = g_k^*. \tag{79}$$

To this end, for $R > 0$, define:

$$\tau_R = \inf_{r \in [0,T]} \left\{ \sup_{s \in [0,r]} |u|^2 + \sup_{s \in [0,r]} |\partial_z u|^2 + \int_0^r \|u\|^2 + \|\partial_z u\|^2 \, ds > R \right\} \wedge T. \tag{80}$$

Notice that τ_R is increasing as a function of $R > 0$ and that moreover:

$$\mathbb{P}(\tau_R \leq T) \leq \mathbb{P}\left(\sup_{s \in [0,T]} |u|^2 + \sup_{s \in [0,T]} |\partial_z u|^2 + \int_0^T \|u\|^2 + \|\partial_z u\|^2 \, ds \geq R \right). \tag{81}$$

Thus, as a consequence of (52) $\tau_R \to T$, almost surely.

We now fix R and show that τ_R satisfies (67). Since by the dominated convergence theorem:

$$\lim_{n \to \infty} \int_0^{\tau_R} \|u - P_n u\|^2 \, dt = 0 \tag{82}$$

it is sufficient to compare $P_n u$ and $u^{(n)}$. The difference of these terms satisfies the differential:

$$\begin{aligned} &d(P_n u - u^{(n)}) + [\nu A(P_n u - u^{(n)}) + P_n B^* - P_n B(u^{(n)})]dt \\ =&\sum_{k=1}^{\infty} P_n(g_k^* - g_k(u^{(n)}))d\beta_k. \end{aligned}$$

By applying Itô's lemma we find that:

$$\begin{aligned} &d(|P_n u - u^{(n)}|^2 e^{\psi}) + 2\nu \|P_n u - u^{(n)}\|^2 e^{\psi} dt \\ =&2\langle B(u^{(n)}) - B^*, P_n u - u^{(n)} \rangle e^{\psi} dt \\ &+ 2\sum_{k=1}^{\infty} \langle g_k^*(t) - g_k(u_n, t), P_n u - u^{(n)} \rangle e^{\psi} d\beta_k \\ &+ \sum_{k=1}^{\infty} |P_n(g_k^*(t) - g_k(u_n, t))|^2 e^{\psi} dt \\ &+ \psi' |P_n u - u^{(n)}|^2 e^{\psi} dt. \end{aligned} \tag{83}$$

As in the a priori estimates for Lemma 4.2, ψ is a C^1 function chosen further on to cancel off terms. The first term on the right hand side of (83) is estimated using the cancellation property, (13) and (15):

$$
\begin{aligned}
&\langle B(u^{(n)}) - B^*, P_n u - u^{(n)}\rangle \\
=&\langle B(u^{(n)} - P_n u, P_n u) + (B(P_n u) - B(u)) + (B(u) - B^*), P_n u - u^{(n)}\rangle \\
\leq& C(|u^{(n)} - P_n u|\|u^{(n)} - P_n u\|\|P_n u\| \\
&\quad + |u^{(n)} - P_n u|^{1/2}\|u^{(n)} - P_n u\|^{3/2}|\partial_z P_n u|) \\
&+ (|B(P_n u - u, P_n u)|_{V'} + |B(u, P_n u - u)|_{V'})\|P_n u - u^{(n)}\| \\
&+ \langle B(u) - B^*, P_n u - u^{(n)}\rangle \\
\leq& \nu\|P_n u - u^{(n)}\|^2 + C_1(\nu)|u^n - P_n u|^2(\|u\|^2 + |\partial_z u|^4) \\
&+ C_2(\nu)(|B(P_n u - u, u)|^2_{V'} + |B(u, P_n u - u)|^2_{V'}) \\
&+ \langle B(u) - B^*, P_n u - u^{(n)}\rangle.
\end{aligned}
\tag{84}
$$

For the last term involving g and g^*, using the basic properties of the ℓ^2 inner product and the Lipschitz conditions for g we infer:

$$
\begin{aligned}
&|P_n(g^* - g(u^{(n)}))|^2_{\ell^2(H)} \\
=&2(g^* - g(u), P_n(g^* - g(u^{(n)})))_{\ell^2(H)} \\
&+ |P_n(g(u) - g(u^{(n)}))|^2_{\ell^2(H)} - |P_n(g^* - g(u))|^2_{\ell^2(H)} \\
\leq& 2(g^* - g(u), P_n(g^* - g(u^{(n)})))_{\ell^2(H)} + 4K_H^2(|P_n u - u^{(n)}|^2 + |P_n u - u|^2).
\end{aligned}
\tag{85}
$$

With estimates (84) and (85) in mind we now take:

$$
\psi(t) = -C_1 \int_0^t (\|u\|^2 + |\partial_z u|^4)ds - 4K_H^2 t \tag{86}
$$

where C_1 is the constant from the last inequality in (84) and K_H is the Lipschitz constant associated with g. By examining (80) we notice:

$$
e^{\psi(\tau_R)} \geq e^{-C_1(R+R^2) - 4K_H^2 T} \qquad \text{a.s.} \tag{87}
$$

The estimates given in (84) and (85) can now be applied to (83). After integrating up to the stopping time τ_R, taking expectations and rearranging one finds:

$$
\begin{aligned}
&\mathbb{E}\left(\int_0^{\tau_R} \|P_n u - u^{(n)}\|^2 dt\right) \\
\leq& C\mathbb{E}\int_0^{\tau_R} \left(\langle B(u) - B^*, P_n u - u^{(n)}\rangle + (g^* - g(u), P_n[g^* - g(u^n)])_{\ell^2}\right) e^{\psi} dt \\
&+ C\mathbb{E}\int_0^{\tau_R} (|Q_n u|^2 + |B(P_n u - u, u)|^2_{V'} + |B(u, P_n u - u)|^2_{V'})dt
\end{aligned}
\tag{88}
$$

where the numerical constant C depends on R and T. Note that since:

$$
\mathbb{E}\left(e^{\psi(\tau_R)} \int_0^{\tau_R} \|P_n u - u^{(n)}\|^2 dt\right) \leq \mathbb{E}\int_0^{\tau_R} e^{\psi(t)}\|P_n u - u^{(n)}\|^2 dt. \tag{89}
$$

and taking into account (87) we see that the term e^{ϕ} can be absorbed into the constants on the right hand side of (88).

Due to (55) one infers that $P_n u - u^{(n)} \rightharpoonup^* 0$ in $L^p(\Omega; L^2(0,T;V))$. Similarly (62) implies that $P_n g^* - P_n g(u^{(n)}) \rightharpoonup 0$ in the space $L^2(\Omega; L^2(0,T;\ell^2(H)))$. As such, the first terms on the right hand side of (88) vanish in the limit as $n \to \infty$. The term involving $|Q_n u|$ approaches zero in this limit as a consequence of the dominated convergence theorem. For the final terms we apply the estimate on B in V' given in (15) and make further use of τ_R to conclude:

$$\begin{aligned}
&\mathbb{E}\int_0^{\tau_R} |B(P_n u - u, u)|_{V'}^2 dt \\
\leq &C\mathbb{E}\int_0^{\tau_R} (|\partial_z (P_n u - u)|^2 + |P_n u - u|^2)\|u\|^2 dt \\
&+ C\mathbb{E}\int_0^{\tau_R} |(P_n u - u)|^2 \|\partial_z u\|^2 dt.
\end{aligned} \tag{90}$$

Moreover:

$$\begin{aligned}
&\mathbb{E}\int_0^{\tau_R} |B(u, P_n u - u)|_{V'}^2 dt \\
\leq &C\mathbb{E}\int_0^{\tau_R} \big((|\partial_z u|^2 + |u|^2)|\partial_x (P_n u - u)|^2 + |u|^2 \|\partial_z (P_n u - u)\|^2 dt\big) \\
\leq &C\mathbb{E}\left((\sup_{t\in[0,\tau_R]} |\partial_z u|^2 + \sup_{t\in[0,\tau_R]} |u|^2) \int_0^{\tau_R} \|(P_n u - u)\|^2 dt \right) \\
&+ C\mathbb{E}\left(\sup_{t\in[0,\tau_R]} |u|^2 \int_0^{\tau_R} \|\partial_z (P_n u - u)\|^2 dt \right) \\
\leq &C(R)\mathbb{E}\left(\int_0^{\tau_R} (\|P_n u - u\|^2 + \|\partial_z (P_n u - u)\|^2)\, dt \right).
\end{aligned} \tag{91}$$

Thus, again by the dominated convergence theorem, the final two terms also converge to zero as $n \to \infty$. We can now conclude that τ_R satisfies Lemma 5.2.

Fixing arbitrary (t, ω) off of an appropriately chosen set of measure zero, there is an R so that $t \leq \tau_R(\omega)$. As such (79) for holds the given pair, completing the proof. □

Having established existence we next turn to the question of uniqueness:

Theorem 5.4 (Uniqueness)**.** *Suppose that u_1 and u_2 are* weak-strong *solutions in the sense of Definition 3.3 and that $u_1(0) = u_2(0)$ a.s. in H. Assume moreover that:*

$$\partial_z u_2 \in L^4([0, T]; H) \qquad a.s. \tag{92}$$

Then:

$$\mathbb{P}(u_1(t) = u_2(t) \forall t \in [0, T]) = 1. \tag{93}$$

In particular, the solutions constructed in Theorem 5.3 are unique since they belong (almost surely) to $L^\infty([0, T]; H)$ which is included in $L^4([0, T]; H)$

Proof. Subtracting we find the $u_1 - u_2$ satisfies the differential:

$$\begin{aligned}
&d\langle u_1 - u_2, v\rangle + \langle \nu A(u_1 - u_2), v\rangle dt \\
=&\langle B(u_1, u_2 - u_1) + B(u_2 - u_1, u_2), v\rangle dt \\
&+ \sum_k \langle g_k(u_1, t) - g_k(u_2, t), v\rangle d\beta_k \\
&\langle u_1(0) - u_2(0), v\rangle = \langle u_{0,1} - u_{0,2}, v\rangle.
\end{aligned} \tag{94}$$

We apply Itô's lemma and deduce:

$$\begin{aligned} &d(|u_1 - u_2|^2 e^{\phi}) + 2\nu \|u_1 - u_2\|^2 e^{\phi} dt \\ =&2\langle B(u_1 - u_2, u_2), u_2 - u_1\rangle e^{\phi} dt \\ &+ \sum_k \langle g_k(u_1, t) - g_k(u_2, t), u_1 - u_2\rangle e^{\phi} d\beta_k \\ &+ \sum_k |g_k(u_1, t) - g_k(u_2, t)|^2 e^{\phi} dt \\ &+ \phi' |u_1 - u_2|^2 e^{\phi}\, dt. \end{aligned} \tag{95}$$

Once again ϕ will be chosen judiciously to cancel off terms below. Applying (13) with Young's inequality one finds:

$$\begin{aligned} &|\langle B(u_1 - u_2, u_2), u_2 - u_1\rangle| \\ \leq&\nu \|u_1 - u_2\|^2 + C(\nu)|u_1 - u_2|^2 (\|u_2\|^2 + |\partial_z u_2|^4). \end{aligned} \tag{96}$$

Taking K_H to be the Lipschitz constant associated with g and $C(\nu)$ from the preceding inequality, set:

$$\phi(t) = -C(\nu) \int_0^t (\|u_2\|^2 + |\partial_z u_2|^4)\, dt - K_H^2 t. \tag{97}$$

Given the regularity conditions assumed for u_2 we infer, as in Theorem 5.3 that $e^{\phi(t)} > 0$ almost surely. Integrating (95) up to t taking expected values and making use of the estimates on B and g one concludes that for any $t \in [0, T]$:

$$\mathbb{E}(|u_1(t) - u_2(t)|^2 e^{\phi(t)}) \leq \mathbb{E}(|u_1(0) - u_2(0)|^2) = 0. \tag{98}$$

This implies:

$$\mathbb{P}(u_1(t) = u_2(t), \forall t \in [0, T] \cap \mathbb{Q}) = 1.$$

However since both u_1 and u_2 take values in $C([0, T]; H)$ (93) follow immediately. $\square$

Acknowledgments. We are grateful to the anonymous referees for their many valuable comments and suggestions. This work has been supported in part by the NSF grant DMS-0505974.

REFERENCES

[1] A. Bensoussan and J. Frehse, *Local solutions for stochastic Navier Stokes equations*, M2AN Math. Model. Numer. Anal., **34** (2000), 241–273.

[2] A. Bensoussan and R. Temam, *Équations stochastiques du type Navier-Stokes*, J. Functional Analysis, **13** (1973), 195–222.

[3] H. Breckner, *Galerkin approximation and the strong solution of the Navier-Stokes equation*, J. Appl. Math. Stochastic Anal., **13** (2000), 239–259.

[4] D. Bresch, A. Kazhikhov and J. Lemoine, *On the two-dimensional hydrostatic Navier-Stokes equations*, SIAM J. Math. Anal., **36** (2004/05), 796–814.

[5] Z. Brzeźniak and S. Peszat, *Strong local and global solutions for stochastic Navier-Stokes equations*, in "Infinite Dimensional Stochastic Analysis" (eds. P. Clment, F. den Hollander J. van Neerven, and B. de Pagter), R. Neth. Acad. Arts Sci., Amsterdam, (2000), 85–98.

[6] C. Cao and E. Titi, *Global well-posedness of the three-dimensional viscous primitive equations of large scale ocean and atmosphere dynamics*, Ann. of Math. (2), **166** (2007), 245–267.

[7] M. Capiński and D. Gatarek, *Stochastic equations in Hilbert space with application to Navier-Stokes equations in any dimension*, J. Funct. Anal., **126** (1994), 26–35.

[8] A. B. Cruzeiro, *Solutions et mesures invariantes pour des équations d'évolution stochastiques du type Navier-Stokes*, Exposition. Math., **7** (1989), 73–82.

[9] B. Cushman-Roisin and J.-M. Beckers, "Introduction to Geophysical Fluid Dynamics: Physical and Numerical Aspects," To be published by Academic Press.

[10] G. Da Prato and J. Zabczyk, "Stochastic Equations in Infinite Dimensions," Cambridge University Press, Cambridge, 1992.

[11] G. Da Prato and J. Zabczyk, "Ergodicity for Infinite-Dimensional Systems," Cambridge University Press, Cambridge, 1996.

[12] F. Flandoli, "Lectures on 3D Fluid Dynamics," C.I.M.E. Lecture Notes, 2005.

[13] F. Flandoli and D. Gatarek, *Martingale and stationary solutions for stochastic Navier-Stokes equations*, Probab. Theory Related Fields, **102** (1995), 367–391.

[14] F. Guillén-González, N. Masmoudi and M. A. Rodríguez-Bellido, *Anisotropic estimates and strong solutions of the primitive equations*, Differential Integral Equations, **14** (2001), 1381–1408.

[15] C. Hu, R. Temam and M. Ziane, *Regularity results for linear elliptic problems related to the primitive equations*, in "Frontiers in Mathematical Analysis and Numerical Methods" (ed T. Li), World Sci. Publ., River Edge, NJ, (2004), 149–170.

[16] I. Karatzas and S. E. Shreve, "Brownian Motion and Stochastic Calculus," 2^{nd} edition, Springer-Verlag, New York, 1991.

[17] G. M. Kobelkov, *Existence of a solution 'in the large' for the 3D large-scale ocean dynamics equations*, C. R. Math. Acad. Sci. Paris, **343** (2006), 283–286.

[18] I. Kukavica and M. Ziane, *On the regularity of the primitive equations of the ocean*, Nonlinearity, **20** (2007), 2739–2753.

[19] J.-L. Lions, R. Temam and S. Wang, *Models for the coupled atmosphere and ocean*, Comput. Mech. Adv., **1** (1993), 1–120.

[20] J.-L. Lions, R. Temam and S. Wang, *New formulations of the primitive equations of atmosphere and applications*, Nonlinearity, **5** (1992), 237–288.

[21] J.-L. Lions, R. Temam and S. Wang, *On the equations of the large-scale ocean*, Nonlinearity, **5** (1992), 1007–1053.

[22] J.-L. Menaldi and S. S. Sritharan, *Stochastic 2-D Navier-Stokes equation*, Appl. Math. Optim., **46** (2002), 31–53.

[23] R. Mikulevicius and B. L. Rozovskii, *Global L_2-solutions of stochastic Navier-Stokes equations*, Ann. Probab., **33** (2005), 137–176.

[24] R. Mikulevicius and B. L. Rozovskii, *Martingale problems for stochastic PDE's*, in "Stochastic partial differential equations: six perspectives" (eds. R. A. Carmona and B. L. Rozovskii), Amer. Math. Soc., Providence, RI, (1999), 243–325.

[25] R. Mikulevicius and B. L. Rozovskii, *On equations of stochastic fluid mechanics*, in "Stochastics in Finite and Infinite Dimensions" (eds. T. Hida, R. L. Karandikar, H. Kunita, B. S. Rajput, S. Watanabe and J. Xiong), Birkhäuser Boston, Boston, MA, (2001), 285–302.

[26] R. Mikulevicius and B. L. Rozovskii, *Stochastic Navier-Stokes equations for turbulent flows*, SIAM J. Math. Anal., **35** (2004), 1250–1310.

[27] J. Pedlosky, "Geophysical Fluid Dynamics," Springer-Verlag, New York, 1987.

[28] M. Petcu, R. Temam and D. Wirosoetisno, *Existence and regularity results for the primitive equations in two space dimensions*, Commun. Pure Appl. Anal., **3** (2004), 115–131.

[29] B. L. Rozovskii, "Stochastic Evolution Systems," Kluwer Academic Publishers Group, Dordrecht, 1990.

[30] R. Temam and M. Ziane, *Some mathematical problems in geophysical fluid dynamics*, in "Handbook of Mathematical Fluid Dynamics. Vol. III" (eds. S. Friedlander and D. Serre), North-Holland, Amsterdam, (2004), 535–657.

Received August 2007; revised March 2008

E-mail address: glatthol@usc.edu
E-mail address: ziane@usc.edu