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STRONG PATHWISE SOLUTIONS OF THE STOCHASTIC NAVIER-STOKES SYSTEM

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Abstract. We consider the stochastic Navier-Stokes equations forced by a multiplicative white noise on a bounded domain in space dimensions two and three. We establish the local existence and uniqueness of strong or pathwise solutions when the initial data takes values in H^1. In the two-dimensional case, we show that these solutions exist for all time. The proof is based on finite-dimensional approximations, decomposition into high and low modes and pairwise comparison techniques.

1. Introduction

In this article we study the Navier-Stokes equations in space dimension $d = 2, 3$, on a bounded domain $\mathcal{M}$ forced by a multiplicative white noise

$$\partial_t u + (u \cdot \nabla)u - \nu\Delta u + \nabla p = f + g(u)\dot{W}, \tag{1.1a}$$

$$\operatorname{div} u = 0, \tag{1.1b}$$

$$u(0) = u_0, \tag{1.1c}$$

$$u_{|\mathcal{M}} = 0. \tag{1.1d}$$

The system (1.1) describes the flow of a viscous incompressible fluid. Here $u = (u_1, \ldots, u_d)$, p and ν represent the velocity field, the pressure and the coefficient of kinematic viscosity respectively. The addition of the white noise driven terms to the basic governing equations is natural for both practical and theoretical applications. Such stochastically forced terms are used to account for numerical and empirical uncertainties and have been proposed as a model for turbulence.

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The mathematical literature for the stochastic Navier-Stokes equations is extensive and dates back to the early 1970's with the work of Bensoussan and Temam [2]. For the study of well posedness, new difficulties related to compactness often arise due to the addition of the probabilistic parameter. For situations where continuous dependence on initial data remains open (for example in $d = 3$ when the initial data merely takes values in L^2), it has proven fruitful to consider martingale solutions. Here, one constructs a probabilistic basis as part of the solution. For this context we refer the reader to the works of Viot [30], Cruzeiro [10], Capinski and Gatarek [6], Flandoli and Gatarek [15], and Mikulevicius and Rozovskii [25].

On the other hand, when working in spaces where continuous dependence on the initial data can be expected, existence of solutions can sometimes be established on a preordained probability space. Such solutions are referred to in the literature as "pathwise" solutions. In the two-dimensional setting, Da Prato and Zabczyk [12] and later Breckner [4] as well as Menaldi and Sritharan [21] established the existence of pathwise solutions where u takes values in $L^\infty([0,T], L^2)$. On the other hand, Bensoussan and Frehse [3] have established local solutions in 3-d for the class $C^\beta([0,T]; H^{2s})$ where $3/4 < s < 1$ and $\beta < 1 - s$. The existence of pathwise, global solutions for the two-dimensional primitive equations of the ocean with multiplicative noise was recently established by Glatt-Holtz and Ziane in [17], for the case when u and its vertical gradient are initially in L^2. In the works of Brzezniak and Peszat [5] and Mikulevicius and Rozovsky [23], the case of arbitrary space dimensions for local solutions evolving in Sobolev spaces of type $W^{1,p}$ for $p > d$ is addressed. Despite these extensive investigations, to the best of our knowledge, no one has addressed the case of local, pathwise, H^1-valued $(W^{1,2})$ solutions for the 3-d Navier-Stokes equations with multiplicative noise.

As we are working at the intersection of two fields, we should note that the terminology may cause some confusion. In th literature for stochastic differential equations the term "weak solution" is sometimes used synonymously with the term "martingale solution" while the designation "strong solution" may be used for a "pathwise solution". See the introductory text of Øksendal [26] for example. The former terminologies are avoided here because it is confusing in the context of partial differential equations. Indeed, from the partial differential equations point of view, strong solutions are solutions which are uniformly bounded in H^1, while weak solutions are those which are merely bounded in L^2. In this work we are therefore considering solutions which are strong in both the probabilistic and partial differential equations senses, which we shall call "strong pathwise solutions," often dropping the pathwise designation when the context is clear.

The exposition is organized as follows. In the first section we review the basic setting, defining the relevant function spaces and introducing various notions of pathwise solutions. We then turn to the Galerkin scheme which we analyze by modifying a pairwise comparison technique [23]. Key estimates are achieved using decompositions into high and low modes. In this way we are able to extract a locally strongly convergent subsequence and surmount the difficult issue of compactness. In the third section, we establish the existence and uniqueness of a local solution u evolving continuously in H^1 up to a maximal existence time ξ. For samples where ξ is finite we show that, on the one hand, the L^2 norm remains bounded and that on the other hand the H^1 norm of the solution blows up. By showing that certain quantities are under control in the two-dimensional case we are able to use this later blow-up criteria to give the proof for the global existence of strong solutions in the two-dimensional case. In the final section, we formulate and prove some abstract convergence results used in the proof of the main theorem. We believe these results to be more widely applicable for the study of well posedness of other non-linear stochastic partial differential equations and therefore hold independent interest.

2. The abstract functional analytic setting

We begin by reviewing some basic function spaces associated with (1.1). In what follows d is the spatial dimension, the physical cases $d = 2, 3$, being the focus of our attention below. For simplicity, we assume that the boundary $\partial\mathcal{M}$ is smooth. Let

$$\mathcal{V} := \{\phi \in (C_0^\infty(\mathcal{M}))^d : \nabla \cdot \phi = 0\}, \tag{2.1}$$

and

$$H := cl_{L^2(\mathcal{M})}\mathcal{V} = \{u \in L^2(\mathcal{M})^d : \nabla \cdot u = 0, u \cdot n = 0\}. \tag{2.2}$$

Here, n is the outer pointing unit normal to $\partial\mathcal{M}$. On H we take the L^2 inner product and norm

$$(u, v) := \int_{\mathcal{M}} u \cdot v d\mathcal{M}, \quad |u| := \sqrt{(u, u)}. \tag{2.3}$$

The Leray-Hopf projector, P_H, is defined as the orthogonal projection of $L^2(\mathcal{M})^d$ onto H. Define also

$$V := cl_{H^1(\mathcal{M})}\mathcal{V} = \{u \in H_0^1(\mathcal{M})^d : \nabla \cdot u = 0\}. \tag{2.4}$$

On this set we use the H^1 norm and inner products

$$((u,v)) := \int_{\mathcal{M}} \nabla u \cdot \nabla v d\mathcal{M}, \quad \|u\| := \sqrt{((u,u))}. \tag{2.5}$$

Note that, due to the Dirichlet boundary condition (cf. (1.1d)), the Poincaré inequality,

$$|u| \leq C\|u\|, \quad \forall u \in V, \tag{2.6}$$

holds, justifying (2.5) as a norm. Take V' to be the dual of V, relative to H with the pairing notated by $\langle \cdot, \cdot \rangle$.

We next define the Stokes operator A. A is understood as a bounded linear map from V to V' via $\langle Au, v\rangle = ((u,v)) \quad u, v \in V$. A can be extended to an unbounded operator from H to H according to $Au = -P_H \Delta u$ with the domain $D(A) = H^2(\mathcal{M}) \cap V$. By applying the theory of symmetric, compact, operators for A^{-1}, one can prove the existence of an orthonormal basis $\{e_k\}$ for H of eigenfunctions of A. Here, the associated eigenvalues $\{\lambda_k\}$ form an unbounded, increasing sequence $0 < \lambda_1 < \lambda_2 \leq \ldots \leq \lambda_n \leq \lambda_{n+1} \leq \ldots$. We shall also make use of the fractional powers of A. For $u \in H$, we denote $u_k = (u, e_k)$. Given $\alpha > 0$, take

$$D(A^\alpha) = \Big\{ u \in H : \sum_k \lambda_k^{2\alpha} |u_k|^2 < \infty \Big\}, \tag{2.7}$$

and define $A^\alpha u = \sum_k \lambda_k^\alpha u_k e_k, \quad u \in D(A^\alpha)$. We equip $D(A^\alpha)$ with the norm $|u|_\alpha^2 := |A^\alpha u|^2 = \sum_k \lambda_k^{2\alpha} |u_k|^2$. Define $H_n = span\{e_1, \ldots, e_n\}$ and take P_n to be the projection from H onto this space. Let $Q_n = I - P_n$. The following extension of the Poincaré inequality will be used for the estimates below.

Lemma 2.1. *Suppose that $\alpha_1 < \alpha_2$. For any $u \in D(A^{\alpha_2})$,*

$$|Q_n u|_{\alpha_1} \leq \lambda_n^{\alpha_1 - \alpha_2} |Q_n u|_{\alpha_2}, \tag{2.8}$$

$$|P_n u|_{\alpha_2} \leq \lambda_n^{\alpha_2 - \alpha_1} |P_n u|_{\alpha_1}. \tag{2.9}$$

Proof. Working from the definitions,

$$|Q_n u|_{\alpha_1}^2 \leq \sum_{k=n+1}^{\infty} \frac{\lambda_k^{2(\alpha_2-\alpha_1)}}{\lambda_n^{2(\alpha_2-\alpha_1)}} \lambda_k^{2\alpha_1} |u_k|^2 = \frac{1}{\lambda_n^{2(\alpha_2-\alpha_1)}} |Q_n u|_{\alpha_2}^2. \tag{2.10}$$

Similarly,

$$|P_n u|_{\alpha_2}^2 \leq \lambda_n^{2(\alpha_2-\alpha_1)} \sum_{k=1}^{n} \lambda_k^{2\alpha_1} |u_k|^2 = \lambda_n^{2(\alpha_2-\alpha_1)} |P_n u|_{\alpha_1}^2. \tag{2.11}$$

The non-linear portion of (1.1) is given by

$$B(u, v) := P_H(u \cdot \nabla)v = P_H(u_j \partial_j v), \quad u, v \in \mathcal{V}. \tag{2.12}$$

Here and below, we occasionally make use of the Einstein convention of summing repeated indices from 1 to d. For notational convenience we will sometimes write $B(u) := B(u, u)$. For $d = 2, 3$, the non-linear functional B can be shown to be well defined as a map from $V \times V$ to V' according to

$$\langle B(u, v), w\rangle := \int_{\mathcal{M}} (u \cdot \nabla)v \cdot w d\mathcal{M} = \int_{\mathcal{M}} u_j \partial_j v_k w_k d\mathcal{M}. \tag{2.13}$$

We shall need the following classical facts concerning B.

Lemma 2.2. *(i) B is continuous from $V \times V$ to V' with*

$$\langle B(u, v), v\rangle = 0, \tag{2.14}$$

and

$$|\langle B(u, v), w\rangle| \leq C \begin{cases} |u|^{1/2}\|u\|^{1/2}\|v\||w|^{1/2}\|w\|^{1/2} & \text{in } d = 2,\\ |u|^{1/2}\|u\|^{1/2}\|v\|\|w\| & \text{in } d = 3,\\ \|u\|\|v\||w|^{1/2}\|w\|^{1/2} & \text{in } d = 3, \end{cases} \tag{2.15}$$

for all $u, v, w \in V$.

(ii) B is also continuous from $V \times D(A)$ to H. If $u \in V$, $v \in D(A)$, and $w \in H$, then

$$|(B(u, v), w)| \leq C \begin{cases} |u|^{1/2}\|u\|^{1/2}\|v\|^{1/2}|Av|^{1/2}|w| & \text{in } d = 2,\\ \|u\|\|v\|^{1/2}|Av|^{1/2}|w| & \text{in } d = 3. \end{cases} \tag{2.16}$$

(iii) If $u \in D(A)$, then $B(u) \in V$, and

$$\|B(u)\|^2 \leq C\|u\||Au|^3 + |u|^{1/2}|Au|^{7/2} \quad \text{in } d = 2, 3. \tag{2.17}$$

Proof. The items (i) and (ii) are classical and are easily established using Hölder's inequality and the Sobolev embedding theorem (see [29] or [8]). For item (iii), fix $u \in \mathcal{V}$. We have

$$\|B(u)\|^2 \leq \int_{\mathcal{M}} |\partial_m(u_j \partial_j u_k)\partial_m(u_l \partial_l u_k)|\, d\mathcal{M}. \tag{2.18}$$

We prove the case $d = 3$; the case $d = 2$ is similar. We have

$$|\phi|_{L^\infty} \leq C|A\phi|^{3/4}|\phi|^{1/4}, \ \phi \in D(A). \tag{2.19}$$

This estimate and the embedding of H^1 in L^6 implies

$$\begin{aligned} \|B(u)\|^2 &\leq C(|\nabla u|^3_{L^6}\|u\| + |Au||\nabla u|^2_{L^6}|u|_{L^6} + |u|^2_{L^\infty}|Au|^2) \\ &\leq C(|Au|^3\|u\| + |Au|^{7/2}|u|^{1/2}). \end{aligned} \tag{2.20}$$

□

The stochastically driven term in (1.1) can be written formally in the expansion

$$g(u)\dot{W} = \sum_k g_k(u)\dot{\beta}_k, \tag{2.21}$$

where β_k are independent standard Brownian motions. To make this rigorous, we recall some definitions.

Definition 2.1. *A stochastic basis* $\mathcal{S} := (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}, \{\beta_k\}_{k\geq 1})$ *consists of a probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ *equipped with a complete, right-continuous filtration, namely* $\mathbb{P}(A) = 0 \Rightarrow A \in \mathcal{F}_0$, $\mathcal{F}_t = \cap_{s>t}\mathcal{F}_s$ *and a sequence of mutually independent, standard, Brownian motions* β_k *relative to this filtration.*

We also need to define a class of spaces for $g = \{g_k\}_{k\geq 1}$.

Definition 2.2. *Suppose* U *is any (separable) Hilbert space. We define* $\ell^2(U)$ *to be the set of all sequences* $h = \{h_k\}_{k\geq 1}$ *of elements in* U *so that*

$$|h|^2_{\ell^2(U)} := \sum_k |h_k|^2_U < \infty. \tag{2.22}$$

For any normed space Y*, we say that* $h : Y \times [0,T] \times \Omega \to \ell^2(U)$ *is uniformly Lipschitz with constant* K_Y*, if for all* $x, y \in Y$

$$|h(x,t,\omega) - h(y,t,\omega)|_{\ell^2(U)} \leq K_Y |x-y|_Y, \tag{2.23}$$

and

$$|h(x,t,\omega)|_{\ell^2(U)} \leq K_Y(1 + |x|_Y). \tag{2.24}$$

We denote the collection of all such mappings $Lip_u(Y, \ell^2(U))$.

For the analysis below we shall assume that

$$g = \{g_k\} : \Omega \times [0,\infty) \times H \to \ell^2(H), \tag{2.25}$$

and that

$$g \in Lip_u(H, \ell^2(H)) \cap Lip_u(V, \ell^2(V)) \cap Lip_u(D(A), \ell^2(D(A))). \tag{2.26}$$

We shall assume moreover that if $u : [0,T] \times \Omega \to H$ is predictable,[1] then so is $g(u)$. Given an H-valued predictable process $u \in L^2(\Omega; L^2(0,T;H))$,

[1]For a given stochastic basis $\mathcal{S}$, let $\Phi = \Omega \times [0,\infty)$ and take $\mathcal{G}$ to be the σ-algebra generated by sets of the form $(s,t] \times F$, $0 \leq s < t < \infty, F \in \mathcal{F}_s$; $\{0\} \times F$, $F \in \mathcal{F}_0$. Recall that a U valued process u is called predictable (with respect to the stochastic basis $\mathcal{S}$) if it is $(\Phi, \mathcal{G}) - (U, \mathcal{B}(U))$ measurable.

the series expansion (2.21) can be shown to be well defined as a stochastic integral and

$$\Big(\int_0^\tau g(u)dW, v\Big) = \Big(\sum_k \int_0^\tau g_k(u)d\beta_k, v\Big) = \sum_k \int_0^\tau (g_k(u), v)d\beta_k, \quad (2.27)$$

for all $v \in H$ and stopping times τ. See [11] or [27] for detailed constructions.

In order to show that the conditions imposed above for g are not overly restrictive we now consider some examples of stochastic forcing regimes satisfying (2.26).

Example 2.1. (i) (*Independently forced modes*) Suppose $(\kappa_k(t,\omega))$ is any sequence uniformly bounded in $L^\infty([0,T] \times \Omega)$. We force the modes independently, defining $g_k(v,t,\omega) = \kappa_k(t,\omega)(v, e_k)e_k$. In this case the Lipschitz constants can be taken to be

$$K_H = K_V = K_{D(A)} = \sup_{\omega,k,t} |\kappa_k(t,\omega)|. \quad (2.28)$$

(ii) (*Uniform forcing*) Given a uniformly square summable sequence $a_k(t,\omega)$ we can take $g_k(v,t,\omega) = a_k(t,\omega)v$, with

$$K_H = K_V = K_{D(A)} = \Big(\sup_{t,\omega} \sum_k a_k(t,\omega)^2\Big)^{1/2}$$

as the Lipschitz constants.

(iii) (*Additive noise*) We can also include the case when the noise term does not depend on the solution $g_k(v,t,\omega) = g_k(t,\omega)$. Here,

$$K_U := \sup_{t,\omega} \Big(\sum_k |g_k(t,\omega)|_U^2\Big)^{1/2}$$

for $U = H, V, D(A)$ as desired.

With the above framework in place, we next give a variational definition for local pathwise solutions of the stochastic Navier-Stokes equations. Given a Hilbert space X, for $p \in [1,\infty]$, we denote

$$L^p_{loc}([0,\infty); X) = \bigcap_{T>0} L^p([0,T]; X),$$
$$C_w([0,\infty); X) = \{v \in L^\infty_{loc}([0,\infty); X) : (v,x) \in C([0,\infty); \mathbb{R}), \forall x \in X\}.$$

Definition 2.3. (Weak and Strong Pathwise Solutions) *Let $\mathcal{S}$ be a fixed stochastic basis. Assume that u_0 is $\mathcal{F}_0$ measurable with $u_0 \in L^2(\Omega, V)$. Suppose that f and g are V' and $\ell^2(H)$ valued, predictable processes respectively with*

$$f \in L^2(\Omega; L^2([0,\infty); H)), \quad (2.29)$$

$$g \in Lip_u(H, \ell^2(H)) \cap Lip_u(V, \ell^2(V)) \cap Lip_u(D(A), \ell^2(D(A))).$$

(*i*) *We say that the pair* (u, τ) *is a* local weak (pathwise) solution *if* τ *is a strictly positive stopping time and* $u(\cdot \wedge \tau)$ *is a predictable process in* V', *with*

$$u(\cdot \wedge \tau) \in L^2(\Omega; C_w([0, \infty); H)), \quad u 1\!\!1_{t \leq \tau} \in L^2(\Omega; L^2_{loc}([0, \infty); V)), \quad (2.30)$$

and so that for any $t > 0$

$$u(t \wedge \tau) + \int_0^{t\wedge\tau} (\nu A u + B(u))\, dt = u(0) + \int_0^{t\wedge\tau} f\, dt + \int_0^{t\wedge\tau} g(u) dW, \quad (2.31)$$

in V'. *This equality is equivalent to requiring that for all* $v \in V$

$$\langle u(t \wedge \tau), v\rangle + \int_0^{t\wedge\tau} \langle \nu A u + B(u), v\rangle dt \quad (2.32)$$
$$= \langle u(0), v\rangle + \int_0^{t\wedge\tau} \langle f, v\rangle\, dt + \sum_{k=1}^{\infty} \int_0^{t\wedge\tau} \langle g_k(u), v\rangle d\beta_k.$$

(*ii*) *The pair* (u, τ) *is a* local strong (pathwise) solution *if* τ *is strictly positive and* $u(\cdot \wedge \tau)$ *is a predictable process in* H *with*

$$u(\cdot \wedge \tau) \in L^2(\Omega; C([0, \infty); V)), \quad u 1\!\!1_{t \leq \tau} \in L^2(\Omega; L^2_{loc}([0, \infty); D(A))), \quad (2.33)$$

and such that u *satisfies (2.31) as an equation in* H.

(*iii*) *Suppose that* u *is a predictable process in* V' *and that* ξ *is a strictly positive stopping time. The pair* (u, ξ) *is said to be a* maximal (pathwise) strong solution, *if there exists an increasing sequence* τ_n *with*

$$\tau_n \uparrow \xi \quad a.s., \quad (2.34)$$

such that each pair (u, τ_n) *is a local strong solution and so that*

$$\sup_{t \leq \xi} \|u\|^2 + \int_0^{\xi} |Au|^2 dt = \infty, \quad (2.35)$$

on the set $\{\xi < \infty\}$. *If, in addition*

$$\sup_{t \in [0, \tau_n]} \|u\|^2 + \int_0^{\tau_n} |Au|^2 ds = n, \quad (2.36)$$

on the set $\{\xi < \infty\}$, *then we say that* $\{\tau_n\}$ announces ξ.

Remark 2.1. (i) For the “pathwise” solutions we consider, the stochastic basis is given in advance. In particular, solutions corresponding to different initial laws are shown to be driven by the same underlying Wiener process. This is in contrast to the theory of *martingale solutions* considered for many non-linear systems. In that case, the underlying probability space

is constructed as part of the solution. See [11], chapter 8 or [24] and the references in the introduction. Since the context is clear, we will drop the "pathwise" designation for the remainder of the exposition.
(ii) If (u, τ) is a local strong solution, then (2.33) implies that

$$\mathbb{E}\Big(\sup_{t\in[0,\tau]} \|u\|^2 + \int_0^\tau |Au|^2 ds \Big) < \infty. \tag{2.37}$$

So far, we are not able to show that $\mathbb{E}\|u(t)\|^2$ is finite for any fixed (deterministic) $t > 0$. This is the case even in the two-dimensional case where we prove the existence of a global strong solution (cf. Proposition 4.2).
(iii) Suppose that (u, τ) is a local strong solution. By applying an infinite-dimensional version of the Itô lemma (see [28] or [27]) one can show that on the interval $[0, \tau]$, for any $p \geq 2$, $|u|^p$ satisfies

$$\begin{aligned} d|u|^p + p\nu\|u\|^2|u|^{p-2}dt &= p\langle f, u\rangle |u|^{p-2}dt + \frac{p}{2}\sum_{k=1}^\infty |g_k(u)|^2|u|^{p-2}dt \\ &+ \frac{p(p-2)}{2}\sum_{k=1}^\infty \langle g_k(u), u\rangle^2 |u|^{p-4}dt + p\sum_{k=1}^\infty \langle g_k(u), u\rangle |u|^{p-2}d\beta_k. \end{aligned} \tag{2.38}$$

Note that the non-linear term B drops out due to the cancellation property. Similarly for $\|u\|^p$, we have

$$\begin{aligned} &d\|u\|^p + p\nu|Au|^2\|u\|^{p-2}dt \\ &= p\langle f - B(u), Au\rangle \|u\|^{p-2}dt + \frac{p}{2}\sum_{k=1}^\infty \|g_k(u)\|^2\|u\|^{p-2}dt \\ &+ \frac{p(p-2)}{2}\sum_{k=1}^\infty \langle g_k(u), Au\rangle^2 \|u\|^{p-4}dt + p\sum_{k=1}^\infty \langle g_k(u), Au\rangle \|u\|^{p-2}d\beta_k. \end{aligned} \tag{2.39}$$

3. The Galerkin Scheme and Comparison Estimates

The first step to prove the existence of a solution is to approximate the full equations with a sequence of finite-dimensional stochastic differential equations, the Galerkin systems.

Definition 3.1. *An adapted process u^n in $C([0,T]; H_n)$ is a solution to the* Galerkin system of order n *if, for any $v \in H_n$,*

$$\begin{aligned} d\langle u^n, v\rangle + \langle \nu Au^n + B(u^n), v\rangle dt &= \langle f, v\rangle dt + \sum_{k=1}^\infty \langle g_k(u^n), v\rangle d\beta_k, \\ \langle u^n(0), v\rangle &= \langle u_0, v\rangle. \end{aligned} \tag{3.1}$$

We can also write (3.1) *as an equation in* $H_n(\cong \mathbb{R}^n)$

$$du^n + (\nu A u^n + P_n B(u^n))dt = P_n f dt + \sum_{k=1}^{\infty} P_n g_k(u^n) d\beta_k, \qquad (3.2)$$
$$u^n(0) = P_n u_0 := u_0^n.$$

The existence of solutions to (3.1) is classical and relies on a priori bounds that are established using the cancellation property (2.14). See [16] for detailed proofs. Uniqueness, which is not essential for our purposes, is established as below for the full infinite-dimensional system.

We now proceed to establish the main result of the section. Note that the conditions established hereafter are precisely those needed to apply Lemma 5.1 in Proposition 4.2 below.

Proposition 3.1. *Suppose that* $d = 2, 3$ *and let* $\{u^n\}$ *be the sequence of solutions of* (3.1). *We assume that for some* $0 < \tilde{M} < \infty$

$$\|u_0\| \leq \tilde{M} \quad a.s., \qquad (3.3)$$

and that

$$f \in L^2(\Omega; L^2([0,T]; H)),$$
$$g \in Lip_u(H, \ell^2(H)) \cap Lip_u(V, \ell^2(V)) \cap Lip_u(D(A), \ell^2(D(A))), \qquad (3.4)$$

where the spaces for g *and the associated Lipschitz constants used are given as in Definition* 2.2. *Consider the collection of stopping times*

$$\mathcal{T}_n^{M,T} = \left\{ \tau \leq T : \left(\sup_{t\in[0,\tau]} \|u^n\|^2 + \nu \int_0^\tau |Au^n|^2 dt \right)^{1/2} \leq M + \|u_0^n\| \right\}, \qquad (3.5)$$

and take $\mathcal{T}_{m,n}^{M,T} := \mathcal{T}_m^{M,T} \cap \mathcal{T}_n^{M,T}$. *Then*

(i) *For any* $T > 0$ *and* $M > 1$

$$\lim_{n\to\infty} \sup_{m>n} \sup_{\tau \in \mathcal{T}_{m,n}^{M,T}} \mathbb{E}\left(\sup_{t\in[0,\tau]} \|u^m - u^n\|^2 + \nu \int_0^\tau |A(u^m - u^n)|^2 dt \right) = 0. \qquad (3.6)$$

(ii) *Moreover, if for* $n \in \mathbb{N}$, $S > 0$ *and a stopping time* τ, *if*

$$A_n(\tau, S) = \left\{ \sup_{t\in[0,\tau\wedge S]} \|u^n\|^2 + \nu \int_0^{\tau\wedge S} |Au^n|^2 dt > \|u_0^n\|^2 + (M-1)^2 \right\},$$

then

$$\lim_{S\to 0} \sup_n \sup_{\tau\in\mathcal{T}_n^{M,T}} \mathbb{P}(A_n(\tau, S)) = 0. \qquad (3.7)$$

Proof. Given $m > n$, we subtract the equation (3.2) for u^n from that for u^m, and let $U^{m,n} = u^m - u^n$. We find that

$$\begin{aligned} dU^{m,n} + \nu A U^{m,n} dt &= [P_n B(u^n) - P_m B(u^m) + (P_m - P_n) f] dt \\ &\quad + \sum_{k=1}^{\infty} [P_m g_k(u^m) - P_n g_k(u^n)] d\beta_k \\ U^{m,n}(0) &= (P_m - P_n) u_0. \end{aligned} \tag{3.8}$$

By applying the Itô lemma to (3.8), we derive an evolution system for the V norm of this difference[2]

$$\begin{aligned} d\|U^{m,n}\|^2 + 2\nu |AU^{m,n}|^2 dt &= 2\langle P_n B(u^n) - P_m B(u^m), AU^{m,n}\rangle dt \\ + 2\langle (P_m - P_n) f, AU^{m,n}\rangle dt &+ \sum_{k=1}^{\infty} \|P_m g_k(u^m) - P_n g_k(u^n)\|^2 dt \\ + 2\sum_{k=1}^{\infty} \langle P_m g_k(u^m) - P_n g_k(u^n), AU^{m,n}\rangle d\beta_k. \end{aligned} \tag{3.9}$$

Fix an arbitrary $\tau \in \mathcal{T}_{m,n}^{M,T}$, and let the stopping times τ_a and τ_b with $0 \leq \tau_a \leq \tau_b \leq \tau$ be given with $\tau \in \mathcal{T}_{m,n}^{M,T}$; we integrate (3.9) from τ_a to r and take a supremum over $[\tau_a, \tau_b]$. After taking expected values we obtain

$$\begin{aligned} &\mathbb{E}\Big(\sup_{t\in[\tau_a,\tau_b]} \|U^{m,n}\|^2 + 2\nu \int_{\tau_a}^{\tau_b} |AU^{m,n}|^2 dt\Big) \\ &\leq \mathbb{E}\|U^{m,n}(\tau_a)\|^2 + 2\mathbb{E}\int_{\tau_a}^{\tau_b} |\langle (P_m - P_n) f, AU^{m,n}\rangle| dt \\ &+ 2\mathbb{E}\int_{\tau_a}^{\tau_b} |\langle P_m B(u^m) - P_n B(u^n), AU^{m,n}\rangle| dt \\ &+ \mathbb{E}\int_{\tau_a}^{\tau_b} \sum_{k=1}^{\infty} \|P_m g_k(u^m) - P_n g_k(u^n)\|^2 dt \\ &+ \mathbb{E}\Big(\sup_{r\in[\tau_a,\tau_b]} \Big| 2\sum_{k=1}^{\infty} \int_{\tau_a}^{r} \langle P_m g_k(u^m) - P_n g_k(u^n), AU^{m,n}\rangle d\beta_k \Big| \Big). \end{aligned} \tag{3.10}$$

With the aim of employing Lemma 5.3 below, we estimate each of the terms on the right-hand side of (3.10). For the first term, we merely split

$$|2\langle (P_m - P_n) f, AU^{m,n}\rangle| \leq \frac{\nu}{2} |AU^{m,n}|^2 + C_\nu |Q_n f|^2. \tag{3.11}$$

[2] Compare to (2.39) with $p = 2$.

Note that

$$\|P_m g_k(u^m) - P_n g_k(u^n)\|^2 \leq 2\|g_k(u^m) - g_k(u^n)\|^2 + 2\|Q_n g_k(u^n)\|^2.$$

Therefore, using the Lipschitz conditions in (3.4) and Lemma 2.1, we have

$$\begin{aligned} &\sum_{k=1}^{\infty} \|P_m g_k(u^m) - P_n g_k(u^n)\|^2 \\ &\leq C\Big(\|U^{m,n}\|^2 + \frac{1}{\lambda_n}\sum_{k=1}^{\infty}|Ag_k(u^n)|^2\Big) \leq C\Big(\|U^{m,n}\|^2 + \frac{1}{\lambda_n}(1 + |Au^n|^2)\Big). \end{aligned} \tag{3.12}$$

For the stochastically forced term, we apply the Burkholder-Davis-Gundy inequality (c.f. [19])

$$\begin{aligned} &\mathbb{E}\Big(\sup_{r\in[\tau_a,\tau_b]}\Big|2\sum_{k=1}^{\infty}\int_{\tau_a}^{r}\langle P_m g_k(u^m) - P_n g_k(u^n), AU^{m,n}\rangle d\beta_k\Big|\Big) \\ &\leq C\mathbb{E}\Big(\int_{\tau_a}^{\tau_b}\sum_{k=1}^{\infty}\langle P_m g_k(u^m) - P_n g_k(u^n), AU^{m,n}\rangle^2\, ds\Big)^{1/2} \\ &\leq C\mathbb{E}\Big(\int_{\tau_a}^{\tau_b}\|U^{m,n}\|^2\sum_{k=1}^{\infty}\|P_m g_k(u^m) - P_n g_k(u^n)\|^2\, ds\Big)^{1/2} \\ &\leq C\mathbb{E}\Big(\int_{\tau_a}^{\tau_b}\|U^{m,n}\|^2\Big(\|U^{m,n}\|^2 + \tfrac{1}{\lambda_n}(1+|Au^n|^2)\Big)\, ds\Big)^{1/2} \\ &\leq \frac{1}{2}\mathbb{E}\Big(\sup_{t\in[\tau_a,\tau_b]}\|U^{m,n}\|^2\Big) + C\mathbb{E}\int_{\tau_a}^{\tau_b}\Big(\|U^{m,n}\|^2 + \tfrac{1}{\lambda_n}(1+|Au^n|^2)\Big)\, ds, \end{aligned} \tag{3.13}$$

where the constant C is independent of n, m, τ_a, and τ_b. It remains to study the non-linear term which we split as follows:

$$\begin{aligned} &\langle P_m B(u^m) - P_n B(u^n), AU^{m,n}\rangle \\ &= \langle B(U^{m,n}, u^m) + B(u^n, U^{m,n}) + (P_m - P_n)B(u^n), AU^{m,n}\rangle \\ &:= T_1 + T_2 + T_3. \end{aligned} \tag{3.14}$$

For T_1 in (3.14), we apply (2.16). If d equals either 2 or 3 we estimate

$$|T_1| \leq C\|U^{m,n}\||Au^m||AU^{m,n}| \leq \frac{\nu}{6}|AU^{m,n}|^2 + C_\nu\|U^{m,n}\|^2|Au^m|^2. \tag{3.15}$$

Regarding T_2, (2.16) yields

$$|T_2| \leq \|u^n\|\|U^{m,n}\|^{1/2}|AU^{m,n}|^{3/2} \leq \frac{\nu}{6}|AU^{m,n}|^2 + C_\nu\|U^{m,n}\|^2\|u^n\|^4. \tag{3.16}$$

For the final term T_3, we apply (2.17) in conjunction with Lemma 2.1, and infer $(m > n)$

$$T_3 \leq \frac{\nu}{6}|AU^{m,n}|^2 + C_\nu |Q_n B(u^n)|^2 \leq \frac{\nu}{6}|AU^{m,n}|^2 + \frac{C_\nu}{\lambda_n}\|Q_n B(u^n)\|^2 \quad (3.17)$$
$$\leq \frac{\nu}{6}|A(u^m - u^n)|^2 + \frac{C_\nu}{\lambda_n}(\|u^n\||Au^n|^3 + |u^n|^{1/2}|Au^n|^{7/2})$$
$$\leq \frac{\nu}{6}|A(u^m - u^n)|^2 + \frac{C_\nu}{\lambda_n^{1/4}}\|u^n\|^2|Au^n|^2,$$

which is once again valid for $d = 2, 3$. Combining the estimates (3.15),(3.16) and (3.17) and using the fact that $\tau_a, \tau_b \in \mathcal{T}_m^{M,T} \cap \mathcal{T}_n^{M,T}$ (c.f. (3.5)) we infer that $\|u^n\| \leq M + \tilde{M}$ on the intervals under consideration, and therefore

$$\mathbb{E}\int_{\tau_a}^{\tau_b} |\langle P_m B(u^m) - P_n B(u^n), AU^{m,n}\rangle| dt \leq \nu \mathbb{E}\int_{\tau_a}^{\tau_b} |AU^{m,n}|^2 ds \quad (3.18)$$
$$+ C_\nu \mathbb{E}\int_{\tau_a}^{\tau_b} \Big(\|U^{m,n}\|^2(|Au^m|^2 + \|u^n\|^4) + \lambda_n^{-1/4}\|u^n\|^2|Au^n|^2\Big) ds$$
$$\leq \nu \mathbb{E}\int_{\tau_a}^{\tau_b}|AU^{m,n}|^2 ds + C\mathbb{E}\int_{\tau_a}^{\tau_b}\Big(\|U^{m,n}\|^2(1 + |Au^m|^2) + \lambda_n^{-\frac{1}{4}}(1 + |Au^n|^2)\Big)ds.$$

Applying the estimates (3.11)–(3.18) and rearranging gives

$$\mathbb{E}\Big(\sup_{t\in[\tau_a,\tau_b]} \|U^{m,n}\|^2 + \nu\int_{\tau_a}^{\tau_b} |AU^{m,n}|^2 dt\Big) \quad (3.19)$$
$$\leq C\mathbb{E}\|U^{m,n}(\tau_a)\|^2 + C\mathbb{E}\Big(\int_{\tau_a}^{\tau_b}(1 + |Au^m|^2)\|u^m - u^n\|^2 dt\Big)$$
$$+ C\mathbb{E}\int_{\tau_a}^{\tau_b}\Big(\frac{1}{\lambda_n^{\frac{1}{4}}}(1 + |Au^n|^2) + |Q_n f|^2\Big)dt.$$

Note that $C = C_{\nu,M,\tilde{M},K_V,K_{D(A)}}$ does not depend on τ_a and τ_b. Also

$$\int_0^\tau 1 + |Au^m|^2 dt \leq \frac{1}{\nu}(M + \tilde{M})^2 + T \quad a.s. \quad (3.20)$$

We now apply the Gronwall lemma (Lemma 5.3) with $X = \|U^{m,n}\|^2$, $Y = \nu|AU^{m,n}|^2$, $Z = \lambda_n^{-1/4}(1 + |Au^n|^2) + |Q_n f|^2$, and $R = 1 + |Au^m|^2$. We obtain

$$\mathbb{E}\Big(\sup_{t\in[0,\tau]} \|U^{m,n}\|^2 + \nu\int_0^\tau |AU^{m,n}|^2 dt\Big) \quad (3.21)$$
$$\leq C\mathbb{E}\Big(\|Q_n u_0\|^2 + \int_0^T |Q_n f|^2 dt + \frac{1}{\lambda_n^{1/4}}\Big).$$

Observe that the constant $C = C_{\nu,M,\tilde{M},K_V,K_{D(A)},T}$ does not depend on n, m or the choice of $\tau \in \mathcal{T}_m^{M,T} \cap \mathcal{T}_n^{M,T}$. Thus, (3.21) implies (3.6) after taking the supremum over $\tau \in \mathcal{T}_{m,n}^{M,T}$, then over $m > n$ and finally taking the limit as $n \to \infty$.

By application of the Itô formula we find an evolution equation for $\|u^n\|^2$. (This is similar to (2.39) but can be justified on more elementary terms since we are in finite dimensions.)

$$d\|u^n\|^2 + 2\nu|Au^n|^2 dt = \Big(2\langle f - B(u^n), Au^n\rangle + \sum_{k=1}^{\infty} \|P_m g_k(u^n)\|^2\Big) dt \qquad (3.22)$$
$$+ 2\sum_{k=1}^{\infty} \langle g_k(u^n), Au^n\rangle d\beta_k.$$

Fix $\tau \in \mathcal{T}_n^{M,T}$ and $S > 0$. Integrating (3.22) from 0 to $\tau \wedge S$ yields

$$\sup_{r\in[0,S\wedge\tau]} \|u^n\|^2 + \int_0^{S\wedge\tau} 2\nu|Au^n|^2\, dr \le \|u_0^n\|^2 + \int_0^{S\wedge\tau} 2|\langle f - B(u^n), Au^n\rangle|\, dr$$
$$+ \int_0^{S\wedge\tau} \sum_{k=1}^{\infty} \|P_m g_k(u^n)\|^2\, dr + \sup_{r\in[0,S\wedge\tau]} \Big| \sum_{k=1}^{\infty} \int_0^r 2((g_k(u^n), u^n))\, d\beta_k \Big|. \qquad (3.23)$$

Applying (2.16) we see that in both cases $d = 2, 3$

$$|\langle B(u^n), Au^n\rangle| \le \|u^n\|^{3/2}|Au^n|^{3/2} \le C_\nu \|u^n\|^6 + \frac{\nu}{4}|Au^n|^2. \qquad (3.24)$$

Using this observation and the Lipschitz conditions imposed on g, one finds that

$$\sup_{r\in[0,S\wedge\tau]} \|u^n\|^2 + \int_0^{S\wedge\tau} \nu|Au^n|^2\, dr \qquad (3.25)$$
$$\le \|u_0^n\|^2 + C_{\nu,K_v} \int_0^{S\wedge\tau} (|f|^2 + \|u^n\|^6 + \|u^n\|^2 + 1)\, dr$$
$$+ \sup_{r\in[0,S\wedge\tau]} \Big| \int_0^r 2\sum_{k=1}^{\infty} ((g_k(u^n), u^n))\, d\beta_k \Big|.$$

This implies

$$\mathbb{P}\Big(\sup_{s\in[0,\tau\wedge S]} \|u^n(s)\|^2 + \nu \int_0^{\tau\wedge S} |Au^n|^2 ds > \|u_0^n\|^2 + (M-1)^2 \Big) \qquad (3.26)$$

$$\leq \mathbb{P}\Big(C_{\nu,K_V}\int_0^{S\wedge\tau}(|f|^2+\|u^n\|^6+\|u^n\|^2+1)\,dr > \frac{(M-1)^2}{2}\Big)$$
$$+\,\mathbb{P}\Big(\sup_{r\in[0,S\wedge\tau]}\Big|\int_0^r\sum_{k=1}^{\infty}((g_k(u^n),u^n))\,d\beta_k\Big| > \frac{(M-1)^2}{2}\Big).$$

For the first term on the right-hand side of (3.26), Chebyshev's inequality and the fact that $\tau \in \mathcal{T}_n^{M,T}$ imply

$$\begin{aligned}\mathbb{P}\Big(C_{\nu,K_V}\int_0^{S\wedge\tau}(|f|^2+\|u^n\|^6+\|u^n\|^2+1)\,dr > \frac{(M-1)^2}{2}\Big) \\ \leq \frac{2C_{\nu,K_V}}{(M-1)^2}\mathbb{E}\int_0^{S\wedge\tau}(|f|^2+\|u^n\|^6+\|u^n\|^2+1)\,dr \\ \leq C_{\nu,K_V,M,\tilde{M}}\mathbb{E}\Big(\int_0^S(|f|^2+1)\,dr\Big).\end{aligned} \tag{3.27}$$

Next, by applying Doob's inequality for the second term, we have

$$\begin{aligned}\mathbb{P}\Big(\sup_{r\in[0,S\wedge\tau]}\Big|\sum_{k=1}^{\infty}\int_0^r((g_k(u^n),u^n))\,d\beta_k\Big| > \frac{(M-1)^2}{2}\Big) \\ \leq \frac{4}{(M-1)^4}\mathbb{E}\Big(\int_0^{S\wedge\tau}\|u^n\|^2\sum_{k=1}^{\infty}\|g_k(u^n)\|^2\,dr\Big) \leq C_{M,\tilde{M},K_V}S.\end{aligned} \tag{3.28}$$

Given the integrability assumed for f and noting that the right-hand sides of (3.27) and (3.28) are independent of τ we have now established (3.7). □

4. EXISTENCE AND UNIQUENESS

With the comparison estimates for the Galerkin systems in hand, we next turn to the questions of existence and uniqueness. Since we will need to split the probability space into pieces (see below), it is convenient to first address the question of uniqueness.

Proposition 4.1. (Uniqueness) *Let $\tau > 0$ be a stopping time. Suppose that $(u^{(1)},\tau)$ and $(u^{(2)},\tau)$ are respectively local strong and weak solutions to the stochastic Navier-Stokes equations in $d = 2,3$ (cf. Definition 2.3). Let $u_0^{(1)}$, $u_0^{(2)}$ be the associated initial conditions and assume that*

$$\mathbb{P}(\mathbb{1}_{\Omega_0}u_0^{(1)} = \mathbb{1}_{\Omega_0}u_0^{(2)}) = 1, \tag{4.1}$$

for some $\Omega_0 \in \mathcal{F}_0$. Then

$$\mathbb{P}(\mathbb{1}_{\Omega_0}u^{(1)}(t\wedge\tau) = \mathbb{1}_{\Omega_0}u^{(2)}(t\wedge\tau);\, t\in[0,\infty)) = 1. \tag{4.2}$$

Proof. Let $U = u^{(1)} - u^{(2)}$; we have

$$dU = -[\nu AU + B(u^{(1)}) - B(u^{(2)})]dt + \sum_{k=1}^{\infty}[g_k(u^{(1)}) - g_k(u^{(2)})]d\beta_k. \quad (4.3)$$

The Itô lemma yields, as with (2.38),

$$\begin{aligned} d|U|^2 =& - 2\nu\|U\|^2 dt - 2\langle B(u^{(1)}) - B(u^{(2)}), U\rangle dt \\ &+ \sum_{k=1}^{\infty}|g_k(u^{(1)}) - g_k(u^{(2)})|^2 dt + 2\sum_{k=1}^{\infty}\langle g_k(u^{(1)}) - g_k(u^{(2)}), U\rangle d\beta_k. \end{aligned} \quad (4.4)$$

Given any stopping time $\sigma \leq \tau$, we may integrate (4.4) from 0 to σ, multiply the resulting expression by $\mathbb{1}_{\Omega_0}$ and finally take expectations to conclude

$$\begin{aligned} &\mathbb{E}\mathbb{1}_{\Omega_0}\Big(|U(\sigma)|^2 + 2\nu\int_0^{\sigma}\|U\|^2 dt\Big) \\ &= \mathbb{E}\mathbb{1}_{\Omega_0}\Big(\int_0^{\sigma} 2\langle B(u^{(2)}) - B(u^{(1)}), U\rangle + \sum_{k=1}^{\infty}|g_k(u^{(1)}) - g_k(u^{(2)})|^2 dt\Big). \end{aligned} \quad (4.5)$$

Using the cancellation property (2.14) and (2.15), we have (e.g. [8], [29])

$$|\langle B(u^{(1)}) - B(u^{(2)}), U\rangle| = |\langle B(U, u^{(1)}), U\rangle| \leq \nu\|U\|^2 + C_\nu\|u^{(1)}\|^4|U|^2. \quad (4.6)$$

For $R > 0$, define the stopping times

$$\sigma_R = \inf_{t>0}\{\|u^{(1)}(t)\|^2 > R\} \wedge \tau. \quad (4.7)$$

Fix R and apply (4.6) and the Lipschitz condition on g in H (cf. (2.29)) to (4.5) with the stopping time $\sigma_R \wedge t$. Rearranging, we estimate

$$\begin{aligned} \mathbb{E}\mathbb{1}_{\Omega_0}&\Big(|U(\sigma_R \wedge t)|^2 + \nu\int_0^{\sigma_R\wedge t}\|U(s)\|^2 ds\Big) \\ &\leq C_{\nu,K_H}\mathbb{E}\mathbb{1}_{\Omega_0}\int_0^{\sigma_R\wedge t}(\|u^{(1)}\|^4 + 1)|U(s)|^2 ds \\ &\leq C_{\nu,K_H,R}\mathbb{E}\mathbb{1}_{\Omega_0}\int_0^{\sigma_R\wedge t}|U(s)|^2 ds. \end{aligned} \quad (4.8)$$

By the Gronwall and Poincaré inequalities

$$\mathbb{E}\mathbb{1}_{\Omega_0}\Big(|u^{(1)}(\sigma_R \wedge t) - u^{(2)}(\sigma_R \wedge t)|^2\Big) = 0, \quad (4.9)$$

which implies

$$\mathbb{1}_{\Omega_0}|u^{(1)}(\sigma_R \wedge t) - u^{(2)}(\sigma_R \wedge t)|^2 = 0, \quad \text{a.s.} \quad (4.10)$$

Thus,

$$\begin{aligned}&\mathbb{P}(\mathbb{1}_{\Omega_0}|u^{(1)}(t\wedge\tau)-u^{(2)}(t\wedge\tau)|^2\neq 0)\\&=\mathbb{P}(\{\sigma_R<\tau\}\cap\{\mathbb{1}_{\Omega_0}|u^{(1)}(t\wedge\tau)-u^{(2)}(t\wedge\tau)|^2\neq 0\})\\&+\mathbb{P}(\{\sigma_R=\tau\}\cap\{\mathbb{1}_{\Omega_0}|u^{(1)}(t\wedge\tau)-u^{(2)}(t\wedge\tau)|^2\neq 0\})\\&\leq\mathbb{P}(\sigma_R<\tau)+\mathbb{P}(\mathbb{1}_{\Omega_0}|u^{(1)}(\sigma_R\wedge t)-u^{(2)}(\sigma_R\wedge t)|^2\neq 0)\\&=\mathbb{P}(\sigma_R<\tau).\end{aligned}\tag{4.11}$$

Note that

$$\mathbb{P}(\sigma_R<\tau)\leq\mathbb{P}\Big(\sup_{s\in[0,\tau]}\|u^{(1)}\|^2\geq R\Big)\leq\frac{1}{R}\mathbb{E}\Big(\sup_{s\in[0,\tau]}\|u^{(1)}\|^2\Big)\to 0,\tag{4.12}$$

as $R\to\infty$. So for any t

$$\mathbb{1}_{\Omega_0}|u^{(1)}(t\wedge\tau)-u^{(2)}(t\wedge\tau)|^2=0,\tag{4.13}$$

on a set of full measure which may depend on t. Taking the intersection of such sets corresponding to positive rational times we infer

$$\mathbb{P}(\mathbb{1}_{\Omega_0}u^{(1)}(t\wedge\tau)=\mathbb{1}_{\Omega_0}u^{(2)}(t\wedge\tau);t\in[0,\infty)\cap\mathbb{Q})=1.\tag{4.14}$$

Given the continuity assumption in (2.30) we finally conclude (4.2) from (4.14), completing the proof. □

4.1. **Existence of local strong solutions.**

Proposition 4.2. (Local Existence) *Suppose that* $d=2,3$ *and assume that*

$$\begin{aligned}&u_0\in L^2(\Omega;V),\quad f\in L^2(\Omega;L^2_{loc}([0,\infty);H)),\\&g\in Lip_u(H,\ell^2(H))\cap Lip_u(V,\ell^2(V))\cap Lip_u(D(A),\ell^2(D(A))),\end{aligned}\tag{4.15}$$

for some T *positive. Then, there exists a local strong solution* (u,τ) *in the sense of Definition* 2.3.

Proof. We proceed in two steps. First, we assume that $\|u_0\|\leq\tilde{M}$, almost surely, so that the estimates in Proposition 3.1 apply. Also fix $M>1$ and a positive time T as in (3.5). Take $\{u^n\}$ to be the associated sequence of Galerkin solutions. Due to (3.6) and (3.7), the assumptions for Lemma 5.1, (i) are satisfied for the spaces $B_1=V$ and $B_2=D(A)$ and the sequence $\{X^n\}=\{u^n\}$. We infer the existence of a subsequence $\{u^{n'}\}$, a strictly positive stopping time $\tau\leq T$ and a process $u(\cdot)=u(\cdot\wedge\tau)$, continuous in V, such that

$$\sup_{t\in[0,\tau]}\|u^{n'}-u\|^2+\nu\int_0^\tau|A(u^{n'}-u)|^2ds\to 0\quad\text{a.s.}\tag{4.16}$$

Notice, moreover, that u_0^n satisfies the conditions for Lemma 5.1, (ii) for any $p \in (1, \infty)$. Thus for any such p

$$u(\cdot \wedge \tau) \in L^p(\Omega; C([0,T]; V)), \tag{4.17}$$

and

$$u \mathbb{1}_{t \leq \tau} \in L^p(\Omega; L^2([0,T]; D(A))). \tag{4.18}$$

From Lemma 5.1, (ii) we also obtain a collection of measurable sets $\Omega_{n'} \in \mathcal{F}$ with $\Omega_{n'} \uparrow \Omega$ such that (cf. (5.9))

$$\sup_{n'} \mathbb{E}\Big[\sup_{t \in [0,T]} \|u^{n'}(t \wedge \tau) \mathbb{1}_{\Omega_{n'}}\|^2 + \nu \int_0^\tau |A u^{n'} \mathbb{1}_{\Omega_{n'}}|^2 ds \Big]^{p/2} < \infty. \tag{4.19}$$

Therefore, given (4.16) and (4.19), we apply Lemma 5.2 and infer

$$\mathbb{1}_{\Omega_{n'}, t \leq \tau} u^{n'} \rightharpoonup \mathbb{1}_{t \leq \tau} u \quad \text{ in } L^p(\Omega; L^2([0,T]; D(A))), \tag{4.20}$$

as well as

$$\mathbb{1}_{\Omega_{n'}} u^{n'}(\cdot \wedge \tau) \rightharpoonup^* u \quad \text{ in } L^p(\Omega; L^\infty([0,T]; V)). \tag{4.21}$$

For the non-linear term we apply (2.15) and estimate

$$\begin{aligned} |(P_{n'} B(u^{n'}) - B(u), v)| \leq{}& |\langle B(u^{n'} - u, u^{n'}), P_{n'} v \rangle| \\ & + |\langle B(u, u^{n'} - u), P_{n'} v\rangle| + |\langle B(u), Q_{n'} v \rangle| \\ \leq{}& C(\|u^{n'} - u\|(\|u^{n'}\| + \|u\|)\|v\| + \|u\|^2 |Q_{n'} v|^{1/2} \|v\|^{1/2}) \\ \leq{}& C\Big(\|u^{n'} - u\|(\|u^{n'}\| + \|u\|)\|v\| + \frac{1}{\lambda_{n'}^{1/4}} \|u\|^2 \|v\| \Big). \end{aligned} \tag{4.22}$$

By applying (4.16) with (4.22) we infer that, given any $v \in V$,

$$\mathbb{1}_{t \leq \tau} (P_{n'} B(u^{n'}), v) \rightarrow \mathbb{1}_{t \leq \tau} (B(u), v) \quad \text{ as } n' \rightarrow \infty, \tag{4.23}$$

for almost every $(\omega, t) \in \Omega \times [0,T]$. Furthermore, making use of the uniform bound (4.19) with $p = 4$, one finds

$$\begin{aligned} \sup_{n'} \mathbb{E}\Big(\mathbb{1}_{\Omega_{n'}} \int_0^\tau |P_{n'} B(u^{n'})|^2 \, ds \Big) &\leq C \sup_{n'} \mathbb{E}\Big(\mathbb{1}_{\Omega_{n'}} \int_0^\tau \|u^{n'}\|^3 |A u^{n'}| \, ds \Big) \\ &\leq C \sup_{n'} \mathbb{E}\Big(\mathbb{1}_{\Omega_{n'}} \sup_{t \in [0,\tau]} \|u^{n'}\|^2 \int_0^\tau |A u^{n'}|^2 \, ds \Big) \\ &\leq C \sup_{n'} \mathbb{E} \mathbb{1}_{\Omega_{n'}} \Big(\sup_{t \in [0,\tau]} \|u^{n'}\|^4 + \Big(\int_0^\tau |A u^{n'}|^2 \, ds \Big)^2 \Big) < \infty. \end{aligned} \tag{4.24}$$

With (4.23) and (4.24), and Lemma 5.2, we gather that

$$\mathbb{1}_{\Omega_{n'}, t \leq \tau} P_{n'} B(u^{n'}) \rightharpoonup \mathbb{1}_{t \leq \tau} B(u) \quad \text{ in } L^2(\Omega; L^2([0,T]; H)). \tag{4.25}$$

For the stochastic terms in (3.2), we use (4.15) and obtain

$$
\begin{aligned}
&\Big(\sum_k |P_{n'} g_k(u^{n'}) - g_k(u)|^2\Big)^{1/2} \qquad (4.26)\\
&\le \Big(\sum_k |P_{n'} g_k(u^{n'}) - P_{n'} g_k(u)|^2\Big)^{1/2} + \Big(\sum_k |Q_{n'} g_k(u)|^2\Big)^{1/2}\\
&\le C_{K_H,K_V}\Big(\|u^{n'} - u\| + \frac{1}{\lambda_{n'}}(1+\|u\|)\,ds\Big).
\end{aligned}
$$

With this estimate, (4.16) implies $1\!\!1_{t\le\tau} P_{n'} g(u^{n'}) \to 1\!\!1_{t\le\tau} g(u)$, in $\ell^2(H)$, for almost every $(\omega,t)\in\Omega\times[0,T]$. On the other hand, if $\mathcal{Y}_n = \sum_k |P_{n'} g_k(u^{n'})|^2$, then

$$
\sup_{n'} \mathbb{E}\Big[1\!\!1_{\Omega_{n'}} \int_0^\tau \mathcal{Y}_n(s)\,ds\Big] \le C + C \sup_{n'} \mathbb{E}\Big[1\!\!1_{\Omega_{n'}} \int_0^\tau \|u^{n'}\|^2\,ds\Big] < \infty,
$$

which means that

$$
1\!\!1_{\Omega_{n'},t\le\tau} P_{n'} g(u^{n'}) \rightharpoonup 1\!\!1_{t\le\tau} g(u), \qquad (4.27)
$$

in $L^2(\Omega; L^2([0,T]; \ell^2(H)))$. Therefore, using (4.20) and (4.25), we deduce that for any fixed $v \in H$ (see Remark 4.1, (i))

$$
\begin{aligned}
1\!\!1_{\Omega_{n'}} \int_0^{t\wedge\tau} (Au^{n'}, v)\,ds &\rightharpoonup \int_0^{t\wedge\tau} (Au, v)\,ds, \qquad (4.28)\\
1\!\!1_{\Omega_{n'}} \int_0^{t\wedge\tau} (P_n B(u^{n'}), v)\,ds &\rightharpoonup \int_0^{t\wedge\tau} (B(u), v)\,ds,\\
1\!\!1_{\Omega_{n'}} \sum_k \int_0^{t\wedge\tau} (P_{n'} g_k(u^{n'}), v)\,d\beta_k &\rightharpoonup \sum_k \int_0^{t\wedge\tau} (g_k(u), v)\,d\beta_k,
\end{aligned}
$$

weakly in $L^2(\Omega\times[0,T])$. If $K \subset \Omega\times[0,T]$ is any measurable set then by (4.21) and (4.28)

$$
\begin{aligned}
\mathbb{E}\int_0^T \chi_K (u(t), v)\,dt &= \lim_{n'\to\infty} \mathbb{E}\int_0^T (1\!\!1_{\Omega_{n'}} u^{n'}(t\wedge\tau), \chi_K v)\,dt \qquad (4.29)\\
&= \lim_{n'\to\infty} \Big(\mathbb{E}\int_0^T \chi_K 1\!\!1_{\Omega_{n'}} (P_{n'} u_0, v)\,dt\\
&\quad - \mathbb{E}\int_0^T \chi_K 1\!\!1_{\Omega_{n'}} \Big[\int_0^{t\wedge\tau} (\nu A u^{n'} + P_{n'} B(u^{n'}) - P_{n'} f, v)\,ds\Big] dt\\
&\quad + \mathbb{E}\int_0^T \chi_K 1\!\!1_{\Omega_{n'}} \Big[\sum_k \int_0^{t\wedge\tau} (P_{n'} g_k(u^{n'}), v) d\beta_k\Big] dt\Big)
\end{aligned}
$$

$$= \mathbb{E}\int_0^T \chi_K\Big[(u_0, v) - \int_0^{t\wedge\tau}(\nu A u + B(u) - f, v)ds\Big]dt$$
$$+ \mathbb{E}\int_0^T \chi_K\Big[\sum_k \int_0^{t\wedge\tau}(g_k(u), v)d\beta_k\Big]dt.$$

Since v and K above are arbitrary, we may infer that u satisfies (2.31) as an equation in H (cf. Remark 4.1, (ii)). It follows from (4.17), (4.18) and the fact that $\tau \leq T$ almost surely that u satisfies the regularity conditions (2.33).

The proof is complete for the case when $\|u_0\| \leq \tilde{M}$ almost surely. We suppose now merely that $\mathbb{E}\|u_0\|^2 < \infty$. For $k \geq 0$, take (u_k, τ_k) to be the local strong solutions corresponding to the initial data $u_0 1\!\!1_{k \leq \|u_0\| < k+1}$. Let

$$u = \sum_{k=0}^{\infty} u_k 1\!\!1_{k\leq\|u_0\|<k+1}, \quad \tau = \sum_{k=0}^{\infty} \tau_k 1\!\!1_{k\leq\|u_0\|<k+1}. \tag{4.30}$$

We now show that (u, τ) is a local strong solution with initial data u_0. Since $u_k \in C([0, \tau_k], V)$ almost surely, we infer that $u \in C([0, \tau]; V)$ almost surely. Using the fact that $\mathbb{E}\|u_0\|^2 < \infty$, one infers that $1 = \sum_{k=0}^{\infty} 1\!\!1_{k\leq\|u_0\|<k+1}$ almost surely. Therefore, by applying (5.7) for each u_k, we deduce

$$\begin{aligned}\sup_{t\in[0,\tau]} &\|u\|^2 + \nu\int_0^{\tau}|Au|^2 ds \\ &= \sum_{k=0}^{\infty} 1\!\!1_{k\leq\|u_0\|<k+1}\Big[\sup_{t\in[0,\tau_k]}\|u_k\|^2 + \nu\int_0^{\tau_k}|Au_k|^2 ds\Big] \\ &\leq C\sum_{k=0}^{\infty} 1\!\!1_{k\leq\|u_0\|<k+1}(M^2 + \|u_0\|^2) \leq C(M^2 + \|u_0\|^2).\end{aligned} \tag{4.31}$$

Taking expectations above, one infers (2.33) for u. Furthermore,

$$\begin{aligned}u(t\wedge\tau) &= \sum_{k=0}^{\infty} 1\!\!1_{k\leq\|u_0\|<k+1} u_k(t\wedge\tau_k) \\ &= \sum_{k=0}^{\infty} 1\!\!1_{k\leq\|u_0\|<k+1}\Big[u_0 - \int_0^{t\wedge\tau_k}(\nu A u_k + B(u_k) - f)dt\Big] \\ &\quad + \sum_{k=0}^{\infty} 1\!\!1_{k\leq\|u_0\|<k+1}\Big[\int_0^{t\wedge\tau_k} g(u_k)dW\Big],\end{aligned} \tag{4.32}$$

and theredore,

$$u(t \wedge \tau) = \sum_{k=0}^{\infty} \mathbb{1}_{k \le \|u_0\| < k+1} \mathcal{F}(t \wedge \tau, u) = \mathcal{F}(t \wedge \tau, u),$$

where

$$\begin{aligned}\mathcal{F}(t \wedge \tau, u) &= \Big[u_0 - \int_0^{t\wedge\tau} (\nu A u + B(u) - f)dt + \int_0^{t\wedge\tau} g(u)dW\Big] \\ &= u(0) - \int_0^{t\wedge\tau} \Big(\nu A u + B(u) - f\Big)dt + \int_0^{t\wedge\tau} g(u)dW, \qquad (4.33)\end{aligned}$$

where all equalities are in H. The proof is now complete for the general case. □

Remark 4.1. For the sake of the non-probabilistic reader we recall the following facts used in Proposition 4.1.

(i) Weak convergences are preserved under continuous linear transformations. In the present context, it is sufficient to observe that if T is a bounded linear mapping between two Hilbert spaces H_1 and H_2 then, assuming that $x_n \rightharpoonup x$ (weakly) in H_1, one can infer that $Tx_n \rightharpoonup Tx$ (weakly) in H_2.

(ii) Suppose that X is a Hilbert space and

$$x(t, \omega) \in C([0, T], X), \ \ a.s.\ \omega, \qquad (4.34)$$

and $y(t, \omega)$ and $z(t, \omega)$ are given so that

$$\int_0^t y(s, \omega)ds + \int_0^t z(s, \omega)dW(s, \omega) \in C([0, T], X), \quad a.s.\ \omega. \qquad (4.35)$$

If we show that for every $v \in X$ and every $K \subset \Omega \times [0, T]$, measurable, that[3]

$$\mathbb{E}\int_0^T \chi_K(x(t), v)dt = \mathbb{E}\int_0^T \chi_K\Big(\int_0^t (y(s), v)ds + \int_0^t (z(s), v)dW\Big)dt,$$

then we can infer that

$$x(t, \omega) = \int_0^t y(s, \omega)ds + \int_0^t z(s, \omega)dW(s, \omega), \qquad (4.36)$$

for almost every (t, ω), with the equality making sense in X. It follows, for a dense, countable set $R \subseteq [0, T]$, that there exists $\tilde{\Omega}$ which is of full measure so that for all $t \in R, \omega \in \tilde{\Omega}$

$$x(t, \omega) = \int_0^t y(s, \omega)ds + \int_0^t z(s, \omega)dW(s, \omega). \qquad (4.37)$$

[3] compare with (4.29)

However, since the functions on both side of the equation above are continuous (in time) we may infer that the equality above holds over all $\omega \in \tilde{\Omega}$ and *every* $t \in [0,T]$. In the present context $X = H$, $x(t) = u(t \wedge \tau)$, $y(t) = 1\!\!1_{t\leq\tau} Au(t) + B(u(t))$ and $z(t) = g(u(t))$. Due to (4.17) we have (4.34). As a consequence of (4.20), (4.25) and (4.27) we may infer (4.35).

4.2. **Maximal existence time and blow-up.** In this section we establish the existence of a maximal strong solution (u, ξ). In addition, we show that (u, ξ) is a weak solution even up to the blow up time ξ. The first step is to show that if a strong solution exists up to time τ one can (uniquely) extend this solution up to some stopping time $\sigma > \tau$. This is captured in the following lemma.

Lemma 4.1. *Assume that (u, τ) is a local strong solution of the stochastic Navier-Stokes equations (cf. Definition 2.3) and that τ is finite almost surely. Then, there exists a local strong solution (u_e, σ) such that $\sigma > \tau$ almost surely and such that $\mathbb{P}(u_e(t \wedge \tau) = u(t \wedge \tau); \quad t \in [0, \infty)) = 1$.*

Proof. Define the stochastic basis

$$\begin{aligned} \tilde{\mathcal{S}} :&= (\tilde{\Omega}, \tilde{\mathcal{F}}, \{\tilde{\mathcal{F}}_t\}_{t\geq 0}, \tilde{\mathbb{P}}, \{\tilde{\beta}_k(t)\}_{k\geq 1}) \\ &= (\Omega, \mathcal{F}, \{\mathcal{F}_{t+\tau}\}_{t\geq 0}, \mathbb{P}, \{\beta^k(t+\tau) - \beta(\tau)\}_{k\geq 1}), \end{aligned} \tag{4.38}$$

and let $\tilde{u}_0 = u(\tau), \tilde{f}(t) = f(t+\tau), \tilde{g}(\cdot, t) = g(\cdot, t+\tau)$. Observe that these data satisfy (4.15) for the basis $\tilde{S}$. As such, according to Proposition 4.2, there exists a local strong solution $(\tilde{u}, \tilde{\tau})$ relative to $\tilde{S}$. Define $\sigma = \tau + \tilde{\tau}$ and $u_e(t) = u(t)$ for $t \leq \tau$ and $\tilde{u}(t - \tau)$ for $t > \tau$. It may be checked directly that $(u_e, \tau + \tilde{\tau}) = (u, \sigma)$ is a local strong solution relative to the original stochastic l basis $\mathcal{S}$. □

The next lemma establishes some further estimates on weak solutions (u, τ) in terms of the data f, g and u_0 that do not depend on τ. Note that this lemma will also be employed in the next section for the proof of the global existence in $d = 2$.

Lemma 4.2. *Suppose that in addition to the assumptions of Proposition 4.2 we have, for some $p \geq 2$ and $u_0 \in L^p(\Omega; H)$, $f \in L^p(\Omega; L^2_{loc}([0,\infty); V'))$. If (u, τ) is a local weak solution in the sense of Definition 2.3, then for any $T > 0$*

$$\mathbb{E}\Big[\sup_{t\in[0,\tau\wedge T]} |u|^p + \int_0^{\tau\wedge T} \|u\|^2 |u|^{p-2} dt\Big] \leq C\mathbb{E}|u_0|^p + C\mathbb{E}\Big(\int_0^T |f|^2_{V'} ds\Big)^{p/2}, \tag{4.39}$$

where the constant $C := C_{\nu, K_H, p, T}$ does not depend on τ.

Proof. By (2.38), for any pair of stopping times $0 \le \sigma_a \le \sigma_b \le \tau \wedge T$,

$$\begin{aligned}
&\mathbb{E}\Big(\sup_{s\in[\sigma_a,\sigma_b]} |u|^p + p\nu \int_{\sigma_a}^{\sigma_b} \|u\|^2 |u|^{p-2} dt \Big) \\
\le & C_p \mathbb{E}\Big(|u(\sigma_a)|^p + \int_{\sigma_a}^{\sigma_b} |\langle f, u\rangle| |u|^{p-2} dt + \int_{\sigma_a}^{\sigma_b} \sum_{k=1}^{\infty} |g_k(u)|^2 |u|^{p-2} dt \Big) \\
& + C_p \mathbb{E}\Big(\sup_{s\in[\sigma_a,\sigma_b]} \Big| \sum_{k=1}^{\infty} \int_{\sigma_a}^{s} \langle g_k(u), u\rangle |u|^{p-2} d\beta_k \Big| \Big).
\end{aligned} \tag{4.40}$$

The first term on the right-hand side of (4.40) is estimated by

$$\begin{aligned}
& C_p \int_{\sigma_a}^{\sigma_b} |\langle f, u\rangle| \; |u|^{p-2} dt \le C_{\nu,p} \int_{\sigma_a}^{\sigma_b} |f|_{V'}^2 |u|^{p-2} \, dt + \frac{\nu p}{2} \int_{\sigma_a}^{\sigma_b} \|u\|^2 |u|^{p-2} \, dt \\
& \le C_{\nu,p} \Big(\sup_{t\in[\sigma_a,\sigma_b]} |u|^{p-2} \Big) \int_{\sigma_a}^{\sigma_b} |f|_{V'}^2 dt + \frac{\nu p}{2} \int_{\sigma_a}^{\sigma_b} \|u\|^2 |u|^{p-2} \, dt \\
& \le \frac{1}{6} \sup_{t\in[\sigma_a,\sigma_b]} |u|^p + \frac{p\nu}{2} \int_{\sigma_a}^{\sigma_b} \|u\|^2 |u|^{p-2} \, dt + C_{\nu,p} \Big(\int_{\sigma_a}^{\sigma_b} |f|_{V'}^2 ds \Big)^{p/2}.
\end{aligned} \tag{4.41}$$

Next, using the Lipschitz assumption for g we have

$$\begin{aligned}
\sum_{k=1}^{\infty} \int_{\sigma_a}^{\sigma_b} |g_k(u)|^2 |u|^{p-2} dt \le & C \int_{\sigma_a}^{\sigma_b} (1 + |u|^2) |u|^{p-2} dt \\
\le & \tfrac{1}{6} \sup_{t\in[0,T]} |u|^p + C \Big(1 + \int_{\sigma_a}^{\sigma_b} |u|^p \, dt \Big).
\end{aligned} \tag{4.42}$$

Finally,

$$\begin{aligned}
& C_p \mathbb{E}\Big(\sup_{s\in[\sigma_a,\sigma_b]} \Big| \sum_{k=1}^{\infty} \int_0^s \langle g_k(u), u\rangle |u|^{p-2} d\beta_k \Big| \Big) \\
& \qquad \le C_p \mathbb{E}\Big(\int_{\sigma_a}^{\sigma_b} \sum_{k=1}^{\infty} \langle g_k(u), u\rangle^2 |u|^{2(p-2)} dt \Big)^{1/2} \\
& \qquad \le C_{K_H,p} \mathbb{E}\Big(\int_{\sigma_a}^{\sigma_b} (1 + |u|^2) |u|^{2(p-1)} dt \Big)^{1/2} \\
& \qquad \le \frac{1}{6} \mathbb{E}\Big(\sup_{t\in[\sigma_a,\sigma_b]} |u|^p \Big) + C_{p,T,K_H} \mathbb{E}\Big(1 + \int_{\sigma_a}^{\sigma_b} |u|^p dt \Big).
\end{aligned} \tag{4.43}$$

From estimates (4.41), (4.42) and (4.43) applied to (4.40) we infer

$$\begin{aligned}&\mathbb{E}\Big[\sup_{t\in[\sigma_a,\sigma_b]}|u|^p+\int_{\sigma_a}^{\sigma_b}\|u\|^2|u|^{p-2}dt\Big]\\&\leq C\mathbb{E}\Big(1+|u(\sigma_a)|^p+\int_{\sigma_a}^{\sigma_b}|u|^p\,dt+\Big(\int_{\sigma_a}^{\sigma_b}|f|_{V'}^2ds\Big)^{p/2}\Big).\end{aligned}\tag{4.44}$$

We may now employ Lemma 5.3 to conclude (4.39). □

We now have everything we need to establish the existence of a maximal solution and a sequence of stopping times announcing any finite time blow-up. The argument is adapted from Jacod [18] and Mikulevicius and Rozovskii [23].

Theorem 4.1. *Given the assumptions in Proposition* 4.2, *there exists a unique maximal solution* (u,ξ) *and a sequence* ρ_n *announcing* ξ. *In addition, the pair* (u,ξ) *is a weak solution.*

Proof. Consider the set $\mathcal{L}$ of all stopping times such that $\tau\in\mathcal{L}$ if and only if there exists a process u such that (u,τ) is a local strong solution. Notice that

$$\sigma_1,\sigma_2\in\mathcal{L}\Rightarrow\sigma_1\vee\sigma_2\in\mathcal{L},\tag{4.45}$$

and that

$$\sigma\in\mathcal{L}\Rightarrow\rho\wedge\sigma\in\mathcal{L},\tag{4.46}$$

for any stopping time ρ. Let $\xi=\sup\mathcal{L}$ (see [13], Chapter 5, Section 18). Using (4.45) we can choose an increasing sequence $\sigma_k\in\mathcal{L}$ such that σ_k converges to ξ almost surely. For each σ_k denote by u_k the process such that (u_k,σ_k) is a local strong solution. Let

$$\Omega_{k,k'}=\{u_k(t\wedge\sigma_k\wedge\sigma_{k'})=u_{k'}(t\wedge\sigma_k\wedge\sigma_{k'});t\in[0,\infty)\}.\tag{4.47}$$

By (4.46) and uniqueness (c.f. Proposition 4.1) we have that $\tilde{\Omega}=\bigcap_{k,k'}\Omega_{k,k'}$ is a set of full measure. For fixed ω on this set and every $t>0$ the sequence $\{u_k(t\wedge\sigma_k)\mathbb{1}_{t<\xi}\}$ is Cauchy in V. Let $\tilde{u}(t)=\lim_{k\to\infty}u_k(t\wedge\sigma_k)\mathbb{1}_{t<\xi}$ almost surely. By Lemma 4.2 and the monotone convergence theorem, for any $T>0$

$$\mathbb{E}\Big[\sup_{t\in[0,\xi\wedge T]}|\tilde{u}|^2+\int_0^{\xi\wedge T}\|\tilde{u}\|^2dt\Big]<\infty.\tag{4.48}$$

We are therefore justified in defining

$$\langle u(t),v\rangle=\langle u(0),v\rangle-\int_0^{t\wedge\xi}\langle\nu A\tilde{u}+B(\tilde{u})-f,v\rangle\,dt+\sum_{k=1}^{\infty}\int_0^{t\wedge\xi}\langle g_k(\tilde{u}),v\rangle d\beta_k,\tag{4.49}$$

for any $t > 0$, $v \in V$. It is direct to check that for $t < \xi(\omega)$, $u(t, \omega) = \tilde{u}(t, \omega)$ and that u is weakly continuous (almost surely) in H. These observations, with (4.48) and (4.49), imply that (u, ξ) is a local weak solution.

For $R > 0$ define the stopping time

$$\rho_R = \inf_{t \geq 0} \left\{ \sup_{s \in [0,t]} \|u\|^2 + \int_0^t |Au|^2 ds > R \right\} \wedge \xi. \tag{4.50}$$

Clearly, (u, ρ_R) is a local strong solution for any $R > 0$. Suppose, toward a contradiction, that for some R, T sufficiently large, $\mathbb{P}(\rho_R \wedge T = \xi) > 0$. By Lemma 4.1 this would imply the existence of an element $\zeta > \rho_R \wedge T$ almost surely with $\zeta \in \mathcal{L}$. But since ξ is the supremal element of $\mathcal{L}$, we have our desired contradiction. We see moreover that $\{\rho_R\}_{R \geq 0}$ announces ξ. □

4.3. **Global existence in dimension two.** In this section, we prove that if $d = 2$ the maximal solution found in Proposition 4.1 is global.

Theorem 4.2. *Suppose that* $d = 2$ *and that, in addition to the assumptions of Proposition* 4.2, *we have* $u_0 \in L^p(\Omega; H)$ *and* $f \in L^p(\Omega; L^2_{loc}([0, \infty); V'))$, *for some* $p \geq 4$. *Then the maximal solution* (u, ξ) *is global in the sense that* $\xi = \infty$ *almost surely.*

Proof. Let ρ_n be an increasing sequence of stopping times announcing ξ. Observe that

$$\{\xi < \infty\} = \bigcup_{T=1}^{\infty} \{\xi \leq T\} = \bigcup_{T=1}^{\infty} \bigcap_{n=1}^{\infty} \{\rho_n \leq T\}. \tag{4.51}$$

Using the fact that ρ_n is increasing we have

$$\mathbb{P}\Big(\bigcap_{n=1}^{\infty} \{\rho_n \leq T\} \Big) = \lim_{N \to \infty} \mathbb{P}\Big(\bigcap_{n=1}^{N} \{\rho_n \leq T\} \Big) = \lim_{N \to \infty} \mathbb{P}\Big(\rho_N \leq T \Big). \tag{4.52}$$

Thus, to establish the desired result, it is sufficient to show that for any fixed $T < \infty$

$$\mathbb{P}(\rho_N \leq T) \to 0 \text{ as } N \to \infty. \tag{4.53}$$

For $M > 0$, define the stopping time

$$\gamma_M := \inf_{t \geq 0} \left\{ \int_0^{t \wedge \xi} \|u\|^2 |u|^2 ds > M \right\} \wedge 2T. \tag{4.54}$$

We have $\mathbb{P}(\rho_N \leq T) \leq \mathbb{P}(\rho_N \leq T \cap \{\gamma_M > T\}) + \mathbb{P}(\gamma_M \leq T)$. The first term on the right-hand side is bounded by

$$\mathbb{P}\Big(\Big\{ \sup_{t \in [0, \rho_N \wedge T]} \|u\|^2 + \int_0^{\rho_N \wedge T} |Au|^2 ds \geq N \Big\} \cap \{\gamma_M > T\} \Big) \tag{4.55}$$

$$\leq \mathbb{P}\Big(\sup_{t \in [0, \rho_N \wedge \gamma_M]} \|u\|^2 + \int_0^{\rho_N \wedge \gamma_M} |Au|^2 ds \geq N \Big).$$

Fix T, M, N and a pair of stopping times $\tau_a \leq \tau_b \leq \rho_N \wedge \gamma_M$. Integrating (2.39) for the case $p = 2$, and performing some standard manipulations using the weighted Young inequality, one finds

$$\begin{aligned} \mathbb{E}\Big(\sup_{t \in [\tau_a, \tau_b]} \|u\|^2 + \nu \int_{\tau_a}^{\tau_b} |Au|^2 ds \Big) \\ \leq C\mathbb{E}\Big(\|u(\tau_a)\|^2 + \int_{\tau_a}^{\tau_b} (|f|^2 + |\langle B(u), Au\rangle| + \|u\|^2) ds \Big) \\ + \mathbb{E}\Big(\sup_{t \in [\tau_a, \tau_b]} \Big| 2 \sum_{k=1}^{\infty} \int_{\tau_a}^{t} ((g_k(u), u)) d\beta_k \Big| \Big). \end{aligned} \tag{4.56}$$

By making use of (2.16) for $d = 2$

$$|\langle B(u), Au\rangle| \leq |u|^{1/2} \|u\| |Au|^{3/2} \leq C_\nu |u|^2 \|u\|^4 + \frac{\nu}{2} |Au|^2. \tag{4.57}$$

For the last term in (4.56), the Burkholder-Davis-Gundy inequality implies

$$\begin{aligned} \mathbb{E}\Big(\sup_{t \in [\tau_a, \tau_b]} \Big| 2 \sum_{k=1}^{\infty} \int_{\tau_a}^{t} ((g_k(u), u)) d\beta_k \Big| \Big) \leq C\mathbb{E}\Big(\sum_{k=1}^{\infty} \int_{\tau_a}^{\tau_b} ((g_k(u), u))^2 dt \Big)^{1/2} \\ \leq C_{K_V} \mathbb{E}\Big(\int_{\tau_a}^{\tau_b} (1 + \|u\|^2) \|u\|^2 dt \Big)^{1/2} \\ \leq \mathbb{E}\Big(\frac{1}{2} \sup_{t \in [\tau_a, \tau_b]} \|u\|^2 + C_{K_V, \nu} \int_{\tau_a}^{\tau_b} (1 + \|u\|^2) dt \Big). \end{aligned} \tag{4.58}$$

Applying (4.57) and (4.58) to (4.56) and rearranging implies

$$\begin{aligned} \mathbb{E}\Big(\sup_{t \in [\tau_a, \tau_b]} \|u(t)\|^2 + \nu \int_{\tau_a}^{\tau_b} |Au|^2 ds \Big) \\ \leq C\mathbb{E}\Big(\|u(\tau_a)\|^2 + \int_{\tau_a}^{\tau_b} \Big((|u|^2 \|u\|^2 + 1) \|u\|^2 + |f|^2 \Big) ds \Big), \end{aligned} \tag{4.59}$$

where $C = C_{\nu, K_V}$. The expression on the right-hand side of (4.59) can be bounded independently of τ_a, τ_b. Also, by definition

$$\int_0^{\gamma_M} |u|^2 \|u\|^2 ds \leq M, \quad \text{a.s.} \tag{4.60}$$

Lemma 5.3 then implies

$$\mathbb{E}\Big(\sup_{t\in[0,\rho_N\wedge\gamma_M]}\|u(t)\|^2+\nu\int_0^{\rho_N\wedge\gamma_M}|Au|^2ds\Big) \leq C_{\nu,K_v,M,T}\mathbb{E}\Big(\|u_0\|^2+\int_0^{2T}(|f|^2+1)ds\Big). \tag{4.61}$$

Note that $C_{\nu,K_v,M,T}$ does not depend on N. Using (4.55) with (4.61) we infer

$$\mathbb{P}(\rho_N<T)\leq\frac{C}{N}\mathbb{E}\Big(\|u_0\|^2+\int_0^{2T}(|f|^2+1)ds\Big)+\mathbb{P}(\gamma_M\leq T). \tag{4.62}$$

Thus, for any fixed M

$$\lim_{N\to\infty}\mathbb{P}(\rho_N<T)\leq\mathbb{P}(\gamma_M\leq T). \tag{4.63}$$

Finally, by applying Lemma 4.2 we find that

$$\mathbb{P}(\gamma_M\leq T)\leq\mathbb{P}\Big(\int_0^{T\wedge\xi}\|u\|^2|u|^2dt\geq M\Big)\leq\frac{1}{M}\mathbb{E}\Big(\int_0^{T\wedge\xi}\|u\|^2|u|^2dt\Big),$$

which goes to zero as $M\to\infty$, and (4.53) follows. □

5. ABSTRACT RESULTS

In this section we formulate and prove a collection of abstract lemmas which are employed above to circumvent the key difficulties related to compactness for the Galerkin scheme. As such we believe that these results could prove useful for the study of local well posedness for other non-linear stochastic partial differential equations.

5.1. **A pairwise comparison theorem.** We have made use in Proposition 4.2 of the following abstract comparison lemma. The formulation and proof extends Lemma 20 in [23]. Let $(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\geq},\mathbb{P})$ be a fixed, filtered probability space. Suppose that B_1 and B_2 are Banach spaces with $B_2\subset B_1$ with continuous embedding. We denote the associated norms by $|\cdot|_i$. Define $\mathcal{E}(T):=C([0,T];B_1)\cap L^2([0,T];B_2)$ with the norm

$$|Y|_{\mathcal{E}(T)}=\Big(\sup_{t\in[0,T]}|Y(t)|_1^2+\int_0^T|Y(t)|_2^2dt\Big)^{1/2}. \tag{5.1}$$

Let X_n be a sequence of B_2-valued stochastic processes so that for every $T>0$ $X_n\in\mathcal{E}(T)$ almost surely. For $M>1,T>0$ define the collection of stopping times

$$\mathcal{T}_n^{M,T}:=\left\{\tau\leq T;|X_n|_{\mathcal{E}(\tau)}\leq M+|X_n(0)|_1\right\}, \tag{5.2}$$

and let $\mathcal{T}_{n,m}^{M,T} := \mathcal{T}_n^{M,T} \cap \mathcal{T}_m^{M,T}$.

Lemma 5.1. *(i) Suppose that, for some $M > 1$ and T, we have*[4]

$$\lim_{n\to\infty} \sup_{m\geq n} \sup_{\tau\in\mathcal{T}_{m,n}^{M,T}} \mathbb{E}|X_n - X_m|_{\mathcal{E}(\tau)} = 0 \tag{5.3}$$

$$\lim_{S\to 0} \sup_{n} \sup_{\tau\in\mathcal{T}_{n}^{M,T}} \mathbb{P}\left[|X_n|_{\mathcal{E}(\tau\wedge S)} > |X_n(0)|_1 + M - 1\right] = 0. \tag{5.4}$$

Then, there exists a stopping time τ with

$$\mathbb{P}(0 < \tau \leq T) = 1, \tag{5.5}$$

and a process $X(\cdot) = X(\cdot \wedge \tau) \in \mathcal{E}(\tau)$, such that

$$|X_{n_l} - X|_{\mathcal{E}(\tau)} \to 0, \quad a.s. \tag{5.6}$$

for some subsequence $n_l \uparrow \infty$. Moreover,

$$|X|_{\mathcal{E}(\tau)} \leq M + \sup_n |X_n(0)|_1, \quad a.s. \tag{5.7}$$

(ii) If, in addition to the conditions imposed in (i)

$$\sup_n \mathbb{E}|X_n(0)|_1^p < \infty, \tag{5.8}$$

for some $1 \leq p < \infty$, then there exists a sequence of sets $\Omega_l \uparrow \Omega$ such that

$$\sup_l \mathbb{E}1\!\!1_{\Omega_l}|X_{n_l}|^p_{\mathcal{E}(\tau)} < \infty, \tag{5.9}$$

and

$$\mathbb{E}|X|^p_{\mathcal{E}(\tau)} \leq C_q\Big(M^p + \sup_n \mathbb{E}|X_n(0)|_1^p\Big). \tag{5.10}$$

Proof. Our first step will be to identify the convergent subsequence. We proceed by induction on l and start with $l = 0$ and $n_0 = 1$. When n_l is known, we easily find $n_{l+1} > n_l$, thanks to (5.3), satisfying

$$\sup_{\tau\in\mathcal{T}_{n_{l+1},n_l}^{M,T}} \mathbb{E}|X_{n_l} - X_{n_{l+1}}|_{\mathcal{E}(\tau)} \leq 2^{-2l}. \tag{5.11}$$

Next, to find τ, as needed for (5.5) and (5.6), we define

$$\tau_l := \inf_{t>0}\left\{|X_{n_l}|_{\mathcal{E}(t)} > |X_{n_l}(0)|_1 + (M - 1 + 2^{-l})\right\} \wedge T, \tag{5.12}$$

and let

$$\Omega_N = \bigcap_{j=N}^{\infty}\left\{|X_{n_j} - X_{n_{j+1}}|_{\mathcal{E}(\tau_j\wedge\tau_{j+1})} < 2^{-(j+2)}\right\}. \tag{5.13}$$

[4]Compare to (3.6) and (3.7)

Noting that $\tau_l \wedge \tau_{l+1} \in \mathcal{T}^{M,T}_{n_{l+1},n_l}$, we have

$$\mathbb{P}\Big(|X_{n_l} - X_{n_{l+1}}|_{\mathcal{E}(\tau_l\wedge\tau_{l+1})} \geq 2^{-(l+2)}\Big) \leq 2^{l+2}\mathbb{E}|X_{n_l} - X_{n_{l+1}}|_{\mathcal{E}(\tau_l\wedge\tau_{l+1})} \leq 2^{-(l-2)}.$$

By the Borel-Cantelli lemma we infer that

$$\mathbb{P}\Big(\bigcap_{N=1}^{\infty}\bigcup_{j=N}^{\infty}\Big\{|X_{n_j} - X_{n_{j+1}}|_{\mathcal{E}(\tau_j\wedge\tau_{j+1})} \geq 2^{-(j+2)}\Big\}\Big) = 0, \tag{5.14}$$

and therefore that $\tilde{\Omega} := \cup_N \Omega_N$ is a set of full measure.

To show that $\{\tau_l\}$ converges, we establish that

$$\tau_{l+1}(\omega) \leq \tau_l(\omega) \quad \text{for every } \; l \geq N, \omega \in \Omega_N. \tag{5.15}$$

Indeed, given N and $l \geq N$ consider the set $\{\tau_{l+1} > \tau_l\} \cap \Omega_N$. On this set we have, in particular, that $\tau_l < T$. With the continuity of $|X_{n_l}|_{\mathcal{E}(t)}$ in t, this implies (cf. (5.12))

$$|X_{n_l}|_{\mathcal{E}(\tau_l)} = |X_{n_l}(0)|_1 + (M - 1 + 2^{-l}). \tag{5.16}$$

Also, on Ω_N it is clear that

$$\begin{aligned} |X_{n_l}|_{\mathcal{E}(\tau_l\wedge\tau_{l+1})} - |X_{n_{l+1}}|_{\mathcal{E}(\tau_l\wedge\tau_{l+1})} &< 2^{-(l+2)} \\ |X_{n_{l+1}}(0)| - |X_{n_l}(0)| &< 2^{-(l+2)}. \end{aligned} \tag{5.17}$$

Combining these observations, we infer

$$\begin{aligned} |X_{n_{l+1}}|_{\mathcal{E}(\tau_l\wedge\tau_{l+1})} &> |X_{n_l}|_{\mathcal{E}(\tau_l\wedge\tau_{l+1})} - 2^{-(l+2)} = |X_{n_l}|_{\mathcal{E}(\tau_l)} - 2^{-(l+2)} \\ &= |X_{n_l}(0)|_1 + (M - 1 + 2^{-l}) - 2^{-(l+2)} \\ &> |X_{n_{l+1}}(0)|_1 + (M - 1 + 2^{-l}) - 2 \cdot 2^{-(l+2)} \\ &= |X_{n_{l+1}}(0)|_1 + (M - 1 + 2^{-(l+1)}), \end{aligned} \tag{5.18}$$

over $\{\tau_{l+1} > \tau_l\} \cap \Omega_N$. On the other hand on Ω_N

$$|X_{n_{l+1}}|_{\mathcal{E}(\tau_l\wedge\tau_{l+1})} \leq |X_{n_{l+1}}|_{\mathcal{E}(\tau_{l+1})} \leq |X_{n_{l+1}}(0)| + (M - 1 + 2^{-(l+1)}). \tag{5.19}$$

Together (5.18) and (5.19) show that $\{\tau_{l+1} > \tau_l\} \cap \Omega_N$ must be empty. Combining (5.15) and (5.14) justifies taking

$$\tau = \lim_l \tau_l \quad a.s. \tag{5.20}$$

To show that $\tau > 0$ almost surely, we fix $\epsilon > 0$ with $T > \epsilon > 0$. We have

$$\begin{aligned} \{\tau_l < \epsilon\} &\subset \{|X_{n_l}|_{\mathcal{E}(\tau_l\wedge\epsilon)} = |X_{n_l}(0)|_1 + (M - 1 + 2^{-l})\} \\ &\subset \{|X_{n_l}|_{\mathcal{E}(\tau_l\wedge\epsilon)} > |X_{n_l}(0)|_1 + (M - 1)\}. \end{aligned} \tag{5.21}$$

Since

$$\{\tau < \epsilon\} = \bigcup_{l=1}^{\infty}\bigcap_{k=l}^{\infty}\{\tau_k < \epsilon\}, \tag{5.22}$$

we have that

$$\begin{aligned}\mathbb{P}(\tau < \epsilon) = \mathbb{P}\Big(\bigcup_{l=1}^{\infty}\bigcap_{k=l}^{\infty}\{\tau_k < \epsilon\}\Big) &\leq \limsup_{l} \mathbb{P}(\tau_l < \epsilon) \\ &\leq \sup_{l} \mathbb{P}(|X_{n_l}|_{\mathcal{E}(\tau_l \wedge \epsilon)} > |X_{n_l}(0)|_1 + (M-1)).\end{aligned} \tag{5.23}$$

Making use of the condition imposed in (5.4) we see that

$$\mathbb{P}(\tau = 0) = \mathbb{P}(\cap_{\epsilon > 0}\{\tau < \epsilon\}) = \lim_{\epsilon \downarrow 0} \mathbb{P}(\tau < \epsilon) = 0. \tag{5.24}$$

By construction, $\tau \leq T$, so 5.5 now follows.

We may now prove that X_{n_l} is Cauchy in $\mathcal{E}(\tau)$ almost surely. Notice that, due to (5.15), for every $\omega \in \tilde{\Omega}$, one can choose $N = N(\omega)$ so that $\omega \in \Omega_N$ and $\tau(\omega) \leq \tau_{l+1}(\omega) \leq \tau_l(\omega)$ whenever $l \geq N$. As such

$$|X_{n_l}(\omega) - X_{n_{l+1}}(\omega)|_{\mathcal{E}(\tau(\omega))} \leq |X_{n_l}(\omega) - X_{n_{l+1}}(\omega)|_{\mathcal{E}(\tau_l \wedge \tau_{l+1}(\omega))} < 2^{-(l+2)}, \tag{5.25}$$

where the final inequality is due to (5.13). Therefore, there exists a process $X(\cdot) = X(\cdot \wedge \tau) \in \mathcal{E}(\tau)$ such that $\lim_{l \to \infty} |X_{n_l} - X|_{\mathcal{E}(\tau)} = 0$ almost surely.

To establish (5.7), (5.9) and (5.10) take Ω_l as in (5.13). Again using the fact that $\tau_l \geq \tau_{l+1} \geq \tau$ on Ω_l,

$$\begin{aligned}\mathbb{1}_{\Omega_l}|X_{n_l}|_{\mathcal{E}(\tau)} &\leq 2^{-(l+2)} + \mathbb{1}_{\Omega_l}|X_{n_{l+1}}|_{\mathcal{E}(\tau)} \\ &\leq |X_{n_{l+1}}(0)|_1 + M \leq \sup_n |X_n(0)|_1 + M,\end{aligned} \tag{5.26}$$

which yields (5.7). The second inequality in (5.26) implies

$$\mathbb{E}\Big(\mathbb{1}_{\Omega_l}|X_{n_l}|^p_{\mathcal{E}(\tau)}\Big) \leq C_p(M^p + \mathbb{E}|X_{n_l}(0)|_1^p). \tag{5.27}$$

The bound (5.9) follows from (5.27) as a consequence of (5.8). By applying Fatou's lemma we infer (5.10). □

5.2. **Weak convergence lemmas.** We next establish a general result concerning weak convergence in Banach spaces that is used to uniquely identify certain limits that one infers from Lemma 5.1.

Lemma 5.2. *Suppose that X is a separable Banach space and let $D \subset X$ be a dense subset. Let X^* be the dual of X and denote the dual pairing between X and X^* by $\langle \cdot, \cdot \rangle$. Assume that $(E, \mathcal{E}, \mu)$ is a finite measure space and that*

$p \in (1,\infty)$. Assume that $u, u^n \in L^p(E, X^*)$ with $\{u^n\}$ uniformly bounded in $L^p(E, X^*)$ and

$$\langle u^n, y\rangle \to \langle u, y\rangle \quad \mu - a.e. \tag{5.28}$$

for all $y \in D$. Then

$$u^n \rightharpoonup^* u, \tag{5.29}$$

in $L^p(E, X^*)$.

Proof. Fix $y \in \mathcal{D}$ and define

$$E_N := \{\omega \in E : |\langle u^m(\omega) - u(\omega), y\rangle| \leq 1, \text{ for every } m \geq N\}. \tag{5.30}$$

Let χ_N be the indicator function associated to E_N. Note that as a consequence of (5.28)

$$1 - \chi_N \to 0 \text{ as } N \to \infty, \tag{5.31}$$

μ-almost surely.

Let $F \in \mathcal{E}$ and $N \geq 1$ be given. We observe that, due to the dominated convergence theorem,

$$\lim_{n\to\infty} \Big| \int_F \chi_N \langle u^n - u, y\rangle d\mu\Big| = 0. \tag{5.32}$$

As such

$$\begin{aligned}\limsup_{n\to\infty} \Big| \int_F \langle u^n - u, y\rangle d\mu\Big| &\leq \limsup_{n\to\infty} \Big| \int_F (1-\chi_N)\langle u^n - u, y\rangle d\mu\Big| \\ &\leq \limsup_{n\to\infty} \|y\|_X \Big(\int_F \|u^n - u\|^p_{X^*} d\mu\Big)^{1/p} \Big(\int_F |1-\chi_N|^{p'} d\mu\Big)^{1/p'} \\ &\leq C\Big(\int_F |1-\chi_N|^{p'} d\mu\Big)^{1/p'}.\end{aligned}$$

For the last bound above we use only the uniform bounds on u^n in $L^p(E, X^*)$ so that C can be chosen independently of N. Taking $N \to \infty$ and applying the dominated convergence theorem with (5.31), we finally conclude that

$$\lim_{n\to\infty} \Big| \int_F \langle u^n - u, y\rangle d\mu\Big| = 0. \tag{5.33}$$

Observe that (cf. [31], Chapter 5)

$$\mathcal{S} := \{s = \sum_{k=1}^{d} y_k : \chi_{F_k}, y_k \in \mathcal{D}, F_k \in \mathcal{E}, d < \infty\} \tag{5.34}$$

is a dense subset of $L^{p'}(E, X)$. As such, a simple density argument establishes (5.29) from (5.33). □

5.3. **A gronwall lemma for stochastic processes.** In this paper, we used the following lemma several times in an analogous manner to the classical Gronwall lemma.

Lemma 5.3. *Fix $T > 0$. Assume that $X, Y, Z, R : [0, T) \times \Omega \to \mathbb{R}$ are real-valued, non-negative stochastic processes. Let $\tau < T$ be a stopping time so that*

$$\mathbb{E} \int_0^\tau (RX + Z)\, ds < \infty. \tag{5.35}$$

Assume, moreover, that for some fixed constant κ,

$$\int_0^\tau R\, ds < \kappa, \quad a.s. \tag{5.36}$$

Suppose that for all stopping times $0 \le \tau_a < \tau_b \le \tau$

$$\mathbb{E}\Big(\sup_{t \in [\tau_a, \tau_b]} X + \int_{\tau_a}^{\tau_b} Y\, ds \Big) \le C_0 \mathbb{E}\Big(X(\tau_a) + \int_{\tau_a}^{\tau_b} (RX + Z)\, ds \Big), \tag{5.37}$$

where C_0 is a constant independent of the choice of τ_a, τ_b. Then

$$\mathbb{E}\Big(\sup_{t \in [0, \tau]} X + \int_0^\tau Y\, ds \Big) \le C \mathbb{E}\Big(X(0) + \int_0^\tau Z\, ds \Big), \tag{5.38}$$

where $C = C(C_0, T, \kappa)$.

Proof. Choose a sequence of stopping times $0 = \tau_0 < \tau_1 < \ldots < \tau_N < \tau_{N+1} = \tau$, so that

$$\int_{\tau_{k-1}}^{\tau_k} R\, ds < \frac{1}{2C_0} \quad a.s. \tag{5.39}$$

For each pair τ_{k-1}, τ_k take $\tau_a = \tau_{k-1}$ and $\tau_b = \tau_k$ in (5.37). By making use of (5.39) and rearranging we deduce

$$\mathbb{E}\Big(\sup_{t \in [\tau_{k-1}, \tau_k]} X + \int_{\tau_{k-1}}^{\tau_k} Y\, ds \Big) \le C\mathbb{E}X(\tau_{k-1}) + C\mathbb{E} \int_{\tau_{k-1}}^{\tau_k} Z\, ds. \tag{5.40}$$

Assuming, by induction on j, that

$$\mathbb{E}\Big(\sup_{t \in [0, \tau_j]} X + \int_0^{\tau_j} Y\, ds \Big) \le C\mathbb{E}X(0) + C\mathbb{E} \int_0^{\tau_j} Z\, ds, \tag{5.41}$$

then

$$\begin{aligned} &\mathbb{E}\Big(\sup_{t \in [0, \tau_{j+1}]} X + \int_0^{\tau_{j+1}} Y\, ds \Big) \\ &\le C\mathbb{E}X(0) + C\mathbb{E} \int_0^{\tau_j} Z\, ds + C\mathbb{E}\Big(\sup_{t \in [\tau_j, \tau_{j+1}]} X + \int_{\tau_j}^{\tau_{j+1}} Y\, ds \Big) \end{aligned} \tag{5.42}$$

$$\leq C\mathbb{E}X(0) + C\mathbb{E}\int_0^{\tau_{j+1}} Z\,ds + C\mathbb{E}X(\tau_j) \leq C\mathbb{E}X(0) + C\mathbb{E}\int_0^{\tau_{j+1}} Z\,ds.$$

Hence (5.38) follows. □

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