

## SINGULAR PERTURBATION SYSTEMS WITH STOCHASTIC FORCING AND THE RENORMALIZATION GROUP METHOD

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**ABSTRACT.** This article examines a class of singular perturbation systems in the presence of a small white noise. Modifying the renormalization group procedure developed by Chen, Goldenfeld and Oono [6], we derive an associated reduced system which we use to construct an approximate solution that separates scales. Rigorous results demonstrating that these approximate solutions remain valid with high probability on large time scales are established. As a special case we infer new small noise asymptotic results for a class of processes exhibiting a physically motivated cancellation property in the nonlinear term. These results are applied to some concrete perturbation systems arising in geophysical fluid dynamics and in the study of turbulence. For each system we exhibit the associated renormalization group equation which helps decouple the interactions between the different scales inherent in the original system.

**1. Introduction.** Perturbation theory has long played an important role in applied analysis. Perturbed systems can provide the relevant setting for the study of physical phenomena exhibiting multiple spatial and temporal scales. In the field of climatology, for example, the basic momentum equations naturally exhibit multiple orders of magnitude, with the largest order being driven by the Coriolis term arising from the earth's rotation.

A basic challenge in the study of perturbed dynamical systems is that the unperturbed problem may exhibit fundamentally different quantitative and qualitative behaviors, particularly for large time scales or near certain boundaries. Such difficulties were appreciated as early as the 18th century in the study of multi-body problems in celestial mechanics. Needless to say such “singular” systems are notoriously challenging to analyze.

A variety of asymptotic methods have been developed, each germane to different types of singular perturbation problems. These methods seek to find approximate solutions to perturbed systems that separate scales and remain valid over long time intervals. See the recent texts of Verhulst [21], [22] for a broad introductory survey

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for deterministic systems. For the stochastic setting see Freidlin and Wentzell [11] or Skorokhod, Hoppensteadt and Habib [18] for an overview.

One approach, the renormalization group method has enjoyed considerable success in recent investigations of singularly perturbed systems. The method was first developed by Chen, Goldenfeld and Oono in the context of perturbative quantum field theory. See [6] or [7] and references therein. Subsequently the work of Ziane [23] and later the work of DeVille, Harkin, Holzer, Josić, and Kaper [9] put the subject on a firm mathematical foundation. Using this technique, Moise, Simonnet, Temam, and Ziane [15] conducted a series of Numerical simulations of ordinary differential equations arising in geophysical fluid dynamics. In the context of partial differential equations the method has been applied to Navier-Stokes type systems by Moise and Ziane [17] and Moise and Temam [16]. The two-dimensional Navier-Stokes equations perturbed by a small additive white noise was addressed using this method by Blömker, Gugg, and Maier-Paape [4]. We should also mention a related work to ours which concerns the stochastic normal form by Arnold and Imkeller [1], Arnold, Namachchivay, and Shenk-Hoppé [2] and Namachchiavayv and Lin [19], see also the book by Arnold [3] and the references therein. These authors seek coordinate transformations that decouple slow and fast modes, and the emphasis is on computing Lyapunov exponents as opposed to pathwise estimates considered in our article.

In any physical system uncertainties (measurement error, unresolved scales or interactions, numerical inaccuracies, etc.) arise that are hard to account for in the basic model. One would like to be able to quantify the robustness of a model, particularly one involving singular perturbations, in the presence of these uncertainties. As such, it is natural to incorporate small white noise driven perturbations into existing singular models. As far as we are aware [4] is the only investigation to apply Renormalization group techniques to a stochastically driven system. In particular the crucial case of highly oscillatory systems remains unaddressed up to the present article.

In this work we investigate a class of stochastic equations taking the form

$$dX^\epsilon + \frac{1}{\epsilon}AX^\epsilon d\tau = F(X^\epsilon)d\tau + \epsilon^m G(\tau, \tau/\epsilon)dW, \quad (1)$$

where  $A$  is a linear operator which is assumed to be either symmetric positive semidefinite or antisymmetric.  $F$  is a nonlinear operator, but may exhibit important cancellation properties that arise physically.  $dW$  is a white noise process in the appropriate sense. The parameter  $m$  is a real number measuring the strength of the noise term. In most cases we restrict  $m > 0$ . In the case of strictly dissipative systems we are also able to address the case of a noise term of moderate strength,  $-1/2 < m \leq 0$ .

Following the classical techniques we attempt to find an approximate solution via a naive perturbation expansion of the solution, setting  $u^\epsilon \approx u^{(0)} + \epsilon u^{(1)}$ . This can (and usually does) break down due to resonances between the nonlinear term  $F$  and the semi-group generated by  $A$ . Additionally we face the new difficulty of accounting for intermediate scale diffusion introduced by the small noise term. To compensate for this we derive the renormalized system

$$dV^\epsilon = R(V^\epsilon)d\tau + \epsilon^m H(\tau, \tau/\epsilon)dW. \quad (2)$$

The solution  $V^\epsilon$  of (2) defines an approximate solution  $\bar{X}^\epsilon = e^{-\tau/\epsilon A} V^\epsilon$ . Indeed, we will prove that if the behavior of  $V^\epsilon$  is reasonable, namely that if for some  $K > 0$

$$\mathbb{P} \left( \sup_{\tau \in [0, T]} \|V^\epsilon(\tau)\| > K \right) \xrightarrow{\epsilon \rightarrow 0} 0, \quad (3)$$

then  $\bar{X}^\epsilon$  is a valid approximation in the sense that

$$\mathbb{P} \left( \sup_{\tau \in [0, T]} \|\bar{X}^\epsilon(\tau) - X^\epsilon(\tau)\| > C\epsilon^\gamma \right) \xrightarrow{\epsilon \rightarrow 0} 0. \quad (4)$$

Here,  $\gamma > 0$ , and depends on  $m$  as well as the structure of  $A$  and  $H$ . Note that beyond establishing convergence in probability, typical for such results, we have also managed to establish a rate of convergence. The system (2) greatly simplifies the original equations. The structure of  $H$  depends on  $A$  and the desired rate of convergence for the approximate solution. In particular, we show that we can take  $H$  to be identically zero at the cost of a reduced rate of convergence. Even for cases where  $H$  is non-zero, sending  $\epsilon$  to zero reduces to a more tractable small noise asymptotic problem. One interesting consequence of these results is that when  $A \equiv 0$  in the original system (1), (4) can be interpreted as a small noise asymptotic result. While such results are classical for systems with Lipschitz nonlinear terms (see [11] or [8]) our result allows us to address the physically important case when one can only expect cancellations in the nonlinear portion of the equation.

We next turn to some concrete examples. In particular we consider stochastic versions of a meteorological model developed by Lorenz in [14] and of a simple model for turbulent flow proposed by Temam in [20]. In each case we exhibit the renormalization group which decouples the two scale inherent in the original system. Current work in preparation by the first author seeks to apply the results in this work to a series of numerical studies of these and other systems.

The final section collects some general results concerning slowly varying stochastic processes. The estimates that we establish in this section form the analytical core of the main approximation results and may hold independent interest for other studies of stochastic singular perturbation systems. Appendices collect further small noise asymptotic results and as well as some mostly classical estimates on stochastic convolutions terms arising in the proof of the main theorems.

**2. The stochastic singular perturbation system and the derivation of the renormalization group.** For  $\alpha \in \mathbb{R}$ , we consider the singularly perturbed system given by the stochastic differential equation in  $\mathbb{C}^n$

$$\begin{aligned} dY^\epsilon(\tau) + \frac{1}{\epsilon} \bar{A} Y^\epsilon(\tau) d\tau &= \bar{F}(Y^\epsilon(\tau)) d\tau + \epsilon^\alpha \bar{G}(\tau, \tau/\epsilon) dW, \\ Y^\epsilon(0) &= Y_0^\epsilon. \end{aligned} \quad (5)$$

We assume that  $\bar{A}$  is a diagonalizable matrix. Below we will analyze both the case when  $\bar{A}$  is antisymmetric and when  $\bar{A}$  is positive semidefinite. The nonlinear term  $\bar{F}$  is assumed, for the sake of simplicity, to be a polynomial.  $W = (W^1, \dots, W^n)^\perp$  is a standard  $n$  dimensional Brownian motion relative to some underlying filtered probability space  $(\Omega, \mathcal{F}, (\mathcal{F}_\tau)_{\tau \geq 0}, \mathbb{P})$ .  $\bar{G}$  takes values in  $\mathbb{M}^{n \times n}$  and is bounded in the Frobenius norm independently of  $\tau$  and  $\tau/\epsilon$

$$\sup_{\tau \geq 0} \|\bar{G}(\tau, \tau/\epsilon)\|^2 = \sup_{\tau \geq 0} \sum_{j,k} |\bar{G}_{j,k}(\tau, \tau/\epsilon)|^2 < \infty. \quad (6)$$

Further assumptions will be imposed on  $\bar{G}$  when we consider the case when  $\bar{A}$  is positive semidefinite. In what follows  $T$  is a fixed large time. The goal will be to study the behavior of  $X^\epsilon$  on the time interval  $[0, T]$  when  $\epsilon$  is small and in the limit as  $\epsilon \rightarrow 0$ .

For the analysis below we shall work in a different basis. Let  $Q$  be an orthogonal matrix diagonalizing  $\bar{A}$ , and let

$$X^\epsilon(\tau) := QY^\epsilon(\tau), \quad F(x) := Q\bar{F}(Q^*x), \quad G(\tau, \tau/\epsilon) := Q\bar{G}(\tau, \tau/\epsilon). \quad (7)$$

Multiplying (5) by  $Q$  gives the following evolution system for  $X^\epsilon$

$$\begin{aligned} dX^\epsilon(\tau) + \frac{1}{\epsilon}AX^\epsilon(\tau) d\tau &= F(X^\epsilon(\tau)) d\tau + \epsilon^\alpha G(\tau, \tau/\epsilon) dW, \\ X^\epsilon(0) &= X_0^\epsilon, \end{aligned} \quad (8)$$

where  $A$  is a diagonal matrix. Note that in this basis  $X^\epsilon$  may evolve in  $\mathbb{C}^n$ . The results below are easily translated back to the original coordinate frame for the system (5). See Remark 1.

The next step is to write  $X^\epsilon$  in a naive perturbation expansion in powers of  $\epsilon$ . The goal is to derive a reduced system that approximates (8) on large time intervals and that separates scales. To this end, we consider (8) on a new time scale  $t = \tau/\epsilon$

$$\begin{aligned} dX^\epsilon(t) + AX^\epsilon(t) dt &= \epsilon F(X^\epsilon(t)) dt + \epsilon^{m+1/2} G(\epsilon t, t) d\tilde{W}, \\ X^\epsilon(0) &= X_0^\epsilon. \end{aligned} \quad (9)$$

Below we set  $\beta = m + 1/2$ . Note that  $\tilde{W}$ , also a standard Brownian motion, is a rescaling of  $W$  given by

$$\tilde{W}(t) = \frac{1}{\sqrt{\epsilon}}W(\epsilon t) = \frac{1}{\sqrt{\epsilon}}W(\tau). \quad (10)$$

When  $\beta < 2$ , the perturbation expansion has the form<sup>1</sup>

$$\begin{aligned} X^\epsilon(t) &= X^{(0)}(t) + \epsilon^\beta X^{(\beta)} + \epsilon X^{(1)}(t) + \mathcal{O}(\epsilon^2), \\ X_0^\epsilon &= X^{(0)}(0). \end{aligned} \quad (11)$$

We plug (11) into (9) and match power of  $\epsilon$  to formally derive the coupled system of equations

$$\begin{aligned} dX^{(0)} &= -AX^{(0)} dt; & X^{(0)}(0) &= X_0^\epsilon, \\ dX^{(\beta)} &= -AX^{(\beta)} dt + G d\tilde{W}; & X^{(\beta)}(0) &= 0, \\ dX^{(1)} &= \left(-AX^{(1)} + F(X^{(0)})\right) dt; & X^{(1)}(0) &= 0, \end{aligned}$$

which admits the solution

$$\begin{aligned} X^{(0)}(t) &= e^{-At} X_0^\epsilon, \\ X^{(\beta)}(t) &= \int_0^t e^{-A(t-s)} G(\epsilon s, s) d\tilde{W}, \\ X^{(1)}(t) &= \int_0^t e^{-A(t-s)} F(e^{-As} X_0^\epsilon) ds. \end{aligned} \quad (12)$$

<sup>1</sup>Note however that no such restriction on  $\beta$  is needed for Proposition 1 or Proposition 2 below.

We have therefore derived the approximate solution

$$\begin{aligned} \bar{X}^\epsilon(t) &= e^{-At} \left( X_0^\epsilon + \epsilon \int_0^t e^{As} F(e^{-As} X_0^\epsilon) ds + \epsilon^\beta \int_0^t e^{As} G(\epsilon s, s) d\tilde{W} \right) + \mathcal{O}(\epsilon^2) \\ &= e^{-At} \left( X_0^\epsilon + \epsilon \int_0^t e^{As} F(e^{-As} X_0^\epsilon) ds \right) \\ &\quad + \epsilon^\beta \int_0^t e^{-A(t-s)} G(\epsilon s, s) d\tilde{W} + \mathcal{O}(\epsilon^2). \end{aligned} \tag{13}$$

The approximation (13) assumes that  $X^{(1)}$  remains  $\mathcal{O}(1)$  with high probability on the time scale  $1/\epsilon$  (see Definition (2.1) below). This may breakdown in  $X^{(1)}$  due to resonances with  $F$ , as analyzed in [23]. In contrast to previous investigations we also need to account for the role of the noise term in (13). As we shall see below, when the noise is “small” (i.e. when  $m > 0$  in the original system (5) or (8)) a approximate solution without stochastic terms can be shown to be valid on the time scale  $1/\epsilon$ . In the case of strictly dissipative systems where  $A$  is (strictly) positive definite we shall see that the noise can be even taken to be of intermediate strength (for example  $m = 0$ ). On the other hand we also show that including some or all of the noise terms in the approximate solution leads to a faster rate of convergence between the real and the approximate solutions. We end this section by formalizing the  $\mathcal{O}(\cdot)$  statement.

**Definition 2.1.** Suppose that  $\delta_1(\cdot), \delta_2(\cdot) : [0, \epsilon_0] \rightarrow [0, \infty)$  are monotonically decreasing functions. We say that a collection of processes  $X^\epsilon(\cdot) : \Omega \times [0, \infty) \rightarrow \mathbb{C}^n$  are  $\mathcal{O}(\delta_1(\epsilon))$  with high probability on time scale  $1/\delta_2(\epsilon)$  if for all  $T > 0$ , there exists a  $K > 0$ , so that

$$\mathbb{P} \left( \sup_{t \in [0, T/\delta_2(\epsilon)]} \|X^\epsilon(t)\| > K\delta_1(\epsilon) \right) \xrightarrow{\epsilon \downarrow 0} 0. \tag{14}$$

**2.1. Resonance analysis for the nonlinear term.** We split  $F$  into its resonant and non-resonant parts with respect to the semi-group generated by  $A$ . Since  $F$  is a polynomial of degree  $d$ , we can write

$$F(u) = \sum_{j=1}^n \left( \sum_{|\alpha| \leq d} C_\alpha^j u^\alpha \right) e_j.$$

Here  $e_j$  the  $j$ th standard basis element in  $\mathbb{C}^n$ . Let  $\Lambda = \{\lambda_1, \dots, \lambda_n\}$  be the eigenvalues of  $A$ . Given any multindex  $\alpha = (\alpha_1, \dots, \alpha_n)$ , we adopt the notation  $(\Lambda, \alpha) := \sum_{i=1}^n \alpha_i \lambda_i$ . Let

$$N_r^j = \{\alpha \in \mathbb{N}^n : |\alpha| \leq d, (\Lambda, \alpha) = \lambda_j\}. \tag{15}$$

The resonant terms in  $F$  are given by

$$R(u) = \sum_{j=1}^n \left( \sum_{\alpha \in N_r^j} C_\alpha^j u^\alpha \right) e_j, \tag{16}$$

and note that for any  $u \in \mathbb{C}^n$

$$e^{At} R(e^{-At} u) = R(u). \tag{17}$$

Let

$$F_{NR}(u) = F(u) - R(u) = \sum_{j=1}^n \left( \sum_{\alpha \notin N_r^j} C_\alpha^j u^\alpha \right) e_j \tag{18}$$

and

$$S(t, u) = e^{tA} F_{NR}(e^{-tA} u) = \sum_{j=1}^n \left( \sum_{\alpha \notin N_r^j} C_\alpha^j e^{t(\lambda_j - \langle \Lambda, \alpha \rangle)} u^\alpha \right) e_j. \tag{19}$$

We have the decompositions

$$\begin{aligned} e^{At} F(e^{-At} u) &= R(u) + S(t, u), \\ F(u) &= e^{-At} R(e^{At} u) + F_{NR}(u) = R(u) + F_{NR}(u). \end{aligned} \tag{20}$$

Applying (20) to (13), we find

$$\begin{aligned} \bar{X}^\epsilon &= e^{-At} \left( X_0^\epsilon + \epsilon t R(X_0^\epsilon) + \epsilon \int_0^t S(s, X_0^\epsilon) ds + \epsilon^\beta \int_0^t e^{As} G(\epsilon s, s) d\tilde{W} \right) + \mathcal{O}(\epsilon^2) \\ &= e^{-At} \left( X_0^\epsilon + \epsilon t R(X_0^\epsilon) + \epsilon \int_0^t S(s, X_0^\epsilon) ds \right) \\ &\quad + \epsilon^\beta \int_0^t e^{-A(t-s)} G(\epsilon s, s) d\tilde{W} + \mathcal{O}(\epsilon^2). \end{aligned} \tag{21}$$

In order to estimate terms arising in the expression of  $\bar{X}^\epsilon$ , and to show that the last term is nonresonant (bounded) in the probabilistic sense, we need the following lemma whose proof is an easy application of Ito’s formula, Burkholder-Davis-Gundy inequality, and therefore will be given in an Appendix for the sake of completeness.

**Lemma 2.2.** *Suppose that  $H$  takes values in  $\mathbb{M}^{n \times n_1}$ , that  $W = (W_1, \dots, W_{n_1})^T$  is a standard  $n_1$  dimensional Brownian motion and that  $A$  in  $\mathbb{M}^{n \times n}$  is symmetric non-negative definite or antisymmetric. Assume that  $H$  is uniformly bounded in time*

$$\sup_{t>0} \|H(t)\| = \sup_{t>0} \left( \sum_{j,k} |H_{k,j}(t)|^2 \right)^{1/2} < \infty. \tag{22}$$

Given constants  $K, T_0 > 0$

$$\mathbb{P} \left( \sup_{t \in [0, T_0]} \left\| \int_0^t e^{-A(t-s)} H(s) dW \right\| > K \right) \leq \frac{CT_0}{K^2}, \tag{23}$$

where  $C := C(\|H\|)$ .

Suppose now that  $\beta > 1/2$  (equivalently  $m > 0$  in the original time scale). For  $\gamma < \beta - 1/2$ , Lemma 2.2 implies that

$$\mathbb{P} \left( \sup_{t \in [0, T/\epsilon]} \left\| \epsilon^\beta \int_0^t e^{-A(t-s)} G(\epsilon s, s) d\tilde{W} \right\| > \epsilon^\gamma \right) \leq CT \epsilon^{2(\beta-\gamma)-1} \xrightarrow{\epsilon \downarrow 0} 0. \tag{24}$$

As such  $\epsilon^\beta \int_0^t e^{-A(t-s)} G(\epsilon s) d\tilde{W}$  is  $\mathcal{O}(\epsilon^\gamma)$  with high probability on the time scale  $1/\epsilon$ . Referring back to (21) we arrive at the renormalized system

$$\begin{aligned} dV^\epsilon(t) &= \epsilon R(V^\epsilon) dt, \\ V^\epsilon(0) &= V_0^\epsilon, \end{aligned} \tag{25}$$

with the associated approximate solution taking the form

$$\bar{X}^\epsilon(t) = e^{-At} \left( V^\epsilon(t) + \epsilon \int_0^t S(s, V^\epsilon(s)) ds \right). \tag{26}$$

This is exactly the form the approximate solution takes in other works ([23], [17], [15], [4] etc.) However by applying Proposition 3, one can show that

$$\int_0^t e^{-At} S(s, V^\epsilon) ds = \int_0^t e^{-A(t-s)} F_{NR}(e^{-sA} V^\epsilon) ds \tag{27}$$

remains  $\mathcal{O}(1)$  with high probability on time scale  $1/\epsilon$  as long as  $V^\epsilon$  remains  $\mathcal{O}(1)$  with high probability on this time scale. We are therefore justified in simplifying the approximate solution to

$$\bar{X}^\epsilon(t) = e^{-At} V^\epsilon(t). \tag{28}$$

This approximate solution satisfies

$$\begin{aligned} d\bar{X}^\epsilon + A\bar{X}^\epsilon dt &= \epsilon R(\bar{X}^\epsilon) dt, \\ \bar{X}^\epsilon(0) &= \bar{X}_0^\epsilon = V_0^\epsilon. \end{aligned} \tag{29}$$

In the original time scale, the renormalization group equation takes the form

$$\begin{aligned} dV^\epsilon(\tau) &= R(V^\epsilon(\tau)) d\tau, \\ V^\epsilon(0) &= V_0^\epsilon, \end{aligned} \tag{30}$$

and the approximate solution is given by

$$\bar{X}^\epsilon(\tau) = e^{-(\tau/\epsilon)A} V^\epsilon(\tau) \tag{31}$$

which solves

$$\begin{aligned} d\bar{X}^\epsilon + \frac{1}{\epsilon} A\bar{X}^\epsilon d\tau &= R(\bar{X}^\epsilon) d\tau \\ \bar{X}^\epsilon(0) &= \bar{X}_0^\epsilon = V_0^\epsilon. \end{aligned} \tag{32}$$

Note that the approximate solution in (31) splits the dynamics into two scales with the (possibly) intermediate scale introduced by the diffusion term of no consequence to the asymptotic validity of the approximation. Employing the estimate (24) we are able to establish that  $\bar{X}^\epsilon$  converges to  $X^\epsilon$  with high probability on time scale 1 with a convergence rate  $\epsilon^\gamma$  for any  $\gamma < m$ . More precisely we show that

$$\mathbb{P} \left( \sup_{\tau \in [0, T]} \|\bar{X}^\epsilon(\tau) - X^\epsilon(\tau)\| > C\epsilon^\gamma \right) \xrightarrow{\epsilon \rightarrow 0} 0, \tag{33}$$

for any such  $\gamma < m$ . In order to improve these convergence rates we may wish to include some stochastic terms in the renormalization group. The analysis must consider the antisymmetric and positive semidefinite cases separately.

**2.2. The antisymmetric case: Improved convergence rates.** Suppose that  $A$  is antisymmetric along with the standing assumption that  $\beta > 1/2$  (i.e.  $m > 0$ ). If we include all of the noise in the renormalized system we obtain

$$\begin{aligned} dV^\epsilon(t) &= \epsilon R(V^\epsilon) dt + \epsilon^\beta e^{tA} G(et, t) d\tilde{W}, \\ V^\epsilon(0) &= V_0^\epsilon, \end{aligned} \tag{34}$$

which reads as

$$\begin{aligned} dV^\epsilon(\tau) &= R(V^\epsilon) d\tau + \epsilon^\alpha e^{(\tau/\epsilon)A} G(\tau, \tau/\epsilon) dW, \\ V^\epsilon(0) &= V_0^\epsilon, \end{aligned} \tag{35}$$

in the original time scale. Since  $A$  is antisymmetric, (6) implies that

$$\sup_{\tau > 0} \|e^{(\tau/\epsilon)A}G(\tau, \tau/\epsilon)\| < \infty.$$

As such the small noise asymptotic results established below (c.f. Remark 3) show that for a large class of physically motivated systems, if the deterministic renormalized system in (30) is  $\mathcal{O}(1)$  with high probability on time scale 1 then so is the solution of (35). We are justified, as in the previous case, in assuming that  $V^\epsilon$  is a slowly varying process in the sense of (111), (112) and (113). Once again Proposition 3 justifies simplifying the approximate solution  $\bar{X}^\epsilon$  according to (28) (or (31) in the original time scale). In this case  $\bar{X}^\epsilon$  satisfies

$$\begin{aligned} d\bar{X}^\epsilon + A\bar{X}^\epsilon dt &= \epsilon R(\bar{X}^\epsilon)dt + \epsilon^\beta G(\epsilon t, t)d\tilde{W}, \\ \bar{X}^\epsilon(0) &= \bar{X}_0^\epsilon = V_0^\epsilon, \end{aligned} \quad (36)$$

and in time scale  $\tau$ , it written in the form

$$\begin{aligned} d\bar{X}^\epsilon + \frac{1}{\epsilon}A\bar{X}^\epsilon d\tau &= R(\bar{X}^\epsilon)d\tau + \epsilon^m G(\tau, \tau/\epsilon)dW, \\ \bar{X}^\epsilon(0) &= \bar{X}_0^\epsilon = V_0^\epsilon. \end{aligned} \quad (37)$$

When comparing this system to (8), the stochastic terms cancel. The approximate solution thus enjoys a faster rate of convergence to the original system (8). In particular we prove below that

$$\mathbb{P}\left(\sup_{\tau \in [0, T]} \|\bar{X}^\epsilon(\tau) - X^\epsilon(\tau)\| > C\epsilon\right) \xrightarrow{\epsilon \rightarrow 0} 0. \quad (38)$$

**Remark 1.** In some applications, particularly for the antisymmetric case, the governing equations (5) are given in a non-diagonalized basis, relative to the dominant linear term. The renormalized systems defined by (30) or (35) can be translated back to this original basis. Let

$$U^\epsilon = Q^*V^\epsilon, \quad \bar{R}(\cdot) = Q^*R(Q\cdot), \quad \bar{Y}^\epsilon = Q^*\bar{X}^\epsilon. \quad (39)$$

If  $V^\epsilon$  is defined by (30), then in the original basis  $U^\epsilon$  solves

$$\begin{aligned} dU^\epsilon(\tau) &= \bar{R}(U^\epsilon(\tau))d\tau, \\ U^\epsilon(0) &= U_0^\epsilon. \end{aligned} \quad (40)$$

On the other hand if we include stochastic forcing in the renormalized system as in (35) then

$$\begin{aligned} dU^\epsilon(\tau) &= \bar{R}(U^\epsilon(\tau))d\tau + \epsilon^\alpha e^{\tau/\epsilon\bar{A}}\bar{G}(\tau, \tau/\epsilon)dW, \\ U^\epsilon(0) &= U_0^\epsilon. \end{aligned} \quad (41)$$

In both cases, the approximate solution is given by

$$\bar{Y}^\epsilon(\tau) = e^{-(\tau/\epsilon)\bar{A}}U^\epsilon(\tau). \quad (42)$$

In the first case  $\bar{Y}^\epsilon$  is the solution of

$$\begin{aligned} d\bar{Y}^\epsilon + \frac{1}{\epsilon}\bar{A}\bar{Y}^\epsilon d\tau &= \bar{R}(\bar{Y}^\epsilon)d\tau, \\ \bar{Y}^\epsilon(0) &= \bar{Y}_0^\epsilon = U_0^\epsilon, \end{aligned} \quad (43)$$



whereas

$$\begin{aligned} d\bar{Y}^\epsilon + \frac{1}{\epsilon} \bar{A} \bar{Y}^\epsilon d\tau &= \bar{R}(\bar{Y}^\epsilon) d\tau + \epsilon^\alpha \tilde{G}(\epsilon t, t) dW, \\ \bar{Y}^\epsilon(0) &= \bar{Y}_0^\epsilon = U_0^\epsilon, \end{aligned} \tag{44}$$

in the second case. The results in Proposition 1 apply when we replace  $V^\epsilon$ ,  $X^\epsilon$  and  $\bar{X}^\epsilon$  with  $U^\epsilon$ ,  $Y^\epsilon$  and  $\bar{Y}^\epsilon$  respectively.

**2.3. The positive semidefinite case: Improved convergence rates and the case of large noise.** We next consider the case when  $A$  is positive semidefinite. Here, beyond the uniform time bounds in (6) we assume that  $G$  is in  $\mathbb{M}^{n \times n}$  and diagonal. Let  $\mathcal{M}$ ,  $\mathcal{N}$  be the projections onto  $\ker(A)$  and  $\ker(A)^\perp$  respectively. Applying this decomposition to (21) we obtain

$$\begin{aligned} \bar{X}^\epsilon &= e^{-At} \left( X_0^\epsilon + \epsilon t R(X_0^\epsilon) + \epsilon \int_0^t S(s, X_0^\epsilon) ds + \epsilon^\beta \int_0^t \mathcal{M} G(\epsilon s, s) d\tilde{W} \right) \\ &+ \epsilon^\beta \int_0^t e^{-A(t-s)} \mathcal{N} G(\epsilon s, s) d\tilde{W} + \mathcal{O}(\epsilon^2). \end{aligned} \tag{45}$$

In order to control the last term in the expression above we need the following lemma whose proof is given in an Appendix.

**Lemma 2.3.** *Suppose that  $A$  and  $H(\cdot)$  are diagonal with  $\sup_{t>0} \|H(t)\| < \infty$ , and the spectrum of  $A$  real and non-negative. For  $q \geq 2$  we have*

$$\mathbb{P} \left( \sup_{t \in [0, T_0]} \left\| \int_0^t e^{-A(t-s)} \mathcal{N} H(s) dW \right\| > K \right) \leq \frac{CT_0}{K^q}, \tag{46}$$

where  $C := C(q, H)$  and  $\mathcal{N}$  is the projection onto  $\ker(A)^\perp$ .

Suppose that  $\beta > 0$  (i.e.  $m > -1/2$  in the original time scale). By applying Lemma 2.3, we have the estimate

$$\mathbb{P} \left( \sup_{t \in [0, T/\epsilon]} \left\| \epsilon^\beta \int_0^t e^{-A(t-s)} \mathcal{N} G(\epsilon s, s) dW \right\| > \epsilon^\gamma \right) \leq C(q) T \epsilon^{q(\beta-\gamma)-1}, \tag{47}$$

for any  $0 < \gamma < \beta$ . By choosing  $q > \max\{(\beta - \gamma)^{-1}, 2\}$ , we see that the stochastic integral  $\epsilon^\beta \int_0^t e^{-A(t-s)} G(\epsilon s) dW$  remains  $\mathcal{O}(\epsilon^\gamma)$  with high probability on a time scale of order  $1/\epsilon$ . With this in mind we define the renormalization group equation

$$\begin{aligned} dV^\epsilon(t) &= \epsilon R(V^\epsilon(t)) dt + \epsilon^\beta \mathcal{M} G(\epsilon t, t) d\tilde{W}, \\ V^\epsilon(0) &= V_0^\epsilon, \end{aligned} \tag{48}$$

which gives

$$\begin{aligned} dV^\epsilon(\tau) &= R(V^\epsilon(\tau)) d\tau + \epsilon^m \mathcal{M} G(\tau, \tau/\epsilon) dW, \\ V^\epsilon(0) &= V_0^\epsilon, \end{aligned} \tag{49}$$

in the original time scale. Note that in order to apply either Proposition 3 or Remark 3 to this renormalized system, we must require either that  $\mathcal{M} = 0$  (equivalently that  $A$  is strictly positive definite) or that  $m > 0$ . The approximate solution is defined in either case by  $\bar{X}^\epsilon(t) = e^{-tA} V^\epsilon$  which is the solution of

$$\begin{aligned} d\bar{X}^\epsilon + A \bar{X}^\epsilon dt &= \epsilon R(\bar{X}^\epsilon) dt + \epsilon^\beta \mathcal{M} G(\epsilon t, t) d\tilde{W}, \\ \bar{X}^\epsilon(0) &= \bar{X}_0^\epsilon = V_0^\epsilon \end{aligned} \tag{50}$$

Returning to the original time scale we have  $\bar{X}^\epsilon(\tau) = e^{-(\tau/\epsilon)A}V^\epsilon(\tau)$  which solves

$$\begin{aligned} d\bar{X}^\epsilon + \frac{1}{\epsilon}A\bar{X}^\epsilon d\tau &= R(\bar{X}^\epsilon)d\tau + \epsilon^m \mathcal{M}G(\tau, \tau/\epsilon)dW, \\ \bar{X}^\epsilon(0) &= \bar{X}_0^\epsilon = V_0^\epsilon. \end{aligned} \tag{51}$$

Relying on the stronger estimates available for the stochastic convolution due to the dissipation in the system (c.f. (47) above) we are able to show that

$$\mathbb{P}\left(\sup_{\tau \in [0, T]} \|\bar{X}^\epsilon(\tau) - X^\epsilon(\tau)\| > C\epsilon^\gamma\right) \xrightarrow{\epsilon \rightarrow 0} 0, \tag{52}$$

where  $\gamma < m + 1/2$ .

**3. Rigorous approximation results.** We now rigorously state and prove the main results legitimating  $\bar{X}^\epsilon$  as an approximate solution of (8). We start with the highly oscillatory case.

**Proposition 1.** *Assume that  $X^\epsilon$  solves (8) with  $A$  antisymmetric and  $m > 0$ . Suppose that  $V^\epsilon$  solves either (30) or (35), and that for any  $T > 0$  there exists  $K > 0$  such that*

$$\mathbb{P}\left(\sup_{\tau \in [0, T]} \|V^\epsilon(\tau)\| > K\right) := \phi_1(\epsilon) \xrightarrow{\epsilon \rightarrow 0} 0, \tag{53}$$

Additionally, we assume that

$$\mathbb{P}(\|V_0^\epsilon - X_0^\epsilon\| > K\epsilon) := \phi_2(\epsilon) \xrightarrow{\epsilon \rightarrow 0} 0. \tag{54}$$

(i) *Let  $V^\epsilon$  be as in (30) and therefore  $\bar{X}^\epsilon$  is as in (31), then for any  $\gamma < m$*

$$\mathbb{P}\left(\sup_{\tau \in [0, T]} \|\bar{X}^\epsilon(\tau) - X^\epsilon(\tau)\| > C\epsilon^\gamma\right) \xrightarrow{\epsilon \rightarrow 0} 0, \tag{55}$$

where  $C = C(K, F)$ .

(ii) *If  $V^\epsilon$  solves (35) and  $\bar{X}^\epsilon$  is given by (37) then*

$$\mathbb{P}\left(\sup_{\tau \in [0, T]} \|\bar{X}^\epsilon(\tau) - X^\epsilon(\tau)\| > C\epsilon\right) \xrightarrow{\epsilon \rightarrow 0} 0. \tag{56}$$

**Remark 2.** As in [4],  $T$  can be replaced with  $T_\epsilon := T \log(1/\epsilon)$  in (53), (55) and (56) (or in (87), (89), (90) below) with no additional complications.

*Proof.* Throughout the following, we work on the time scale  $t$ . Let  $Z^\epsilon(t) = e^{At}X^\epsilon(t)$ . This process satisfies

$$dZ^\epsilon = \epsilon e^{At}F(e^{-At}Z^\epsilon) + \epsilon^\beta e^{At}G(\epsilon t, t)d\tilde{W}; \quad Z^\epsilon(0) = X_0^\epsilon. \tag{57}$$

Since  $A$  is antisymmetric

$$\|V^\epsilon(t) - Z^\epsilon(t)\| = \|\bar{X}^\epsilon(t) - X^\epsilon(t)\|, \quad t \geq 0. \tag{58}$$

Thus, we are justified to consider the error process

$$\mathcal{E}^\epsilon(t) := V^\epsilon(t) - Z^\epsilon(t). \tag{59}$$

For the case (i), using (20)

$$\begin{aligned} \mathcal{E}^\epsilon(t) &= \mathcal{E}_0^\epsilon + \epsilon \int_0^t (R(V^\epsilon) - e^{As}F(e^{-As}Z^\epsilon)) ds - \epsilon^\beta \int_0^t e^{As}G(\epsilon s, s)d\tilde{W} \\ &= \mathcal{E}_0^\epsilon + \epsilon \int_0^t (e^{As}F(e^{-As}V^\epsilon) - e^{As}F(e^{-As}(V^\epsilon - \mathcal{E}^\epsilon))) ds \\ &\quad - \epsilon \int_0^t S(s, V^\epsilon)ds - \epsilon^\beta \int_0^t e^{As}G(\epsilon s, s)d\tilde{W}. \end{aligned} \tag{60}$$

However, in the case (ii), there is no stochastic integral term, and we have

$$\begin{aligned} \mathcal{E}^\epsilon(t) &= \mathcal{E}_0^\epsilon + \epsilon \int_0^t (R(V^\epsilon) - e^{As}F(e^{-As}Z^\epsilon)) ds \\ &= \mathcal{E}_0^\epsilon + \epsilon \int_0^t (e^{As}F(e^{-As}V^\epsilon) - e^{As}F(e^{-As}(V^\epsilon - \mathcal{E}^\epsilon))) ds \\ &\quad - \epsilon \int_0^t S(s, V^\epsilon)ds. \end{aligned} \tag{61}$$

Notice that in both cases  $V^\epsilon$  has the form (111) and satisfies the conditions (112) and (113). For  $\alpha \notin N_R^j$ , let  $K_{\alpha,j}$  be the constant arising in Proposition 3 with  $f(u) = C_\alpha^j u^\alpha$ . Define

$$C_1 = \sum_{j=1}^n \sum_{\alpha \notin N_R^j} K_{\alpha,j}. \tag{62}$$

Applying Proposition 3, we estimate

$$\begin{aligned} &\mathbb{P}\left(\sup_{t \in [0, T/\epsilon]} \epsilon \left\| \int_0^t S(s, V^\epsilon)ds \right\| > C_1 \epsilon\right) \\ &\leq \sum_{\substack{1 \leq j \leq n \\ \alpha \notin N_R^j}} \mathbb{P}\left(\sup_{t \in [0, T/\epsilon]} \left\| \int_0^t e^{t(\lambda_j - (\Lambda, \alpha))} C_\alpha^j (V^\epsilon)^\alpha ds \right\| > K_{\alpha,j}\right) \xrightarrow{\epsilon \downarrow 0} 0. \end{aligned} \tag{63}$$

For case (i) we need to estimate the stochastic integral term in (60). Lemma 2.2 implies

$$\begin{aligned} &\mathbb{P}\left(\sup_{t \in [0, T/\epsilon]} \left\| \epsilon^\beta \int_0^t e^{As}G(\epsilon s, s)d\tilde{W} \right\| > \epsilon^\gamma\right) \\ &= \mathbb{P}\left(\sup_{t \in [0, T/\epsilon]} \left\| \int_0^t e^{-A(t-s)}G(\epsilon s, s)d\tilde{W} \right\| > \epsilon^{\gamma-\beta}\right) \\ &\leq CT\epsilon^{2(\beta-\gamma)-1} = CT\epsilon^{2(m-\gamma)} \xrightarrow{\epsilon \downarrow 0} 0. \end{aligned} \tag{64}$$

By the mean value theorem, we have

$$\begin{aligned} &\|F(e^{-At}V^\epsilon) - F(e^{-At}V^\epsilon(t) - e^{-At}\mathcal{E}^\epsilon(t))\| \\ &\leq \sup_{\sigma \in [0,1]} \|D(F)(e^{-At}V^\epsilon(t) - \sigma e^{-At}\mathcal{E}^\epsilon(t))\| \|\mathcal{E}^\epsilon(t)\|. \end{aligned} \tag{65}$$

Let

$$\eta_\epsilon(t) = \sup_{\sigma \in [0,1]} \|D(F)(e^{-At}V^\epsilon(t) - \sigma e^{-At}\mathcal{E}^\epsilon(t))\|. \tag{66}$$

Note that

$$\left\| \int_0^t e^{As} (F(e^{-As}V^\epsilon) - F(e^{-As}(V^\epsilon - \mathcal{E}^\epsilon))) ds \right\| \leq \Xi^\epsilon(t), \tag{67}$$

where

$$\Xi^\epsilon(t) = \int_0^t \eta_\epsilon(s) \|\mathcal{E}^\epsilon(s)\| ds. \tag{68}$$

Define

$$\begin{aligned} \mathcal{G}_\epsilon = \{ \|\mathcal{E}_0^\epsilon\| \leq K\epsilon \} \cap & \left\{ \sup_{t \in [0, T/\epsilon]} \epsilon \left\| \int_0^t S(s, V^\epsilon) ds \right\| \leq C_1\epsilon \right\} \\ & \cap \left\{ \sup_{t \in [0, T/\epsilon]} \left\| \epsilon^\beta \int_0^t e^{As} G(\epsilon s, s) d\tilde{W} \right\| \leq \epsilon^\gamma \right\} \end{aligned} \tag{69}$$

in case (i) and

$$\mathcal{G}_\epsilon = \{ \|\mathcal{E}_0^\epsilon\| \leq K\epsilon \} \cap \left\{ \sup_{t \in [0, T/\epsilon]} \epsilon \left\| \int_0^t S(s, V^\epsilon) ds \right\| \leq C_1\epsilon \right\} \tag{70}$$

for (ii). Due to (54) and (63) and in the first case (64), we have

$$\mathbb{P}(\mathcal{G}_\epsilon^C) \xrightarrow{\epsilon \rightarrow 0} 0. \tag{71}$$

We complete the proof for either (i) or (ii) using a maximal argument. We give the details for (i), the other case is nearly identical. Let  $C_2 = K + C_1 + 1$ . We have for any sample  $\omega$  coming from  $\mathcal{G}_\epsilon$

$$\|\mathcal{E}^\epsilon(t, \omega)\| \leq \epsilon^\gamma C_2 + \epsilon \Xi(t, \omega). \tag{72}$$

Thus

$$\frac{d\Xi^\epsilon(t, \omega)}{dt} = \eta_\epsilon(t, \omega) \|\mathcal{E}^\epsilon(t, \omega)\| \leq \epsilon^\gamma (C_2 + \Xi^\epsilon(t, \omega)) \eta_\epsilon(t, \omega). \tag{73}$$

The Gronwall's inequality implies

$$\Xi^\epsilon(t, \omega) \leq \epsilon^\gamma C_2 \exp\left(\int_0^t \epsilon^\gamma \eta(s, \omega) ds\right) \int_0^t \eta(s, \omega) ds. \tag{74}$$

For  $\omega \in \mathcal{G}_\epsilon$ , define  $I^\epsilon(\omega) \subset [0, T/\epsilon]$  to be the maximal interval so that  $\|\mathcal{E}^\epsilon(s, \omega)\| \leq 1$  for every  $s \in I^\epsilon(\omega)$ . Note that for  $\epsilon < K^{-1}$ , the definition of  $\mathcal{G}_\epsilon$  insures that this interval is nontrivial for almost every  $\omega \in \mathcal{G}_\epsilon$ . Let

$$C_3 := \sup_{\substack{t \in \mathbb{R}^+ \\ \|x\| \leq K, \|y\| \leq 1}} \left( \sup_{\sigma \in [0, 1]} \|D(F)(e^{-At}(x - \sigma y))\| \right). \tag{75}$$

By using the estimate (74) we obtain

$$\|\mathcal{E}^\epsilon(s, \omega)\| \leq \epsilon^\gamma (C_2 e^{\epsilon^\gamma T C_3} T C_3) \leq \epsilon^\gamma C_4, \tag{76}$$

for any  $s \in I(\omega)$ . Note that  $C_4$  can be chosen independently of  $\omega$  and  $\epsilon$ . As such for  $\epsilon < \left(\frac{1}{C_4}\right)^{1/\gamma}$ , it follows that  $I^\epsilon = [0, T/\epsilon]$ . The proof is complete.  $\square$

A special case of Proposition 1, (i) can be interpreted as a small noise asymptotic result

**Corollary 1.** *Suppose that for  $\epsilon > 0$ ,  $X^\epsilon$  solves*

$$dX^\epsilon = F(X^\epsilon)d\tau + \epsilon^m G(\tau, \tau/\epsilon)dW; \quad X^\epsilon(0) = X_0^\epsilon. \tag{77}$$

*and that  $x^\epsilon$  is the solution of the associated deterministic system*

$$dx^\epsilon = F(x^\epsilon)d\tau; \quad x^\epsilon(0) = x_0^\epsilon. \tag{78}$$

*Assume that  $G(\cdot, \cdot)$  is uniformly bounded as in (6), that*

$$\langle F(u), u \rangle \leq 0 \quad \text{for all } u \in \mathbb{R}^n, \tag{79}$$

*that*

$$\mathbb{P}(\|X_0^\epsilon - x_0^\epsilon\| > K\epsilon^m) \xrightarrow{\epsilon \rightarrow 0} 0, \tag{80}$$

*and finally that*

$$\mathbb{P}(\|x_0^\epsilon\| > K) \xrightarrow{\epsilon \rightarrow 0} 0. \tag{81}$$

*Then for any  $\gamma < m$*

$$\mathbb{P}\left(\sup_{\tau \in [0, T]} \|X^\epsilon(\tau) - x^\epsilon(\tau)\| > C\epsilon^\gamma\right) \xrightarrow{\epsilon \rightarrow 0} 0. \tag{82}$$

*Where  $C := C(F, K)$ .*

*Proof.* Notice that (78) is in the form of (5)<sup>2</sup> for  $A \equiv 0$ . Defining  $R$  according to (16) we find that  $F(u) = R(u)$  and that the renormalization group derived in Section 2.1 is given by (78). Due to (79) we have

$$\|x^\epsilon(\tau)\|^2 = \|x_0^\epsilon\|^2 + 2 \int_0^\tau \langle F(x^\epsilon), x^\epsilon \rangle d\tau \leq \|x_0^\epsilon\|^2. \tag{83}$$

By applying (81) we find that the condition (53) holds for  $x^\epsilon$ . The conclusion (82) therefore follows from Proposition 1, (i). □

**Remark 3.**

- (i) One important class of systems that satisfy (79) arise when

$$F(u) = Lu + B(u, u), \tag{84}$$

where  $L$  is linear and either antisymmetric or positive semidefinite and  $B$  is a bilinear form satisfying the cancellation property  $\langle B(u, v), v \rangle = 0$ .

- (ii) Small noise asymptotic results involving Lipschitz continuous nonlinear terms are classical and have been studied by many authors (see [11] and [8]). As noted, Corollary 1 covers the physically important case when we can only expect cancellations in the nonlinear portion of the equation. In Section 7 below we provide a different proof of Corollary 1 that covers the case of multiplicative noise and also establishes convergence to the deterministic limit system in  $L^p(\Omega)$  for  $p \geq 2$ .
- (iii) Corollary 1 can sometimes be used to verify (53) for  $V^\epsilon$ , the solution of (35). Suppose that according to (16)  $R(u) = Lu + B(u, u)$ , so that  $\langle R(u), u \rangle \leq 0$ . Take

$$dv^\epsilon = R(v^\epsilon)d\tau; \quad v^\epsilon(0) = v_0^\epsilon. \tag{85}$$

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<sup>2</sup>In this case, (5) and (8) are identical

If  $V_0^\epsilon - v_0^\epsilon$  satisfies (80) and  $v_0^\epsilon$  fulfills (81), each with constant  $\kappa$ , then Corollary 1 and the calculation in (83) imply

$$\begin{aligned} & \mathbb{P}\left(\sup_{\tau \in [0, T]} |V^\epsilon(\tau)| > \kappa + 1\right) \\ & \leq \mathbb{P}\left(\sup_{\tau \in [0, T]} |V^\epsilon(\tau) - v^\epsilon(\tau)| > 1\right) + \mathbb{P}\left(\sup_{\tau \in [0, T]} |v^\epsilon(\tau)| > \kappa\right) \xrightarrow{\epsilon \downarrow 0} 0. \end{aligned} \tag{86}$$

In a similar manner one may verify (87) for  $V^\epsilon$  satisfying (49) in Proposition 2 below.

We next address the positive semidefinite case.

**Proposition 2.** *Assume that  $X^\epsilon$  solves (8) with  $A$  positive semidefinite. Suppose that  $V^\epsilon$  is a solution of either (30) or (49) so that for  $T > 0$  there is a constant  $K > 0$  such that*

$$\mathbb{P}\left(\sup_{\tau \in [0, T]} \|V^\epsilon(\tau)\| > K\right) := \phi_1(\epsilon) \xrightarrow{\epsilon \rightarrow 0} 0. \tag{87}$$

Additionally, suppose that the initial data  $V_0^\epsilon$  approximates  $X_0^\epsilon$  with high probability for small  $\epsilon$

$$\mathbb{P}(\|V_0^\epsilon - X_0^\epsilon\| > K\epsilon) := \phi_2(\epsilon) \xrightarrow{\epsilon \rightarrow 0} 0. \tag{88}$$

(i) *If  $m > 0$  in (8) and  $V^\epsilon$  solves (30) then there is a positive constant  $C = C(K, F)$  so that whenever  $\gamma \leq m$*

$$\mathbb{P}\left(\sup_{\tau \in [0, T]} \|\bar{X}^\epsilon(\tau) - X^\epsilon(\tau)\| > C\epsilon^\gamma\right) \xrightarrow{\epsilon \rightarrow 0} 0, \tag{89}$$

where  $\bar{X}^\epsilon(\tau) := e^{-\left(\frac{\tau}{\epsilon}\right)A}V^\epsilon(\tau)$ .

(ii) *In addition to the uniform bound (6), suppose that  $G(\cdot, \cdot)$  is diagonal. If  $m > 0$  in (8) and  $V^\epsilon$  is the solution of (49) then*

$$\mathbb{P}\left(\sup_{\tau \in [0, T]} \|\bar{X}^\epsilon(\tau) - X^\epsilon(\tau)\| > C\epsilon^\gamma\right) \xrightarrow{\epsilon \rightarrow 0} 0, \tag{90}$$

for any  $\gamma < m + 1/2$ .

(iii) *If, moreover,  $A$  is (strictly) positive definite (i.e.  $\sigma(A)$  is strictly positive) then if  $m > -1/2$ , (90) holds for  $\gamma < m + 1/2$ .*

*Proof.* As in the proof of Proposition 1 we work on time scale  $t$ . Here we define the error process by  $\mathcal{E}^\epsilon(t) = \bar{X}^\epsilon(t) - X^\epsilon(t)$ . Subtracting (29) from (9) in case (i), or (50) from (9) in cases (ii) and (iii), we arrive at the system

$$d\mathcal{E}^\epsilon + A\mathcal{E}^\epsilon dt = \epsilon (R(\bar{X}^\epsilon) - F(X^\epsilon)) dt - \epsilon^\beta \mathcal{P}G(et, t)dW; \quad \mathcal{E}^\epsilon(0) = V_0^\epsilon - X_0^\epsilon. \tag{91}$$

In case (i)  $\mathcal{P} = I$  while in the later cases  $\mathcal{P} = \mathcal{N}$ , the projection on  $\ker(A)^\perp$ . We have

$$\begin{aligned} \mathcal{E}^\epsilon(t) &= e^{-At} \mathcal{E}_0^\epsilon + \epsilon \int_0^t e^{-A(t-s)} (R(\bar{X}^\epsilon) - F(X^\epsilon)) ds \\ &\quad - \epsilon^\beta \int_0^t e^{-A(t-s)} \mathcal{P}G(\epsilon t, t) dW \\ &= e^{-At} \mathcal{E}_0^\epsilon + \epsilon \int_0^t e^{-A(t-s)} (e^{-As} R(e^{As} \bar{X}^\epsilon) - F(X^\epsilon)) ds \\ &\quad - \epsilon^\beta \int_0^t e^{-A(t-s)} \mathcal{P}G(\epsilon t, t) dW \\ &= e^{-At} \mathcal{E}_0^\epsilon + \epsilon \int_0^t e^{-A(t-s)} (F(\bar{X}^\epsilon) - F(\bar{X}^\epsilon - \mathcal{E}^\epsilon)) ds \\ &\quad - \epsilon \int_0^t e^{-A(t-s)} F_{NR}(\bar{X}^\epsilon) ds - \epsilon^\beta \int_0^t e^{-A(t-s)} \mathcal{P}G(\epsilon t, t) dW. \end{aligned} \tag{92}$$

First we estimate the term

$$\begin{aligned} \int_0^t e^{-A(t-s)} F_{NR}(\bar{X}^\epsilon) ds &= \sum_{k=1}^n \sum_{\alpha \notin N_r^j} \left( \int_0^t e^{-\lambda_j(t-s)} C_\alpha^j(\bar{X}^\epsilon)^\alpha ds \right) e_j \\ &= \sum_{k=1}^n \sum_{\alpha \notin N_r^j} \left( \int_0^t e^{-\lambda_j(t-s)} e^{-s(\Lambda, \alpha)} C_\alpha^j(V^\epsilon)^\alpha ds \right) e_j. \end{aligned} \tag{93}$$

Here  $\lambda_j = A_{jj} \geq 0$  (since  $A$  is assumed diagonal, this is the  $j$ th eigenvalue of  $A$ ) and  $(\Lambda, \alpha) = \sum_k \alpha_k \lambda_k$ . Recalling (15) we have  $\lambda_j \neq (\Lambda, \alpha)$ , for each  $\alpha \notin N_r^j$ . Notice that for cases (i) and (iii)  $V^\epsilon$  takes the form (111) trivially since both (25) and (48) are deterministic systems (since  $\sigma(A) > 0$  in the later case  $\mathcal{M} = 0$ ). On the other hand, for case (ii),  $\beta > 1/2$  in (48). Combining these observations with the boundedness condition (87) we see that each of the terms in (93) satisfy Lemma 3 (ii). Similarly to (62) and (63) we infer a constant  $C_1$ , depending on  $C_\alpha^j u^\alpha$  and  $\lambda_j$  in (93), so that

$$\mathbb{P} \left( \sup_{t \in [0, T/\epsilon]} \left\| \epsilon \int_0^t e^{-A(t-s)} F_{NR}(\bar{X}^\epsilon) ds \right\| > C_1 \epsilon \right) \xrightarrow{\epsilon \downarrow 0} 0. \tag{94}$$

We next address the stochastic integral terms. For case (i) we estimate

$$\mathbb{P} \left( \sup_{t \in [0, T/\epsilon]} \left\| \epsilon^\beta \int_0^t e^{-A(t-s)} G(\epsilon t, t) dW \right\| > \epsilon^\gamma \right) \leq CT \epsilon^{2(\beta-\gamma)-1} \xrightarrow{\epsilon \downarrow 0} 0, \tag{95}$$

using Lemma 2.2. On the other hand for cases (ii) and (iii) we apply Lemma 2.3, choosing  $q > \max\{(\beta - \gamma)^{-1}, 2\}$  so that

$$\mathbb{P} \left( \sup_{t \in [0, T/\epsilon]} \left\| \epsilon^\beta \int_0^t e^{-A(t-s)} \mathcal{N}G(\epsilon t, t) dW \right\| > \epsilon^\gamma \right) \leq C(q) T \epsilon^{q(\beta-\gamma)-1} \xrightarrow{\epsilon \downarrow 0} 0. \tag{96}$$

The final step is to estimate the term

$$\epsilon \int_0^t e^{-A(t-s)} (F(\bar{X}^\epsilon) - F(\bar{X}^\epsilon - \mathcal{E}^\epsilon)) ds. \tag{97}$$

By the mean value theorem

$$\begin{aligned} & \|F(\bar{X}^\epsilon) - F(\bar{X}^\epsilon - \mathcal{E}^\epsilon)\| \\ & \leq \sup_{\sigma \in [0,1]} \|D(F)(\bar{X}^\epsilon(s) - \mathcal{E}^\epsilon(s))\| \|\mathcal{E}^\epsilon(t)\|. \end{aligned} \tag{98}$$

Let

$$\eta_\epsilon(t) = \sup_{\sigma \in [0,1]} \|D(F)(e^{-As}V^\epsilon(s) - \sigma\mathcal{E}^\epsilon(s))\|, \tag{99}$$

and as in the proof of Proposition 1, define

$$\Xi^\epsilon(t) = \int_0^t \eta_\epsilon(s) \|\mathcal{E}^\epsilon(s)\| ds \tag{100}$$

and

$$\begin{aligned} \mathcal{G}_\epsilon = & \left\{ \sup_{t \in [0, T/\epsilon]} \|e^{-At}\mathcal{E}_0^\epsilon\| \leq K\epsilon \right\} \\ & \cap \left\{ \sup_{t \in [0, T/\epsilon]} \epsilon \left\| \int_0^t e^{-A(t-s)} F_{NR}(\bar{X}^\epsilon) ds \right\| \leq C_1\epsilon \right\} \\ & \cap \left\{ \sup_{t \in [0, T/\epsilon]} \left\| \epsilon^\beta \int_0^t e^{-A(t-s)} \mathcal{N}G(\epsilon s) d\tilde{W} \right\| \leq \epsilon^\gamma \right\}, \end{aligned} \tag{101}$$

or

$$\begin{aligned} \mathcal{G}_\epsilon = & \left\{ \sup_{t \in [0, T/\epsilon]} \|e^{-At}\mathcal{E}_0^\epsilon\| \leq K\epsilon \right\} \\ & \cap \left\{ \sup_{t \in [0, T/\epsilon]} \epsilon \left\| \int_0^t e^{-A(t-s)} F_{NR}(\bar{X}^\epsilon) ds \right\| \leq C_1\epsilon \right\} \\ & \cap \left\{ \sup_{t \in [0, T/\epsilon]} \left\| \epsilon^\beta \int_0^t e^{-A(t-s)} G(\epsilon s) d\tilde{W} \right\| \leq \epsilon^\gamma \right\}. \end{aligned} \tag{102}$$

The maximal argument is then employed exactly as in Proposition 1 to complete the proof.  $\square$

**4. Applications: Some examples from fluid dynamics.** With the results developed above in hand, we now consider several examples of stochastically perturbed multi-scale systems. These simple models were first considered in the deterministic setting in [15] and are motivated by the numerical study of geophysical fluid flow and turbulence. For each example below observe that the renormalized system decouples the fast and slow components of the original system. The stochastically forced renormalized systems are seen to satisfy (53) (or (87)) for a large class of initial conditions by Remark 3. We further remark that for the first two examples the perturbation systems are written in the non-diagonalized form (5). As such we exhibit the deterministic and stochastic renormalized system according to (40) and (41) respectively. See Remark 1. A future article which develops applications of the results in this work to the numerical integration of singular stochastic systems will consider the examples below in more detail.



**Example 1** (Linear Renormalization Group Equation). For the first example we consider a nonlinear equation that exhibits a linear renormalization group equation.

$$\begin{aligned} dY_1 + (\lambda_1 Y_1 - \frac{1}{\epsilon} Y_2 - Y_1 Y_3) d\tau &= \sqrt{\epsilon} dW_1, \\ dY_2 + \left( \lambda_2 Y_2 + \frac{1}{\epsilon} Y_1 + Y_2 Y_3 \right) d\tau &= \sqrt{\epsilon} dW_2, \\ dY_3 + (\lambda_3 Y_3 + Y_1^2 - Y_2^2) d\tau &= \sqrt{\epsilon} dW_3, \end{aligned} \quad (103)$$

where  $\lambda_1, \lambda_2, \lambda_3$  are fixed constants. After some routine computations we conclude that the renormalization group equation has the form

$$\begin{aligned} dU_1 + \frac{1}{2}(\lambda_1 + \lambda_2)U_1 d\tau &= 0, \\ dU_2 + \frac{1}{2}(\lambda_1 + \lambda_2)U_2 d\tau &= 0, \\ dU_3 + \lambda_3 U_3 d\tau &= 0. \end{aligned} \quad (104)$$

The stochastic counterpart is given by

$$\begin{aligned} dU_1 + \frac{1}{2}(\lambda_1 + \lambda_2)U_1 d\tau &= \sqrt{\epsilon} \cos(\tau/\epsilon) dW_1 - \sqrt{\epsilon} \sin(\tau/\epsilon) dW_2, \\ dU_2 + \frac{1}{2}(\lambda_1 + \lambda_2)U_2 d\tau &= \sqrt{\epsilon} \sin(\tau/\epsilon) dW_1 + \sqrt{\epsilon} \cos(\tau/\epsilon) dW_2, \\ dU_3 + \lambda_3 U_3 d\tau &= \sqrt{\epsilon} dW_3. \end{aligned} \quad (105)$$

**Example 2** (The Lorenz System). We next consider a system introduced by E. Lorenz (see [14]) with a small additive noise. Let  $\lambda_1, \lambda_2$  and  $b$  be fixed positive parameters.

$$\begin{aligned} dY_1 &= (-\lambda_1 Y_1 - Y_2 Y_3 + b Y_2 Y_5) d\tau + \sqrt{\epsilon} dW_1, \\ dY_2 &= (-\lambda_1 Y_2 + 2Y_1 Y_3 - 2b Y_1 Y_5) d\tau + \sqrt{\epsilon} dW_2, \\ dY_3 &= (-\lambda_1 Y_3 - Y_1 Y_2) d\tau + \sqrt{\epsilon} dW_3, \\ dY_4 &= \left( -\lambda_2 Y_4 - \frac{1}{\epsilon} Y_5 \right) d\tau + \sqrt{\epsilon} dW_4, \\ dY_5 &= \left( -\lambda_2 Y_4 + \frac{1}{\epsilon} Y_4 + b Y_1 Y_2 \right) d\tau + \sqrt{\epsilon} dW_5. \end{aligned} \quad (106)$$

In this case the renormalization group equation is given by

$$\begin{aligned} dU_1 &= (-\lambda_1 U_1 - U_2 U_3) d\tau, \\ dU_2 &= (-\lambda_1 U_2 + 2U_1 U_3) d\tau, \\ dU_3 &= (-\lambda_1 U_3 - U_1 U_2) d\tau, \\ dU_4 &= -\lambda_2 U_4 d\tau, \\ dU_5 &= -\lambda_2 U_5 d\tau \end{aligned} \quad (107)$$

and by

$$\begin{aligned}
 dU_1 &= (-\lambda_1 U_1 - U_2 U_3) d\tau + \sqrt{\epsilon} dW_1, \\
 dU_2 &= (-\lambda_1 U_2 + 2U_1 U_3) d\tau + \sqrt{\epsilon} dW_2, \\
 dU_3 &= (-\lambda_1 U_3 - U_1 U_2) d\tau + \sqrt{\epsilon} dW_3, \\
 dU_4 &= -\lambda_2 U_4 d\tau + \sqrt{\epsilon} \cos(\tau/\epsilon) dW_4 - \sqrt{\epsilon} \sin(\tau/\epsilon) dW_5, \\
 dU_5 &= -\lambda_2 U_5 d\tau + \sqrt{\epsilon} \sin(\tau/\epsilon) dW_4 + \sqrt{\epsilon} \cos(\tau/\epsilon) dW_5.
 \end{aligned}
 \tag{108}$$

**Example 3** (A Symmetric Positive Semidefinite). The final example addresses the dissipative case.

$$\begin{aligned}
 dX_1 + (X_1 + X_1 X_2) d\tau &= \sqrt{\epsilon} dW_1, \\
 dX_2 + \left( \frac{1}{\epsilon} X_2 - X_1^2 \right) &= \sqrt{\epsilon} dW_2.
 \end{aligned}
 \tag{109}$$

This system was studied in [20] as a simple model for the numerical simulation of turbulent fluid flows. Here we have replaced the external forcing terms with a small white noise. The renormalized system has the form

$$\begin{aligned}
 dV_1 + V_1 d\tau &= \sqrt{\epsilon} dW_1, \\
 dV_2 &= 0.
 \end{aligned}
 \tag{110}$$

**5. Large time estimates for slowly varying processes.** The following general result establishes that time integrals of slowly varying processes against oscillatory or dissipative terms remain  $\mathcal{O}(1)$  with high probability on large time scales.

**Proposition 3.** *Suppose that, for  $\epsilon > 0$ ,  $Z^\epsilon \in \mathbb{C}^n$  solves*

$$dZ^\epsilon = \epsilon \Phi(t, Z^\epsilon) dt + e^\beta \Psi(t, Z^\epsilon) dW; \quad Z^\epsilon(0) = Z_0^\epsilon.
 \tag{111}$$

*Here we suppose that  $\beta > 1/2$ ,  $\Phi$  and  $\Psi$  are uniformly bounded in time for bounded subsets of  $\mathbb{C}^n$  in the sense that, for any  $R > 0$*

$$\sup_{t>0, \|x\| \leq R} \|\Phi(t, x)\| < \infty, \quad \sup_{t>0, \|x\| \leq R} \|\Psi(t, x)\| < \infty,
 \tag{112}$$

*and assume that for some constant  $K > 0$*

$$\mathbb{P} \left( \sup_{t \in [0, T/\epsilon]} \|Z^\epsilon(t)\| > K \right) := \phi(\epsilon) \xrightarrow{\epsilon \downarrow 0} 0.
 \tag{113}$$

*Then*

(i) *For any  $\sigma \in \mathbb{R} \setminus \{0\}$  and any analytic function  $f : \mathbb{C}^n \rightarrow \mathbb{C}$*

$$\mathbb{P} \left( \sup_{t \in [0, T/\epsilon]} \left| \int_0^t e^{i\sigma s} f(Z^\epsilon) ds \right| > C \right) \xrightarrow{\epsilon \downarrow 0} 0,
 \tag{114}$$

*where  $C := C(f, \sigma, T)$ .*

(ii) *Suppose that  $Z^\epsilon$  take values in  $\mathbb{R}^n$ , and assume that  $\lambda, \sigma \geq 0$  with  $\sigma \neq \lambda$ . Then for any  $C^2$  function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$*

$$\mathbb{P} \left( \sup_{t \in [0, T/\epsilon]} \left| \int_0^t e^{-\lambda(t-s)} e^{-\sigma s} f(Z^\epsilon) ds \right| > C \right) \xrightarrow{\epsilon \downarrow 0} 0,
 \tag{115}$$

*where  $C := C(f, \sigma, T)$ .*

*Proof.* In both cases the Itô formula gives

$$\begin{aligned}
 df(Z^\epsilon) &= \epsilon \sum_{k=1}^n \frac{\partial f(Z^\epsilon)}{\partial z_k} \Phi_k(t, Z^\epsilon) dt + \epsilon^\beta \sum_{k=1}^n \frac{\partial f(Z^\epsilon)}{\partial z_k} (\Psi(t, Z^\epsilon) dW)_k \\
 &\quad + \frac{\epsilon^{2\beta}}{2} \sum_{k,l,j=1}^n \left( \frac{\partial^2 f(Z^\epsilon)}{\partial z_k \partial z_l} \Psi_{k,j}(t, Z^\epsilon) \Psi_{l,j}(t, Z^\epsilon) \right) dt \\
 &:= \epsilon H_D(t, Z^\epsilon) dt + \epsilon^\beta H_S(t, Z^\epsilon) dW + \epsilon^{2\beta} H_C(t, Z^\epsilon) dt.
 \end{aligned} \tag{116}$$

For (i), integrating by parts and using (116), we have

$$\begin{aligned}
 \int_0^t e^{i\sigma s} f(Z^\epsilon) ds &= \frac{e^{i\sigma t} f(Z^\epsilon(t)) - f(Z^\epsilon(0))}{i\sigma} \\
 &\quad + \epsilon \int_0^t \frac{e^{i\sigma s}}{i\sigma} (H_D(s, Z^\epsilon) + \epsilon^{2\beta-1} H_C(s, Z^\epsilon)) ds \\
 &\quad + \epsilon^\beta \int_0^t \frac{e^{i\sigma s}}{i\sigma} H_S(s, Z^\epsilon) dW.
 \end{aligned} \tag{117}$$

Define

$$\begin{aligned}
 C_1 &:= \frac{2}{|\sigma|} \sup_{\|z\| \leq K} |f(z)|, \\
 C_2 &:= \frac{T}{|\sigma|} \sup_{t>0, \|z\| \leq K} (|H_D(t, z)| + |H_C(t, z)|), \\
 C_3 &:= \frac{1}{|\sigma|} \sup_{t>0, \|z\| \leq K} \|H_S(t, Z^\epsilon)\|.
 \end{aligned} \tag{118}$$

Note that (112) assures that these constants are finite. For the first term in (117), we have

$$\mathbb{P} \left( \sup_{t \in [0, T/\epsilon]} \left| \frac{e^{i\sigma t} f(Z^\epsilon(t)) - f(Z^\epsilon(0))}{i\sigma} \right| > C_1 \right) \leq \phi(\epsilon). \tag{119}$$

Next

$$\begin{aligned}
 &\mathbb{P} \left( \sup_{t \in [0, T/\epsilon]} \left| \epsilon \int_0^t \frac{e^{i\sigma s}}{i\sigma} (H_D(s, Z^\epsilon) + \epsilon^{2\beta-1} H_C(s, Z^\epsilon)) ds \right| > C_2 \right) \\
 &\leq \mathbb{P} \left( \sup_{t \in [0, T/\epsilon]} \frac{\epsilon}{|\sigma|} \int_0^t (|H_D(s, Z^\epsilon)| + |H_C(s, Z^\epsilon)|) ds > C_2 \right) \\
 &\leq \mathbb{P} \left( \sup_{t \in [0, T/\epsilon]} \frac{T(|H_D(t, Z^\epsilon(t))| + |H_C(t, Z^\epsilon(t))|)}{|\sigma|} > C_2 \right) \leq \phi(\epsilon).
 \end{aligned} \tag{120}$$

For the final term from (117), define the stopping time

$$\xi_{\epsilon, K} := \inf_{t \geq 0} \{\|Z^\epsilon(t)\| > K\}. \tag{121}$$

Splitting the integral and making use of Doob’s inequality

$$\begin{aligned}
 & \mathbb{P} \left( \sup_{t \in [0, T/\epsilon]} \left| \epsilon^\beta \int_0^t \frac{e^{i\sigma s}}{i\sigma} H_S(s, Z^\epsilon) dW(s) \right| > C_3 \right) \\
 & \leq \mathbb{P} \left( \sup_{t \in [0, T/\epsilon]} \left| \epsilon^\beta \int_0^t \frac{e^{i\sigma s}}{i\sigma} H_S(s, Z^\epsilon) \mathbb{1}_{\xi_{\epsilon, K} > s} dW(s) \right| > C_3 \right) \\
 & \quad + \mathbb{P}(\xi_{\epsilon, K} \leq T/\epsilon) \\
 & \leq \frac{\epsilon^{2\beta}}{C_3^2} \mathbb{E} \int_0^{T/\epsilon} \mathbb{1}_{\xi_{\epsilon, K} > s} \left\| \frac{e^{i\sigma s}}{i\sigma} H_S(s, Z^\epsilon) \right\|^2 ds + \psi(\epsilon) \\
 & \leq \frac{\epsilon^{2\beta}}{C_3^2} \mathbb{E} \int_0^{T/\epsilon} \mathbb{1}_{\xi_{\epsilon, K} > s} C_3^2 ds + \psi_1(\epsilon) \leq \epsilon^{2\beta-1} T + \psi(\epsilon).
 \end{aligned} \tag{122}$$

Setting  $C = C_1 + C_2 + C_3$  and applying (119), (120) and (122) with (117), we obtain the desired result.

For item (ii), the integration by parts gives

$$\begin{aligned}
 & e^{-\lambda t} \int_0^t e^{(\lambda-\sigma)s} f(Z^\epsilon) ds \\
 & = \frac{e^{-\sigma t} f(Z^\epsilon(t)) - e^{-\lambda t} f(Z^\epsilon(0))}{\lambda - \sigma} \\
 & \quad + \epsilon \int_0^t \frac{e^{-\lambda(t-s)} e^{-s\sigma}}{\lambda - \sigma} (H_D(s, Z^\epsilon) + \epsilon^{2\beta-1} H_C(s, Z^\epsilon)) ds \\
 & \quad + \epsilon^\beta \int_0^t \frac{e^{-\lambda(t-s)} e^{-s\sigma}}{\lambda - \sigma} H_S(s, Z^\epsilon) dW.
 \end{aligned} \tag{123}$$

We define constants  $C_1$  and  $C_2$  similarly to the previous case

$$\begin{aligned}
 C_1 & := \frac{2}{|\lambda - \sigma|} \sup_{\|z\| \leq K} |f(z)|, \\
 C_2 & := \frac{T}{|\lambda - \sigma|} \sup_{\|z\| \leq K, t > 0} (|H_D(t, Z^\epsilon)| + |H_S(t, Z^\epsilon)|).
 \end{aligned} \tag{124}$$

The estimates for the first two terms on the left hand side of (123) are carried out in the same manner as (119) and (120). For the stochastic integral terms we consider two cases. First when  $\lambda > 0$ , we take  $C_3 = 1$ , and we have

$$\begin{aligned}
 & \mathbb{P} \left( \sup_{t \in [0, T/\epsilon]} \left| \epsilon^\beta \int_0^t \frac{e^{-\lambda(t-s)}}{\lambda - \sigma} e^{-\sigma s} H_S(s, Z^\epsilon) dW(s) \right| > 1 \right) \\
 & \leq \mathbb{P} \left( \sup_{t \in [0, T/\epsilon]} \left| \int_0^t \frac{e^{-\lambda(t-s)}}{\lambda - \sigma} \mathbb{1}_{\xi_{\epsilon, K} > s} e^{-\sigma s} H_S(s, Z^\epsilon) dW(s) \right| > \epsilon^{-\beta} \right) \\
 & \quad + \mathbb{P}(\xi_{\epsilon, K} \leq T/\epsilon) \\
 & \leq CT \epsilon^{2\beta-1} + \phi(\epsilon),
 \end{aligned} \tag{125}$$

where we applied Lemma 2.2 with  $G(t) = \mathbb{1}_{\xi_{\epsilon, K} > t} e^{-\sigma t} H_S(t, Z^\epsilon)$  for the second inequality. On the other hand, if  $\lambda = 0$ , then  $\sigma > 0$  and we take  $C_3$  as in (118). The stochastic integral estimate can be made as in (122).  $\square$

**6. Appendix I: Stochastic convolution estimates.** We give here the proofs of Lemmas (2.2) and(2.3) which were used to estimate terms arising in the proof of Proposition 1, (i) and Proposition 2.

**Proof of Lemma (2.2).** Recall that

$$X(t) = \int_0^t e^{-A(t-s)} H(s) dW \tag{126}$$

is the solution of

$$dX + AXdt = HdW; \quad X(0) = 0. \tag{127}$$

Itô's lemma implies that

$$d\|X\|^2 + 2\langle AX, X \rangle = 2 \sum_j \langle H_{\cdot,j}, X \rangle dW_j + \|H\|^2 dt. \tag{128}$$

Using the Burkholder-Davis-Gundy inequality (c.f. [13], Theorem 3.28), we estimate

$$\begin{aligned} \mathbb{E} \left( \sup_{t \in [0, T_0]} \left| 2 \sum_j \int_0^t \langle H_{\cdot,j}, X \rangle dW_j \right| \right) &\leq C \mathbb{E} \left( \int_0^{T_0} \sum_j \langle H_{\cdot,j}, X \rangle^2 ds \right)^{1/2} \\ &\leq C \mathbb{E} \left( \int_0^{T_0} \|H\|^2 \|X\|^2 ds \right)^{1/2} \\ &\leq \frac{1}{2} \mathbb{E} \sup_{t \in [0, T_0]} \|X\|^2 + C \mathbb{E} \int_0^{T_0} \|H\|^2 ds. \end{aligned} \tag{129}$$

This estimate and (128) imply

$$\mathbb{E} \left( \sup_{t \in [0, T_0]} \|X\|^2 \right) \leq C \int_0^{T_0} \|H\|^2 ds. \tag{130}$$

The Chebyshev inequality therefore implies (23).

**Proof of Lemma (2.3).** Since the proof is very similar to Lemma 5.1 in [5], we shall be brief in details. The factorization method relies on the identity

$$\int_s^t (t - \sigma)^{\theta-1} (\sigma - s)^{-\theta} d\sigma = \frac{\pi}{\sin(\pi\theta)}, \tag{131}$$

which is valid for  $\theta \in (0, 1)$  and for any  $s < t$ . From (131) and the stochastic Fubini Theorem (see [8], Theorem 4.18) we infer

$$\begin{aligned} &\int_0^t e^{-A(t-s)} \mathcal{N}H(s) dW_s \\ &= \frac{\sin(\pi\theta)}{\pi} \int_0^t e^{-A(t-s)} \left( \int_s^t (t - \sigma)^{\theta-1} (\sigma - s)^{-\theta} d\sigma \right) \mathcal{N}H(s) dW_s \\ &= \frac{\sin(\pi\theta)}{\pi} \int_0^t (t - \sigma)^{\theta-1} e^{-A(t-\sigma)} \mathcal{N} \left( \int_0^\sigma (\sigma - s)^{-\theta} e^{-A(\sigma-s)} \mathcal{N}H(s) dW_s \right) d\sigma. \end{aligned} \tag{132}$$

Let  $\lambda_{mp} = \min\{A_{jj} > 0\}$ . By applying the decomposition (132) with the Chebyshev and Hölder inequalities we find that for any  $q > 2$

$$\begin{aligned} & \mathbb{P}\left(\sup_{t \in [0, T_0]} \left\| \int_0^t e^{-A(t-s)} \mathcal{N}H(s) dW \right\| > K\right) \\ & \leq C(\theta, q, \lambda_{mp}) \frac{T_0}{K^q} \left(\sup_{\sigma > 0} \mathbb{E} \left\| \int_0^\sigma (\sigma - s)^{-\theta} e^{-A(\sigma-s)} \mathcal{N}H(s) dW \right\|^q\right), \end{aligned} \tag{133}$$

where we can take

$$C(\theta, q, \lambda_{mp}) = \left(\frac{\sin(\pi\theta)}{\pi}\right)^q \left(\int_0^\infty \frac{e^{-(q'\lambda_{mp})\sigma}}{\sigma^{q'(1-\theta)}} d\sigma\right)^{q/q'}. \tag{134}$$

Note that this constant is finite for any choice of  $\theta > 1/q$ . To complete the proof we estimate the moments of

$$M(\sigma) := \int_0^\sigma (\sigma - s)^{-\theta} e^{-A(\sigma-s)} \mathcal{N}H(s) dW. \tag{135}$$

This process is Gaussian with mean zero and covariance matrix

$$V_{jk}^\theta(\sigma) = \delta_{jk} \int_0^\sigma (\sigma - s)^{-2\theta} e^{-2\lambda_j(\sigma-s)} \mathcal{N}_{jj} H_{jj}(s)^2 ds. \tag{136}$$

Since  $q > 2$ , we can chose  $\theta \in (1/q, 1/2)$  arbitrarily. Let  $\theta$  be fixed in this range, and take,  $V_j(\sigma) = V_{jj}^\theta(\sigma)$ . Notice that

$$V_j^* = \sup_{\sigma > 0} V_j(\sigma) < \infty. \tag{137}$$

For appropriately small values of  $s$ , the moment generating function of  $M(\sigma)$  is given by

$$\begin{aligned} \phi(s) &= \mathbb{E}(\exp(s\|M(\sigma)\|^2)) = \prod_{\lambda_j \neq 0} \mathbb{E}(\exp(s|M_{jj}(\sigma)|^2)) \\ &= \prod_{\lambda_j \neq 0} (2\pi V_j(\sigma))^{-1/2} \int_{-\infty}^\infty \exp\left(-\frac{1 - 2V_j(\sigma)s}{2V_j(\sigma)} x_j^2\right) dx_j \\ &= \exp\left(-\frac{1}{2} \sum_{j=1}^n \sum_{m=1}^\infty \frac{(2sV_j(\sigma))^m}{m}\right). \end{aligned} \tag{138}$$

Differentiating  $\phi(s)$ , we find

$$\mathbb{E}(|M(\sigma)|^{2k}) = \phi^k(0) \leq C(k) \left(\sum_{j=1}^n V_j^*\right)^k. \tag{139}$$

Applying (139) with  $2k > q$  to (133), one infers (46).

**7. Appendix II: Small noise asymptotic results.** In this section we present an alternative proof of the small noise asymptotic result in Corollary 1. Note that while the approach below is closer in spirit to classical work (see [11], for example) we are able to address the case of multiplicative noise.

Consider stochastically perturbed system

$$\begin{aligned} dX^\epsilon + [LX^\epsilon + B(X^\epsilon, X^\epsilon)] dt &= \epsilon^m \sigma(t, \epsilon, X^\epsilon) dW, \\ X^\epsilon(0) &= X_0^\epsilon \end{aligned} \tag{140}$$

where  $m > 0$ . The related deterministic system is given by

$$\begin{aligned} \frac{dx}{dt} + [Lx + B(x, x)] &= 0, \\ x(0) &= x_0. \end{aligned} \tag{141}$$

We assume that  $L$  is linear and either non-negative definite or anti-symmetric.  $B(\cdot, \cdot)$  is a bilinear form with the cancellation property

$$\langle B(y, x), x \rangle = 0. \tag{142}$$

As in the previous sections,  $W = (W^1, \dots, W^m)$  is a standard Brownian motion relative to a filtered probability space  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ . The diffusion term  $\sigma$  takes values in  $\mathbb{M}^{n \times m}$ . On  $\mathbb{M}^{n \times m}$  we use the Frobenius norm  $|A| = \left(\sum_{j,k} A_{j,k}^2\right)^{1/2}$  and impose the uniform Lipschitz condition

$$\begin{aligned} |\sigma(t, \epsilon, x) - \sigma(t, \epsilon, y)| &\leq K|x - y|, \\ |\sigma(t, \epsilon, x)| &\leq K(1 + |x|). \end{aligned} \tag{143}$$

Note that  $K$  is assumed to be independent of both  $\epsilon$  and  $t$ .

**Remark 4.** Under the conditions given for  $L, B$  and  $\sigma$  and assuming that  $X_0^\epsilon$  is  $\mathcal{F}_0$  measurable (140) admits a unique continuous solution. If, in addition, we assume that  $X_0^\epsilon \in L^p(\Omega)$  then

$$\mathbb{E} \int_0^T |X^\epsilon(t)|^p dt < \infty. \tag{144}$$

See [10] for detailed proofs.

We first establish sufficient conditions for convergence in  $L^p(\Omega)$ .

**Proposition 4** ( $L^p(\Omega)$  Convergence). *Let  $T > 0, p \geq 2, X_0^\epsilon \in L^p(\Omega)$  for  $\epsilon > 0$  and  $x_0$  be a fixed element in  $\mathbb{R}^n$ . Assume that for some appropriate constant  $C_0$*

$$\mathbb{E}|X_0^\epsilon - x_0|^p \leq \epsilon^{mp} C_0. \tag{145}$$

Then

$$\mathbb{E} \sup_{t \in [0, T]} (1 + |X^\epsilon(t)|^2)^{p/2} \leq C \tag{146}$$

and for  $\epsilon > 0$

$$\mathbb{E} \sup_{t \in [0, T]} |X^\epsilon(t) - x(t)|^p \leq \epsilon^{mp} C \tag{147}$$

where  $C = C(|x_0|, T, L, B, K, p)$ .

For the proof of Proposition 4 we shall use the following stochastic analogue of the Gronwall Lemma. See [12] for a more general formulation and the proof.

**Lemma 7.1.** *Suppose that  $Y, Z : [0, T] \times \Omega \rightarrow \mathbb{R}^+$  are stochastic processes so that*

$$\mathbb{E} \int_0^T (Y + Z) dt < \infty. \tag{148}$$

Assume that for any  $0 \leq S_a \leq S_b \leq T$

$$\mathbb{E} \left( \sup_{t \in [S_a, S_b]} Y(t) \right) \leq C_0 \mathbb{E} \left( Y(S_a) + \int_{S_a}^{S_b} (Y + Z) dt \right), \tag{149}$$

where  $C_0$  is independent of the choice of  $S_a, S_b$ . Then, there exists  $C = C(C_0, T)$ , such that

$$\mathbb{E} \left( \sup_{t \in [0, T]} Y(t) \right) \leq C \mathbb{E} \left( Y(0) + \int_0^T Z dt \right) \tag{150}$$

*Proof - Proposition 4.* We first establish (146). By applying Itô's lemma to  $1 + |X^\epsilon(t)|^2$ , using the cancellation property for  $B$  and then applying Itô's lemma to  $(1 + |X^\epsilon(t)|^2)^{p/2}$ , we discover

$$\begin{aligned} & d(1 + |X^\epsilon|^2)^{p/2} \\ &= -p(1 + |X^\epsilon|^2)^{p/2-1} \langle LX^\epsilon, X^\epsilon \rangle dt \\ &+ \frac{\epsilon^{2m} p}{2} (1 + |X^\epsilon|^2)^{p/2-1} \sum_{j,k} \sigma_{j,k}(t, \epsilon, X^\epsilon)^2 dt \\ &+ \epsilon^m p (1 + |X^\epsilon|^2)^{p/2-1} \sum_{j,k} X_j^\epsilon \sigma_{j,k}(t, \epsilon, X^\epsilon) dW^k \\ &+ \frac{\epsilon^{2m} p(p-2)}{2} (1 + |X^\epsilon|^2)^{p/2-2} \sum_k \left( \sum_j X_j^\epsilon \sigma_{j,k}(t, \epsilon, X^\epsilon) \right)^2 dt. \end{aligned} \tag{151}$$

Given the assumptions on  $L$  the first term on the right hand side of (151) is non-positive for all  $t > 0$ . Fix arbitrary  $0 \leq S_a < S_b \leq T$ . Integrate (151) between  $S_a$  and  $t$  and take a supremum over  $[S_a, S_b]$ . After taking an expected value we have

$$\begin{aligned} & \mathbb{E} \left( \sup_{t \in [S_a, S_b]} (1 + |X^\epsilon|^2)^{p/2} \right) \\ & \leq \mathbb{E}(1 + |X^\epsilon(S_a)|^2)^{p/2} \\ & + C(p) \mathbb{E} \int_{S_a}^{S_b} (1 + |X^\epsilon|^2)^{p/2-1} \sum_{j,k} \sigma_{j,k}(s, \epsilon, X^\epsilon)^2 ds \\ & + C(p) \mathbb{E} \int_{S_a}^{S_b} (1 + |X^\epsilon|^2)^{p/2-2} \sum_k \left( \sum_j X_j^\epsilon \sigma_{j,k}(s, \epsilon, X^\epsilon) \right)^2 ds \\ & + C(p) \mathbb{E} \sup_{t \in [S_a, S_b]} \left| \sum_{j,k} \int_{S_a}^t (1 + |X^\epsilon|^2)^{p/2-1} X_j^\epsilon \sigma_{j,k}(t, \epsilon, X^\epsilon) dW^k \right|. \end{aligned} \tag{152}$$

By applying the Cauchy-Schwartz inequality and using the Lipschitz assumption, we can write

$$\begin{aligned} & (1 + |X^\epsilon|^2)^{p/2-2} \sum_k \left( \sum_j X_j^\epsilon \sigma_{j,k}(s, \epsilon, X^\epsilon) \right)^2 \\ & \leq (1 + |X^\epsilon|^2)^{p/2-2} |X^\epsilon|^2 \sum_{j,k} \sigma_{j,k}(s, \epsilon, X^\epsilon)^2 \\ & \leq C(1 + |X^\epsilon|^2)^{p/2}. \end{aligned} \tag{153}$$



We estimate the stochastic integral terms using the Burkholder-Davis-Gundy inequality (see [13])

$$\begin{aligned}
 & \mathbb{E} \sup_{t \in [S_a, S_b]} \left| \sum_{j,k} \int_{S_a}^t (1 + |X^\epsilon|^2)^{p/2-1} X_j^\epsilon \sigma_{j,k}(t, \epsilon, X^\epsilon) dW^k \right| \\
 & \leq C \mathbb{E} \left( \int_{S_a}^{S_b} (1 + |X^\epsilon|^2)^{p-2} \sum_k \left( \sum_j X_j^\epsilon \sigma_{j,k}(t, \epsilon, X^\epsilon) \right)^2 dt \right)^{1/2} \\
 & \leq C \mathbb{E} \left[ \sup_{t \in [S_a, S_b]} (1 + |X^\epsilon|^2)^{p/4} \left( \int_{S_a}^{S_b} (1 + |X^\epsilon|^2)^{p/2} dt \right)^{1/2} \right] \\
 & \leq \frac{1}{2} \mathbb{E} \left( \sup_{t \in [S_a, S_b]} (1 + |X^\epsilon|^2)^{p/2} \right) + C \mathbb{E} \left( \int_{S_a}^{S_b} (1 + |X^\epsilon|^2)^{p/2} dt \right).
 \end{aligned} \tag{154}$$

Using the observations in (153) and (154) with (152) and rearranging, we have

$$\begin{aligned}
 & \mathbb{E} \left( \sup_{t \in [S_a, S_b]} (1 + |X^\epsilon|^2)^{p/2} \right) \\
 & \leq C \mathbb{E} \left( (1 + |X^\epsilon(S_a)|^2)^{p/2} + \int_{S_a}^{S_b} (1 + |X^\epsilon|^2)^{p/2} ds \right).
 \end{aligned} \tag{155}$$

Note that the constant  $C = C(K, p)$  above is independent of  $S_a, S_b$ . Applying Lemma 7.1 one infers

$$\mathbb{E} \sup_{t \in [0, T]} (1 + |X^\epsilon(t)|^2)^{p/2} \leq C(K, p, T) \mathbb{E}(1 + |X_0^\epsilon|^2)^{p/2}. \tag{156}$$

Given the assumptions on the initial data (145), we have the uniform bound

$$\mathbb{E}(1 + |X_0^\epsilon|^2)^{p/2} \leq C \mathbb{E}(1 + |x_0|^p) \tag{157}$$

which implies (146).

To establish (147), we again apply Itô's lemma to determine an evolution equation for  $|X^\epsilon - x|^p$

$$\begin{aligned}
 d|X^\epsilon - x|^p &= -p|X^\epsilon - x|^{p-2} \langle L(X^\epsilon - x), X^\epsilon - x \rangle dt \\
 & \quad - p|X^\epsilon - x|^{p-2} \langle B(X^\epsilon, X^\epsilon) - B(x, x), X^\epsilon - x \rangle dt \\
 & \quad + \frac{\epsilon^{2m} p}{2} |X^\epsilon - x|^{p-2} \sum_{j,k} \sigma_{j,k}(t, \epsilon, X^\epsilon)^2 dt \\
 & \quad + \epsilon^m p |X^\epsilon - x|^{p-2} \sum_{j,k} (X_j^\epsilon - x_j) \sigma_{j,k}(t, \epsilon, X^\epsilon) dW^k \\
 & \quad + \frac{\epsilon^{2m} p(p-2)}{2} |X^\epsilon - x|^{p-4} \sum_k \left( \sum_j (X_j^\epsilon - x_j) \sigma_{j,k}(t, \epsilon, X^\epsilon) \right)^2 dt.
 \end{aligned} \tag{158}$$

Using the cancellation assumption (142), one infers

$$\langle B(X^\epsilon, X^\epsilon) - B(x, x), X^\epsilon - x \rangle = \langle B(X^\epsilon - x, x), X^\epsilon - x \rangle. \tag{159}$$

Assumption (142) also allows us to determine an a priori bound for  $x$ , the solution of (141). Taking inner products we have

$$\frac{d}{dt}|x|^2 = -2\langle Lx, x \rangle \leq 0. \tag{160}$$

So, for  $t \in [0, T]$

$$|x(t)| \leq |x_0|. \tag{161}$$

Now fix  $0 \leq S_a \leq S_b \leq T$ . Integrate (158) from  $S_a$  to  $t$ , take a supremum over this time interval and then an expected value

$$\begin{aligned} & \mathbb{E} \left( \sup_{t \in [S_a, S_b]} |X^\epsilon - x|^p \right) \\ & \leq \mathbb{E} |X^\epsilon(S_a) - x(S_a)|^p + C(p, B, |x_0|) \mathbb{E} \int_{S_a}^{S_b} |X^\epsilon - x|^p dt \\ & \quad + \epsilon^{2m} C(K, p) \mathbb{E} \int_{S_a}^{S_b} |X^\epsilon - x|^{p-2} (1 + |X^\epsilon|^2) dt \\ & \quad + \epsilon^m C(p) \mathbb{E} \left( \sup_{t \in [S_a, S_b]} \left| \int_{S_a}^t |X^\epsilon - x|^{p-2} \sum_{j,k} (X_j^\epsilon - x_j) \sigma_{j,k}(t, \epsilon, X^\epsilon) dW^k \right| \right). \end{aligned} \tag{162}$$

Once again we make use of the Burkholder-Davis-Gundy inequality

$$\begin{aligned} & \epsilon^m C \mathbb{E} \sup_{t \in [S_a, S_b]} \left| \sum_{j,k} \int_{S_a}^t |X^\epsilon - x|^{p-2} (X_j^\epsilon - x_j) \sigma_{j,k}(t, \epsilon, X^\epsilon) dW^k \right| \\ & \leq \epsilon^m C \mathbb{E} \left( \int_{S_a}^{S_b} |X^\epsilon - x|^{2(p-2)} \sum_k \left( \sum_j (X_j^\epsilon - x_j) \sigma_{j,k}(t, \epsilon, X^\epsilon) \right)^2 dt \right)^{1/2} \\ & \leq \epsilon^m C(K) \mathbb{E} \left( \int_{S_a}^{S_b} |X^\epsilon - x|^{2(p-1)} (1 + |X^\epsilon|^2) dt \right)^{1/2} \\ & \leq \frac{1}{2} \mathbb{E} \left( \sup_{t \in [S_a, S_b]} |X^\epsilon - x|^p \right) + \epsilon^{mp} C(K, p) \left( \int_{S_a}^{S_b} (1 + |X^\epsilon|^2) dt \right)^{p/2} \\ & \leq \frac{1}{2} \mathbb{E} \left( \sup_{t \in [S_a, S_b]} |X^\epsilon - x|^p \right) + \epsilon^{mp} C(K, p, T) \int_{S_a}^{S_b} (1 + |X^\epsilon|^2)^{p/2} dt. \end{aligned} \tag{163}$$

Using this estimate and (146), we have

$$\begin{aligned} & \mathbb{E} \left( \sup_{t \in [S_a, S_b]} |X^\epsilon - x|^p \right) \\ & \leq 2 \mathbb{E} |X^\epsilon(S_a) - x(S_a)|^p + C \int_{S_a}^{S_b} |X^\epsilon - x|^p dt \\ & \quad + \epsilon^{mp} C \int_{S_a}^{S_b} (1 + |X^\epsilon|^2)^{p/2} dt. \end{aligned} \tag{164}$$

Noting, once again that the constants  $C(K, p, T, |x_0|)$  above are independent of  $S_a$ . We apply Lemma 7.1. together with (145) and (146), we finally deduce

$$\mathbb{E} \left( \sup_{t \in [0, T]} |X^\epsilon - x|^p \right) \leq C \mathbb{E} |X_0^\epsilon - x_0|^p + \epsilon^{mp} C \int_0^T (1 + |X^\epsilon|^2)^{p/2} \leq \epsilon^{mp} C. \quad (165)$$

□

As a corollary of Proposition 4, we infer convergence of  $X^\epsilon$  to  $x$  in probability along with a rate of convergence.

**Proposition 5** (Convergence in Probability). *Let  $T > 0$ ,  $p \geq 2$ ,  $X_0^\epsilon \in L^p(\Omega)$  and  $x_0$  be a fixed element in  $\mathbb{R}^n$ . Suppose that*

$$\mathbb{E} |X_0^\epsilon - x_0|^p \leq \epsilon^{mp} C_0. \quad (166)$$

Then for any  $\gamma < m$

$$\mathbb{P} \left( \sup_{t \in [0, T]} |X^\epsilon - x| > \epsilon^\gamma \right) \leq C \epsilon^{(m-\gamma)p}. \quad (167)$$

*Proof.* The result follows directly from (147) by applying Markov's Inequality. □

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