Products and Permutations

1. Properties of the Product Functor

In each of the following, let A, B, G and H be groups.

a) Prove that $G \times H$ is isomorphic to $H \times G$. Consider the map $\phi\colon G \times H \to H \times G$ given by $(g, h) \mapsto (h, g)$.

This map is a group homomorphism:

$$\phi((g, h)(g', h')) = \phi(gg', hh') = (hh', gg') = (h, g)(h', g') = \phi(g, h)\phi(g', h').$$

It is injective: if $(h, g) = (h', g')$, then $h = h'$ and $g = g'$, so $(g, h) = (g', h')$. It is surjective: every (h, g) is $\phi(g, h)$. So this map ϕ is an isomorphism.

b) Given a homomorphism $f\colon A \to G$, construct a homomorphism from $A \times H$ to $G \times H$. Find its kernel and image. Define the homomorphism $\phi = f \times 1_H$ by $\phi(a, h) = (f(a), h)$. This map is a homomorphism because f is. Its kernel is the set of points (a, h) such that $(f(a), h) = (1, 1)$, which is $\ker f \times 1$. The image is the set of points (g, h) such that $g \in f(A)$ and $h \in H$, or $f(A) \times H$.

c) Prove that $A \times H$ is isomorphic to $G \times H$ if and only if A is isomorphic to G. A homomorphism is an isomorphism if it is injective (has trivial kernel) and surjective (the image is the whole target group). By part b), the kernel of ϕ is 1×1 if and only if the kernel of f is 1, and the image of ϕ is $G \times H$ if and only if the image of f is G.

d) Given homomorphisms $f\colon A \to G$, $g\colon B \to H$, construct a homomorphism from $A \times B$ to $G \times H$. Find its kernel and image. Define the homomorphism $\phi = f \times g$ by $\phi(a, b) = (f(a), g(b))$. This map is a homomorphism because f and g are. Its kernel is the set of pairs (a, b) such that $f(a) = 1$ and $g(b) = 1$, which is $\ker f \times \ker g$. Its image is $f(A) \times f(B)$.

e) Given homomorphisms $f\colon A \to G$, $g\colon A \to H$, construct a homomorphism from A to $G \times H$. Find its kernel and image. Define the homomorphism by $\phi(a) = (f(a), g(a))$. This map is a homomorphism because f and g are. Its kernel is the set of elements a such that $(f(a), g(a)) = (1, 1)$, or the intersection of the kernels of f and g. The image is $f(A) \times f(B)$.

2. Permutations

a) In S_9, write the following in function notation: $p = (12)(45)(78)$,
$q = (135)(79)$, $r = (2468)$, pq, qr, pr.

$$p = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 2 & 1 & 3 & 5 & 4 & 6 & 8 & 7 & 9 \end{pmatrix}$$

$$q = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 3 & 2 & 5 & 4 & 1 & 6 & 9 & 8 & 7 \end{pmatrix}$$

$$r = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 1 & 4 & 3 & 6 & 5 & 8 & 7 & 2 & 9 \end{pmatrix}$$

$$pq = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 3 & 1 & 4 & 5 & 2 & 6 & 9 & 7 & 8 \end{pmatrix}$$

$$qr = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 3 & 4 & 5 & 6 & 1 & 8 & 9 & 2 & 7 \end{pmatrix}$$

$$pr = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 2 & 5 & 3 & 6 & 4 & 7 & 8 & 1 & 9 \end{pmatrix}$$

b) Write out the elements of S_3 in function and cycle notations.
The identity is the empty cycle or

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$$

The transpositions are

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} = (23),$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} = (12),$$

and

$$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} = (13).$$

The 3-cycles are

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = (123)$$

and

$$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} = (132).$$

c) What do you obtain when you multiply two 2-cycles in S_3? Two 3-cycles? A 2-cycle by a 3-cycle?

The product of two 2-cycles is the identity if the cycles are the same, or a 3-cycle if they are different: $(ab)(bc) = (abc)$. The product of two 3-cycles is the identity if they are different, and the other 3-cycle if they are the same: $(abc)^2 = (acb)$. The product of a 2-cycle and a 3-cycle is a 2-cycle: $(ab)(abc) = (bc)$.