APMA 1650 - Spring 2021

Midterm Exam 1 - Solutions

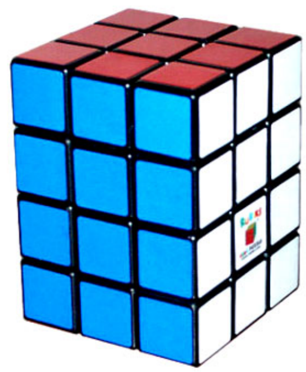


Figure 1: A $4 \times 3 \times 3$ Rubik's cube

Problem 1. (Rubik's rectangular prism) (20pts) Suppose you have a $4 \times 3 \times 3$ Rubik's "cube" that has 4 sides that are 4×3, and 2 sides that are 3×3 (see Figure 1). Every side of the puzzle has a unique color to start with. If you're allowed to remove all the stickers and replace them in any position, how many unique ways are there to color the "cube" assuming each side is distinct (i.e. there is one fixed orientation that determines a given coloring)?

Solution: The four 3×4 sides have $3 \times 4 = 12$ stickers and the two 3×3 sides have $3 \times 3 = 9$ stickers. This gives a total of

$$4 \times 12 + 2 \times 9 = 66 \text{ stickers.}$$

The number of unique ways to color the cube is then just the number of ways to partition 66 squares into 4 groups of 12 and 2 groups of 9 (each group corresponding to a unique color) and is given by the multinomial coefficient

$$\binom{66}{12, 12, 12, 12, 9, 9}.$$

Problem 2. (Campuswire) (20 pts) Campuswire isn't loading and you are running out of patience. Every time you try to load the page, it has a probability of $1/4$ of actually loading. You wake up this morning full of patience and decide to try to load Campuswire. Let X be the number of tries it takes to load.

a. (10 pts) What is the probability that X equals an odd number $1, 3, 5, \ldots$? Write your answer as a fraction in simplest form.

b. (10 pts) Every time a page fails to load you lose patience. Lets assume your patience level starts this morning at the value a and after every failed page-load it gets halved (so after one failed page load you have $a/2$ patience left and after two failed attempts you have $a/4$). What is your expected patience level when the page finally loads? Write your answer as a fraction in simplest form.

Solution:

a. We see that $X \sim \text{Geometric}(1/4)$ being the number of failures before a successful page load. Therefore it has a PMF $P_X(k) = (1/4)(3/4)^{k-1}$ for $k = 1, 2, \ldots$ The probability of being odd is then just

$$
\begin{aligned}
P(X = \text{odd}) &= \sum_{k=0}^{\infty} P_X(2k+1) \\
&= \sum_{k=0}^{\infty} \left(\frac{1}{4}\right)\left(\frac{3}{4}\right)^{2k} \\
&= \frac{1}{4}\sum_{k=0}^{\infty} \left(\frac{9}{16}\right)^{k} \\
&= \frac{1}{4}\frac{1}{1-9/16} \qquad \text{geometric series} \\
&= \frac{16}{4(7)} = \frac{4}{7}
\end{aligned}
$$

b. The number of failures before the page loads is just $X - 1$ and the amount of patience left at this point is $a2^{-(X-1)}$. Therefore using LOTUS, the expected patience left over is just

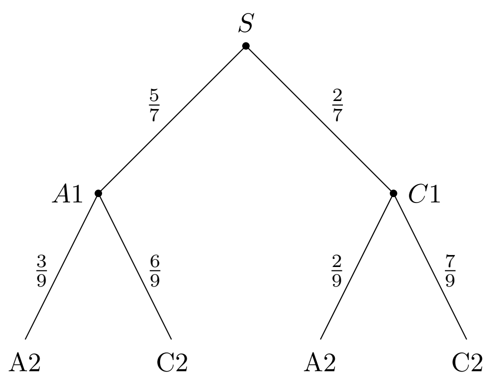
$$
\begin{aligned}
E[a2^{-(X-1)}] &= aE[2^{-(X-1)}] \\
&= a\sum_{k=1}^{\infty} 2^{-(k-1)}\left(\frac{1}{4}\right)\left(\frac{3}{4}\right)^{k-1} \\
&= \frac{a}{4}\sum_{k=1}^{\infty} \left(\frac{1}{2}\right)^{k-1}\left(\frac{3}{4}\right)^{k-1} \\
&= \frac{a}{4}\sum_{k=1}^{\infty} \left(\frac{3}{8}\right)^{k-1} \\
&= \frac{a}{4}\frac{1}{1-3/8} \qquad \text{geometric series} \\
&= \frac{2a}{5}
\end{aligned}
$$

Problem 3. (Recruiters) (20 pts) A job recruiter is looking to hire someone for a fancy data science job. They have two lists: List 1 contains the names of 5 APMA students and 2 CSCI students. List 2 contains the names of 2 APMA students and 6 CSCI students. One of the names is then randomly selected from list 1 and added to list 2. A final name is then randomly selected from the newly augmented list 2. Given that the final name selected was a CSCI student, what is the probability that an APMA student's name was originally selected from list 1 and added to list 2?

Solution: Let $A1$ and $C1$ be the events that and APMA or a CS student was chosen from list 1 respectively. Let $A2$ and $C2$ be the events that an APMA or a CS student was chosen from the new list 2 respectively. We want to compute $P(A1|C2)$. To do this, we use Bayes' rule

$$P(A1|C2) = \frac{P(C2|A1)P(A1)}{P(C1)}.$$

To aid us, we will draw a probability tree



From this, we see that $P(C2|A1) = 6/9$, $P(A1) = 5/7$ and

$$P(C2) = \left(\frac{6}{9}\right)\left(\frac{5}{7}\right) + \left(\frac{7}{9}\right)\left(\frac{2}{7}\right)$$

Therefore we have

$$\begin{aligned} P(A1|C2) &= \frac{\left(\frac{6}{9}\right)\left(\frac{5}{7}\right)}{\left(\frac{6}{9}\right)\left(\frac{5}{7}\right) + \left(\frac{7}{9}\right)\left(\frac{2}{7}\right)} \\ &= \frac{6(5)}{6(5) + 7(2)} \\ &= \frac{15}{22}. \end{aligned}$$

Problem 4. (Conflicts) (20 pts) Suppose m professors randomly choose from n time slots to hold their final exams. If two professors pick the same time slot, we say that they are in conflict. (If three professors all pick the same time slot, that gives three pairs of professors in conflict.) What is the expected number of pairs of professors in conflict? Your answer should depend on m and n.

Solution: Label the professors $1, 2, \dots m$. For any two professors i and j, $i \neq j$ let

$$X_{i,j} = \begin{cases} 1 & \text{if professors } i \text{ and } j \text{ have a conflict} \\ 0 & \text{otherwise} \end{cases}$$

Note that $EX_{i,j}$ is just the probability that any two professors conflict, this is just

$$EX_{i,j} = 1/n$$

since one of the professors has a probability of $1/n$ of choosing what the other one picked. The total number of conflicts is then

$$X = \frac{1}{2} \sum_{\substack{i,j=1 \\ i \neq j}}^{m} X_{i,j}$$

where we have divided by 2 to account for the fact that $X_{i,j}$ and $X_{j,i}$ are the same conflict. By linearity of expectation we have

$$\begin{aligned} EX &= \frac{1}{2} \sum_{\substack{i,j=1 \\ i \neq j}}^{m} EX_{i,j} \\ & \frac{1}{2} \sum_{\substack{i,j=1 \\ i \neq j}}^{m} \frac{1}{n} \\ &= \frac{m(m-1)}{2n}, \end{aligned}$$

where we used the fact that the $\sum_{\substack{i,j=1 \\ i \neq j}}^{n}$ has $m(m-1)$ terms. This can be deduced by

$$\sum_{\substack{i,j=1 \\ i \neq j}}^{n} 1 = \sum_{i=1}^{m} \sum_{\substack{j=1 \\ j \neq i}}^{m} 1 = \sum_{j=1}^{m} (m-1) = m(m-1).$$

or by drawing a grid.

Alternate Solution: We could instead define

$$X_i = \text{ the number of other professors who conflict with professor i}$$

Since every other of the $m-1$ professors has an independent $1/n$ chance of conflicting with a given professor i, then we see that $X_i \sim \text{Binomial}(m-1, 1/n)$ and therefore has

$$EX_i = \frac{m-1}{n}$$

The total number of conflicting pairs is then

$$X = \frac{1}{2}\sum_{i=1}^{m} X_i$$

where we have divided by two to correct for double counting pairs. Using linearity of of expectation, we get that the expected number of conflicting pairs is

$$EX = \frac{1}{2}\sum_{i=1}^{m} EX_i = \frac{1}{2}\sum_{i=1}^{m} \frac{m-1}{n} = \frac{m(m-1)}{2n}.$$

Problem 5. (Powerball) (20 pts) Suppose in a lottery you pick five different numbers from 1 to 90. Then five different winning numbers are drawn equally likely at random from 1 to 90.

a. (10 pts) Let X be the number of winning numbers that you picked. What is the range and PMF of X?

b. (10 pts) If you pick two winning numbers, you win 20 dollars. For three, you win 150 dollars. For four, you win 5,000 dollars. If all five match, you win a million dollars! What are your expected winnings? Give an exact answer to 3 decimal places. (You may use a combinatorial calculator). For partial credit, give a formula that involves the PMF from part a).

Solution:

a. We can think about dividing up the 90 lottery numbers into 5 winning numbers and 85 non-winning numbers. When you select your numbers (or equivalently when the lottery numbers are selected) this is the same as sampling 5 numbers, unordered without replacement from these 90 numbers. The number of winning numbers X has a range $0, 1, 2, 3, 4, 5$ and is given by a hypergeometric distribution with PMF

$$P_X(k) = \frac{\binom{5}{k}\binom{85}{5-k}}{\binom{90}{5}}, \quad k = 0, 1, 2, 3, 4, 5.$$

b. The expected winnings are given by

$$\begin{aligned} EX &= 20P_X(2) + 150P_X(3) + 5,000P_X(4) + 1,000,000P_X(5) \\ &= 20\frac{\binom{5}{2}\binom{85}{3}}{\binom{90}{5}} + 150\frac{\binom{5}{3}\binom{85}{2}}{\binom{90}{5}} + 5,000\frac{\binom{5}{4}\binom{85}{1}}{\binom{90}{5}} + 1,000,000\frac{\binom{5}{5}\binom{85}{0}}{\binom{90}{5}} \\ &\approx 20(0.022473639) + 150(0.0008123) + 5000(0.00000967) + 1000000(0.000000023) \\ &\approx 0.64266778 \end{aligned}$$

where we used a combinatorial calculator. So you should expect to win 64 cents on average... not really a great income strategy if lotto tickets cost more than that (which they do by design).