Chapter 2: Combinatorics

TAs: Jack, Aaron, Linghai

2.1.0 Counting Methods

- **Sampling**: sampling from a set means choosing an element from that set. We often **draw** a sample at random from a given set in which each element of the set has equal chance of being chosen.

- **With or without replacement**: usually we draw multiple samples from a set. If we put each object back after each draw, we call this **sampling with replacement**. In this case a single object can be possibly chosen multiple times. For example, if $A = \{a_1, a_2, a_3, a_4\}$ and we pick 3 elements with replacement, a possible choice might be (a_3, a_1, a_3). Thus "with replacement" means "repetition is allowed." On the other hand, if repetition is not allowed, we call it **sampling without replacement**.

- **Ordered or unordered**: If ordering matters (i.e.: $a_1, a_2, a_3 \neq a_2, a_3, a_1$), this is called **ordered sampling**. Otherwise, it is called **unordered**.

- ordered sampling with replacement
- ordered sampling without replacement
- unordered sampling without replacement
- unordered sampling with replacement

2.1.1 Ordered With Replacement

Here we have a set with n elements (e.g.: $A = \{1, 2, 3, \cdots . n\}$), and we want to draw k samples from the set such that ordering matters and repetition is allowed. For example, if $A = \{1, 2, 3\}$ and $k = 2$, there are 9 different possibilities:

In general, we can argue that there are k positions in the chosen list: (Position 1, Position 2, ..., Position k). There are n options for each position. Thus, when ordering matters and repetition is allowed, the total number of ways to choose k objects from a set with n elements is

$$n \times n \times ... \times n = n^k$$

2.1.2 Ordered Without Replacement

Consider the same setting as above, but now repetition is not allowed. For example, if $A = \{1, 2, 3\}$ and $k = 2$, there are 6 different possibilities:

1. (1,2);
2. (1,3);
3. (2,1);
4. (2,3);
5. (3,1);
6. (3,2).

Any of the chosen lists in the above setting (choose k elements, ordered and no repetition) is called a k-permutation of the elements in set A. We use the following notation to show the number of k-permutations of an n-element set:

$$P_k^n = n \times (n-1) \times \ldots \times (n-k+1).$$

The number of k-permutations of n distinguishable objects is given by

$$P_k^n = \frac{n!}{(n-k)!}, \quad \text{for } 0 \le k \le n.$$

2.1.3 Unordered Without Replacement

We take on the idea from the previous slide. If I have a set of 3 elements {A, B, C}, and we take 2 elements without replacement.
When the sampling is ordered, we have 3P2 = 6 possible outcomes.
When the sampling is not ordered, this means getting AB is the same as getting BA, as we only care about what elements we are getting.

In a general case, we have n elements and we are going to sample k unordered elements.
We denote the number of possible outcomes as $\binom{n}{k}$

For each possible outcome, we can arrange the k elements in k! different ways if we DO care about order.

That is, $P_k^n = \binom{n}{k} \times k!$ Therefore, $\binom{n}{k} = \frac{n!}{k!(n-k)!}$

2.1.4 Unordered With Replacement

If I have a set of 3 elements {A, B, C}, and we take 2 elements with replacement. This means each time we can choose from A,B,& C, rather than the case where once we choose A, we cannot get A again.
For this case, we have 6 possible outcomes: AA, AB, AC, BB, BC, CC.
We can represent these outcomes in a different way: (2,0,0),(1,1,0),(1,0,1),(0,2,0),(0,1,1),(0,0,2).
In fact, finding the number of outcomes is the same as finding the number of distinct solutions to the following function: x+y+z = 2, where x,y,z can only take values from {0,1,2}.

More generally, when we have a set with n elements, and we want to take k elements without order BUT with replacement, we can use the following theorem in textbook:

Theorem 2.1
The number of distinct solutions to the equation

$$x_1 + x_2 + \ldots + x_n = k, \text{ where } x_i \in \{0, 1, 2, 3, \ldots\} \tag{2.3}$$

is equal to

$$\binom{n+k-1}{k} = \binom{n+k-1}{n-1}.$$

The number of ways = the number of distinct solutions

Practice Problems

5. A bag contains 100 balls of 75 white balls and 25 black balls. At each time, we randomly pick a ball from the bag and check the color and throw the ball away. Let's say we repeat this n times.

   (a) What is the probability that the number of observed black balls among $n = 10$ trials is $k = 4$?

(a) What is the probability that the number of observed black balls among $n = 10$ trials is $k = 4$?

We want to pick 4 black balls out of 25 and 6 white balls out of 75. Thus, there are $\binom{25}{4} \cdot \binom{75}{4}$ ways to do this, but there are $\binom{100}{10}$ ways to choose any 10 balls from 100, so this means the probability is

$$P(4 \text{ black balls}) = \frac{\binom{25}{4}\binom{75}{6}}{\binom{100}{10}} = \frac{25! \cdot 75! \cdot 10! \cdot 90!}{4! \cdot 21! \cdot 69! \cdot 6! \cdot 100!} \approx 0.1471$$

(b) 6 points. A group of 4 ECON and 12 APMA students are randomly divided into 4 groups of 4. What is the probability that each group includes an ECON student?

Solution 2 counting

First we count the size of the sample space S. The multinomial coefficient gives us:

$$\binom{16}{4,4,4,4} = \frac{16!}{4!4!4!4!}$$

Each outcome is equally likely. We need to count how many have one ECON student in each group. The four ECON students can be distributed to the four groups in 4! ways. The APMA students can then be distributed in groups of 3:

$$\binom{12}{3,3,3,3} = \frac{12!}{3!3!3!3!}$$

The size of the event over the size of the sample space reduces to:

$$\frac{12 \cdot 8 \cdot 4}{15 \cdot 14 \cdot 13}$$

7. The lottery in the city of Problandia works like this: 100 balls numbered 0-99 are placed in an urn, and 5 balls are drawn, giving an ordered sequence of 5 numbers. Once balls are drawn, they are not replaced before the next one is drawn. Citizens buy tickets with 5 numbers of their choosing. The jackpot is awarded for matching all 5 numbers in the right order.

a) How many possible outcomes of the lottery drawing are there?

b) Mrs. Bernoulli plays 5 numbers corresponding to the days-of-the-month on which each of her 5 children were born (each being born on a different day). She buys several tickets, one for each of the permutations of these 5 numbers. How many tickets does she buy? What is her chance of winning the jackpot?

c) A separate prize is awarded for getting the correct set of 5 numbers, but not necessarily in the right order. What is the probability that any given ticket will win this prize? What is Mrs. Bernoulli's probability of winning this prize?

a) order matters: have 100 #s to choose from & selecting 5 => permutation

$= \boxed{100 \cdot 99 \cdot 98 \cdot 97 \cdot 96}$ or $100P5$ or $\binom{100}{5} 5!$

b) If we think about this interms of Mrs. Bernoulli winning, we need to know the various Events she can win for.

She has 5 dates that are different, and she can arrange them however she wishes: so there are $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$ tickets she can buy.

$\boxed{\text{\# tickets she buys: } 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120}$

$$\text{Prob(win)} = \frac{\text{\# tickets purchase}}{\text{total \# tickets}} = \boxed{\frac{5!}{100 \cdot 99 \cdot 98 \cdot 97 \cdot 96}}$$

c) Now $\binom{100}{5}$ is # of ways to select any 5 #s when order doesn't matter.

$$\boxed{P(\text{win}) = \frac{1}{\binom{100}{5}}}$$

MRS Bernoulli also has $\boxed{P(\text{win}) = \frac{1}{\binom{100}{5}}}$ b/c all of her tickets have the same five numbers.

Exercise 2. A committee of 5 is to be selected from a group of 6 men and 9 women. If the selection is made randomly, what is the probability that the committee consists of 3 men and 2 women?

Exercise 2. A committee of 5 is to be selected from a group of 6 men and 9 women. If the selection is made randomly, what is the probability that the committee consists of 3 men and 2 women?

There are 15 people total, thus the total number of possible ways of selecting 5 people is $\binom{15}{5} = 3003$.

Number of possible ways of choosing 3 men from the 6 men $= \binom{6}{3} = 20$.

Number of possible ways of choosing 2 women from the 9 women $= \binom{9}{2} = 36$.

Number of ways to choose 3 men and 2 women $= 20 \times 36 = 720$

Probability that committee consists of 3 men and 2 women $= \frac{720}{3003} = 0.240$

4. Suppose a class has 300 students.

(a) The professor will pick 100 students equally likely at random and give these 100 students a "free" A. In lecture we computed the probability that a given group of 100 students would be chosen. Suppose you are in the class, what is the probability that you will be one of the 100 students chosen and hence receive a "free" A?

(b) The exams are handed back one by one in a random order (i.e. an ordering of the 300 students is chosen equally likely at random from all orderings and the exams are handed back in this order). If Alice and Bob are two students in class, what is the probability that Alice will get her test back before Bob?

Note: You may have intuition about the solutions to both parts of this problem. Be sure to set up an appropriate probability space, define the relevant events, and justify their probability.

4. (a) Straightforward way: sample space = all configurations with equal probability.

$$\frac{p(\text{config. where I'm chosen})}{p(\text{all config.})} = \frac{\binom{299}{99}}{\binom{300}{100}}.$$

Alternatively, we intuitively know this is $\frac{1}{3}$. To justify, imagine there are 300 slots among which 100 are free A ones that students randomly sit in. Since no one is special and no slot is special,

$$p(\text{me in a free A slot}) = \frac{\#\text{ free A slots}}{\#\text{ of all slots}}.$$

In this point of view the sample space is the set of all possible slots I'm in, each with equal probability, by symmetry (or can be justified by direct computation from the sample space in the first method.)

(b) Straightforward: the sample space = all configurations with equal probability. The number of combinations in which Alice before Bob is $\frac{300!}{2}$. Divided by the number of all combinations, 300!, we get $\frac{1}{2}$.

We can see this faster by symmetry again. Since neither Alice nor Bob has any advantage, they should have the same chance to go first. More specifically the sample space, $S = \{\text{Alice first}\} \amalg \{\text{Bob first}\}$ (disjoint union–meaning the union of two set with empty intersection). By symmetry, $p(Alice) = p(Bob)$ but $p(Alice) + p(Bob) = p(S) = 1$. So $p(Alice) = p(Bob) = 0.5$.

End of chapter Problems

Problem 16

I have 10 red and 10 blue cards. I shuffle the cards and then label the cards based on their orders: I write the number one on the first card, the number two on the second card, and so on. What is the probability that

a. All red cards are assigned numbers less than or equal to 15?
b. Exactly 8 red cards are assigned numbers less than or equal to 15?

(a) P = (10C10) (10C5) / (20C15)
(b) P = (10C8) (10C7) / (20C15)

End of chapter Problems

Problem 17

I have two bags. Bag 1 contains 10 blue marbles, while Bag 2 contains 15 blue marbles. I pick one of the bags at random, and throw 6 red marbles in it. Then I shake the bag and choose 5 marbles (without replacement) at random from the bag. If there are exactly 2 red marbles among the 5 chosen marbles, what is the probability that I have chosen Bag 1?

There is conditional probability involved.
A = the event that there are exactly 2 red among 5 chosen.
Calculate P(A|bag1) and P(A|bag2) separately.
P(A|bag1) = (6C2)(10C3) / (16C5)
P(A|bag2) = (6C2)(15C3) / (21C5)
P(bag1)=P(bag2)=½ because we choose a bag randomly.
Use Bayes' rule.
P(bag1|A) = P(A|bag1)P(bag1) / (P(A|bag1)P(bag1)+P(A|bag2)P(bag2)) -> plug in numbers