Chapter 4

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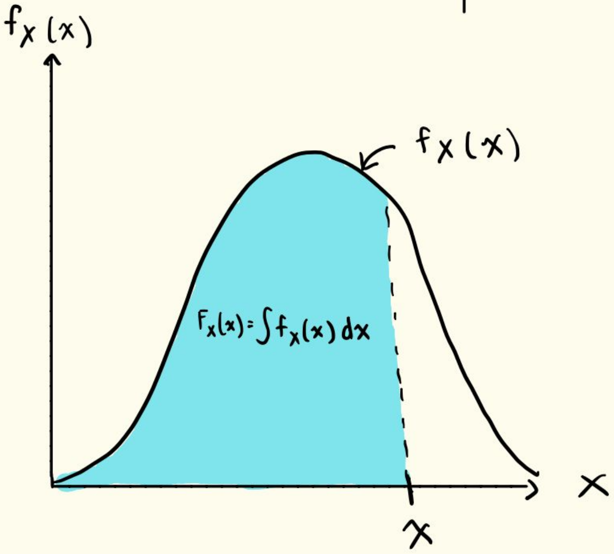
Summary

- 4.1 Continuous Random Variables
  - Solved Problems
- 4.2 Special Distributions
  - Uniform, Exponential, Normal, Gamma, etc.
  - Solved Problems
- 4.3 Mixed Random Variables
  - Solved Problems
- Q&A

Continuous Random Variables

Discrete R.V.	Continuous R.V.
PMF : $P(X = k)$	PDF : $f_X(x)$
CDF : $P(X \le k)$	CDF : $F_X(x) = P(X \le x)$



6-0 Properties of $f_X(x)$:

1) $f_X(x) \ge 0$ for all $x \in \mathbb{R}$
2) $\int_{-\infty}^{\infty} f_X(x)\,dx = 1$
3) $P(a < X \le b) = F_X(b) - F_X(a)$
$= \int_a^b f_X(x)\,dx$

Range : R_X = all possible values of random variable X

Expected Value (for continuous R.V.)

$$\boxed{E[X] = \int_{-\infty}^{\infty} x\, f_X(x)\, dx}$$

ex) X is a continuous R.V. w/ pdf : $f_X(x) = \begin{cases} 2x & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$

$$\begin{aligned} E[X] &= \int_{-\infty}^{\infty} x\, f_X(x)\, dx \\ &= \int_{-\infty}^{0} x(0)\, dx + \int_{0}^{1} x(2x)\, dx + \int_{1}^{\infty} x(0)\, dx \\ &= \int_{0}^{1} 2x^2\, dx = \frac{2}{3} x^3 \Big|_0^1 = \boxed{\frac{2}{3}} \end{aligned}$$

LOTUS : $\boxed{E[g(X)] = \int_{-\infty}^{\infty} g(x)\, f_X(x)\, dx}$

Properties : ① $E[aX+b] = E[aX] + E[b]$

$= a\,E[X] + b$

② $E[X_1 + X_2 + \dots + X_n] = E[X_1] + E[X_2] + \dots + E[X_n]$

Variance

$$\boxed{Var[X] = E[X^2] - (E[X])^2} \quad (*)$$

STEPS : ① Find $E[X]$

② Find $E[X^2]$ (LOTUS)

③ Plug into $(*)$

Properties : $Var[aX + b] = a^2 Var[X]$

$Var[X_1 + X_2 + \ldots + X_n] = Var[X_1] + \ldots + Var[X_n]$

★ ONLY WHEN INDEPENDENT

Functions of Continuous Random Variables

X is a R.V.

$Y = g(X) \rightarrow Y$ is a R.V.

If we know the CDF (or pdf) of X, how can we find the CDF (and pdf of Y)?

1. Find $F_X(x)$ if you are given $f_X(x)$
2. Define $F_Y(y) = P(Y \leq y)$
3. Plug in $g(X)$ for Y: $P(g(X) \leq y)$
4. Isolate X: $P(X \leq g^{-1}(y)) = F_X(g^{-1}(y))$
5. We know $F_X(x)$. Evaluate $F_X(x)$ at $g^{-1}(y)$ and this equals $F_Y(y)$
6. Take the derivative of $F_Y(y)$ to find $f_Y(y)$ (the pdf of Y)

Example: X is a RV w/ pdf: $f_X(x) = \begin{cases} 4x^3 & 0 < x \le 1 \\ 0 & \text{else} \end{cases}$

$Y = \frac{1}{X}$. Find $F_Y(y)$ and $f_Y(y)$

① Find $F_X(x)$ if you are given $f_X(x)$

$F_X(x) = \int_0^x 4x^3 \, dx = x^4$

② Define $F_Y(y) = P(Y \le y)$

$F_Y(y) = P(Y \le y)$

③ Plug in $g(X)$ for Y: $P(g(X) \le y)$

$P(Y \le y) = P\left(\frac{1}{X} \le y\right)$

④ Isolate X: $P(X \le g^{-1}(y)) = F_X(g^{-1}(y))$

$P\left(\frac{1}{X} \le y\right) = P\left(X \le \frac{1}{y}\right) = F_X\left(\frac{1}{y}\right)$

⑤ we know $F_X(x)$. Evaluate $F_X(x)$ at $g^{-1}(y)$

$F_X\left(\frac{1}{y}\right) = \left(\frac{1}{y}\right)^4 = \frac{1}{y^4} = F_Y(y)$

⑥ Take the derivative of $F_Y(y)$ to find $f_Y(y)$ (the pdf of Y)

$f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{d}{dy}\left(\frac{1}{y^4}\right) = \frac{-4}{y^5} \qquad 1 \le y$

Problem 1: Problem 3 from Textbook

Problem 3

Let X be a continuous random variable with PDF

$$f_X(x) = \begin{cases} x^2 + \frac{2}{3} & 0 \leq x \leq 1 \\ 0 & otherwise \end{cases}$$

a. Find $E(X^n)$, for $n = 1, 2, 3, \cdots$.

b. Find the variance of X.

Problem 1 : Problem 3 From Textbook

$$f_X(x) = \begin{cases} x^2 + \frac{2}{3} & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) Find $E(X^n)$ for $n = 1, 2, 3, \ldots$

$$E[X^n] = \int_0^1 x^n \left(x^2 + \frac{2}{3}\right) dx = \int_0^1 x^{n+2} + \frac{2}{3} x^n \, dx$$

$$= \frac{x^{n+3}}{n+3} + \frac{2}{3(n+1)} x^{n+1} \Big|_{x=0}^{1} = \frac{1^{n+3}}{n+3} + \frac{2}{3(n+1)} 1^{n+1}$$

$$= \boxed{\frac{1}{n+3} + \frac{2}{3(n+1)}}$$

(b) Find Var [X]

$$E[X] = E[X^1] = \frac{1}{(1)+3} + \frac{2}{3(1+1)} = \frac{1}{4} + \frac{2}{6} = \frac{7}{12}$$

using $E[X^n] = \frac{1}{n+3} + \frac{2}{3(n+1)}$

$$E[X^2] = \frac{1}{2+3} + \frac{2}{3(2+1)} = \frac{1}{5} + \frac{2}{9} = \frac{9}{45} + \frac{10}{45} = \frac{19}{45}$$

$$Var[X] = E[X^2] - (E[X])^2$$

$$= \frac{19}{45} - \left(\frac{7}{12}\right)^2 = \boxed{0.0819}$$

Problem 2: Problem 5 from Textbook

Problem 5

Let X be a continuous random variable with PDF

$$f_X(x) = \begin{cases} \frac{5}{32}x^4 & 0 < x \leq 2 \\ 0 & otherwise \end{cases}$$

and let $Y = X^2$.

a. Find the CDF of Y.
b. Find the PDF of Y.
c. Find EY.

Problem 2: problem 5 from Textbook

$$f_X(x) = \begin{cases} \frac{5}{32} x^4 & 0 < x \le 2 \\ 0 & \text{else} \end{cases}$$

$$Y = X^2$$

a) Find CDF of Y

$$F_X(x) = \int_0^x \frac{5}{32} x^4 \, dx = \frac{1}{32} x^5 \Big|_0^x = \frac{1}{32} x^5 \quad \text{for} \quad 0 < x \le 2$$

$$F_Y(y) = P(Y \le y) = P(X^2 \le y) = P(X \le \sqrt{y}) = F_X(\sqrt{y})$$

$$F_X(\sqrt{y}) = \frac{1}{32} (\sqrt{y})^5 = \boxed{\frac{1}{32} y^{\frac{5}{2}}}$$

for $0 < y \le 4$

(b) Find the PDF of Y

$$f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{d}{dy} \frac{1}{32} y^{\frac{5}{2}} = \frac{5}{2} \frac{1}{32} y^{\frac{3}{2}} = \frac{5}{64} y^{\frac{3}{2}}$$

for $0 \le y \le 4$

(c) Find $E[Y]$

$$E[Y] = \int_{-\infty}^{\infty} y f_Y(y)\, dy = \int_0^4 y \frac{5}{64} y^{\frac{3}{2}}\, dy$$

$$= \int_0^4 \frac{5}{64} y^{\frac{5}{2}}\, dy = \frac{2}{7} \cdot \frac{5}{64} y^{\frac{7}{2}} \Big|_{y=0}^{4} = \frac{2}{7} \cdot \frac{5}{64} \cdot 128$$

$$= \frac{2}{7} \cdot 10 = \boxed{\frac{20}{7}}$$

Problem 3: Problem 10 from Textbook

Problem 10

Let X be a continuous random variable with PDF

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \qquad \text{for all } x \in \mathbb{R}.$$

and let $Y = \sqrt{|X|}$. Find $f_Y(y)$.

Problem 3: Problem 10 from Textbook

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}} \quad \text{for all } x \in \mathbb{R}$$

$$Y = \sqrt{|X|}$$

Find $f_Y(y)$

1) Find $F_X(x)$ from $f_X(x)$

$$F_X(x) = \int_{-\infty}^{x} f_X(x)\,dx = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\,dx$$

2) Define $F_Y(y)$

$$F_Y(y) = P(Y \leq y)$$

3) Plug in $G(X)$ for Y

$$P(Y \leq y) = P(\sqrt{|X|} \leq y)$$

4) Isolate X

$$P(\sqrt{|X|} \le y) = P(|X| \le y^2) = P(-y^2 \le X \le y^2)$$

5) Evaluate integral to solve for $F_Y(y)$

$$F_Y(y) = P(-y^2 \le X \le y^2) = \int_{-y^2}^{y^2} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\, dx$$

6) Find $f_Y(y)$ by taking the derivative wrt/y of $F_Y(y)$

$$f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{d}{dy} \int_{-y^2}^{y^2} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\, dx$$

$$\frac{d}{dx} \int_{g(x)}^{f(x)} h(t)\, dt = h(f(x))\, f'(x) - h(g(x))\, g'(x)$$

$$h(f(y))\, f'(y) = h(y^2)(2y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{y^4}{2}} (2y)$$

$$h(g(y))\, g'(y) = h(-y^2)(-2y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{y^4}{2}} (-2y)$$

$$f_Y(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{y^4}{2}} \left(2y - (-2y)\right) = \boxed{\frac{4y}{\sqrt{2\pi}} e^{-\frac{y^4}{2}}}$$

4.2.1 Uniform Distributions

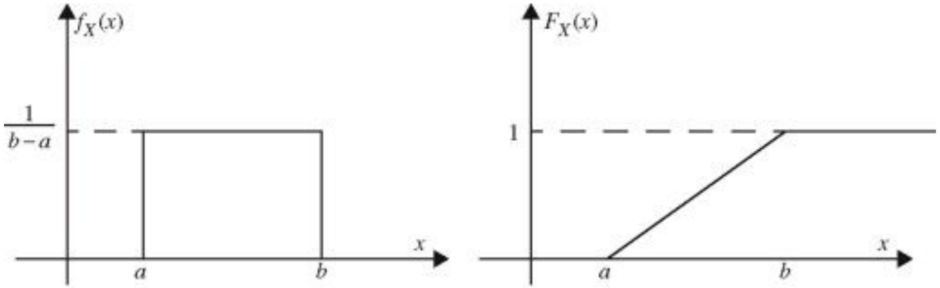
- Uniform random variables work the same way even when you're looking at a joint distribution! You just divide 1 by an area instead of a length.

$$f(x) = \begin{cases} \frac{1}{b-a} & \text{for } a \le x \le b, \\ 0 & \text{for } x < a \text{ or } x > b \end{cases}$$

$$F(x) = \begin{cases} 0 & \text{for } x < a \\ \frac{x-a}{b-a} & \text{for } a \le x \le b \\ 1 & \text{for } x > b \end{cases}$$

$$E(X) = \frac{1}{2}(b+a)$$

$$E(X^2) = \frac{b^3 - a^3}{3b - 3a} \qquad V(X) = \frac{1}{12}(b-a)^2$$



(a) (b)

4.2.2 Exponential Distributions

- Just think of these as the time elapsed between events
- From homework, related to both Geometric *and* Poisson Distributions
- Memoryless Property
  - Constrained to nonnegative real numbers
  - The memoryless distribution is exponential

$$\Pr(X > t + s \mid X > t) = \Pr(X > s).$$

$$f(x;\lambda) = \begin{cases} \lambda e^{-(\lambda x)} & x \geq 0, \\ 0 & x < 0. \end{cases}$$

$$F(x;\lambda) = \begin{cases} 1 - e^{-(\lambda x)} & x \geq 0, \\ 0 & x < 0. \end{cases}$$

$$\mathrm{E}[X] = \frac{1}{\lambda}.$$

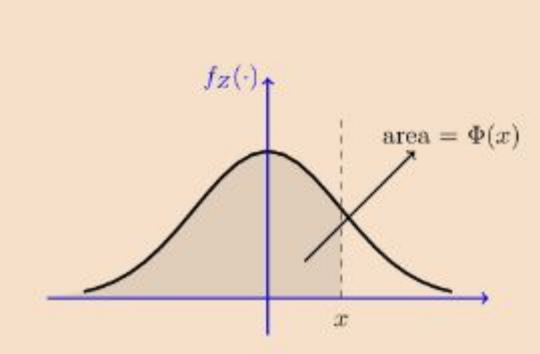
$$\mathrm{Var}[X] = \frac{1}{\lambda^2},$$

$$\mathrm{E}[X^n] = \frac{n!}{\lambda^n}.$$

4.2.3 Normal (Gaussian) Distributions

- If you’re doing anything with stats, this comes up everywhere (too much)
- You can “standardize” by using the formula (X-mu)/sigma

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$



$$F_X(x) = P(X \leq x) = \Phi\left(\frac{x-\mu}{\sigma}\right),$$

$$P(a < X \leq b) = \Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right).$$

1. Alice and Bob each uniformly and independently select a point from the interval $[0, 2]$.

   (a) What is the joint distribution of the two chosen points?

   (b) What is the probability that the distance between these two points is no more than 1?

Solution

(a) Uniform on $R = [0,2] \times [0,2]$, which has area 4. The joint density is $f(a,b) = \frac{1}{4}$ on R, 0 elsewhere.

(b) The region R_0 in R satisfying $|a - b| \leq 1$, or $-1 \leq a - b \leq 1$. Since the distribution is uniform, the probability is the fraction of the area, i.e. $\frac{area(R_0)}{area(R)} = \frac{3}{4}$.

6. Let $X_1, X_2, \ldots X_n$ be a random sample of size n form a uniform distribution on the interval $[\theta_1, \theta_2]$. Let $Y = \min(X_1, X_2, \ldots, X_n)$.

(a) Find the density function for Y. (Hint find the cdf and then differentiate.)

(b) Compute the expectation of Y.

Solution:
That hint seems like a good idea, lets do that. To get the cdf, we have

$$F_Y(y) = \mathbb{P}(Y \leq y) = 1 - \mathbb{P}(Y > y) = 1 - \prod_{i=1}^{n} \mathbb{P}(X_i > y)$$

since $Y > y$ if and only if all of the $X_i > y$, and the X_i are independent. This gives us

$$F_Y(y) = 1 - \left(\frac{\theta_2 - y}{\theta_2 - \theta_1}\right)^n$$

for $y \in [\theta_1, \theta_2]$, and $F_Y(y) = 1$ for $y > \theta_2$ and 0 for $y < \theta_1$.

Differentiating to get the density, we obtain

$$\begin{aligned} f_Y(y) &= \frac{d}{dy} F_Y(y) \\ &= -n \left(\frac{\theta_2 - y}{\theta_2 - \theta_1}\right)^{n-1} \left(\frac{-1}{\theta_2 - \theta_1}\right) \\ &= n \frac{(\theta_2 - y)^{n-1}}{(\theta_2 - \theta_1)^n} \end{aligned}$$

for $y \in [\theta_1, \theta_2]$, and $f_Y(y) = 0$ otherwise.

7. Suppose that X and Y are independent random variables that are exponential with parameter λ and let

$$Z = \frac{X}{X+Y}.$$

Show that for $0 < z < 1$

$$F_z(z) = P(Z \leq z) = z,$$

i.e. the random variable Z is uniformly distributed over (0,1).

Solution: To find Z's distribution, we want to look at $P(Z \le z)$. Let $z \in (0,1)$.

$$\begin{aligned}
P(Z \le z) &= P(\frac{X}{X+Y} \le z) \\
&= P(X \le Xz + Yz) \\
&= P(X - Xz \le Yz) \\
&= P(X \le \frac{Yz}{1-z}) \\
&= P(X\frac{1-z}{z} \le Y) \\
&= P((X,Y) \in \{(x,y) : x(1-z)/z \le y\}) \\
&= \int_0^\infty \int_{x(1-z)/z}^\infty \frac{1}{\beta} e^{-\frac{y}{\beta}} \frac{1}{\beta} e^{-\frac{x}{\beta}} dy dx \\
&= \int_0^\infty -e^{-\frac{y}{\beta}} \Big|_{x(1-z)/z}^\infty \frac{1}{\beta} e^{\frac{-x}{\beta}} dx \\
&= \int_0^\infty \frac{1}{\beta} e^{\frac{-x}{\beta} - \frac{x(1-z)}{\beta z}} dx \\
&= \int_0^\infty \frac{1}{\beta} e^{\frac{-x}{\beta}(1 + \frac{1-z}{z})} dx \\
&= \int_0^\infty \frac{1}{\beta} e^{\frac{-x}{\beta}\frac{1}{z}} dx \\
&= -ze^{\frac{-x}{\beta}\frac{1}{z}} \Big|_0^\infty \\
&= z
\end{aligned}$$

$F_z(z) = z$, for $0 < z < 1$ is the distribution function for a uniform (0,1) random variable.

40. One method of arriving at economic forecasts is to use a consensus approach. A forecast is obtained from each of a large number of analysts; the average of these individual forecasts is the consensus forecast. Suppose that the individual 1996 January prime interest-rate forecasts of all economic analysts are approximately normally distributed with mean 7% and standard deviation 2.6%. If a single analyst is randomly selected from among this group, what is the probability that the analyst's forecast of the prime interest rate will

(a) exceed 11%?

(b) be less than 9%?

We want $P(X > .11)$. Knowing our mean is .07 and our standard deviation is .026, we can standardize. $P(X > .11) = P\Big(\frac{X - .07}{.026} > \frac{.11 - .07}{.026}\Big)$. This gives us $P(Z > 1.54) =$ **.0618**, by looking at the z-table.

Solution: Now we want $P(X < .09)$, we do a similar procedure of standardizing and get $P(Z < .77)$. However, our z-table is written for $P(Z \geq .77)$, so when we look up the value in the table, we need to make sure that we do 1 - the value that we find. $P(Z \geq .77) = .2206$, so $P(Z < .77) = 1 - .2206 =$ **.7794**.

6. Suppose X and Y are independent normal random variables with the same mean $\mu = 0$ and the same variance σ^2. Then the pair (X, Y) gives the coordinates of a random point in the plane. Let $A = A(X, Y)$ denote the area of the circle centered at the origin passing through (X, Y). (That is to say, the circle has center (0,0) and radius $R = \sqrt{X^2 + Y^2}$. Compute the expected value of A.

$E[A]$. Know $A = \pi R^2$ because its the area of a circle with radius R.

$$E[A] = E[\pi R^2] = E[\pi(X^2+Y^2)].$$

so we just need to calculate $E[X^2]$ & $E[Y^2]$ by linearity of expectation.

$$X \sim N(0,\sigma^2) \quad \& \quad Y \sim N(0,\sigma^2).$$

$$V[X] = \sigma^2 = E[X^2] - \underbrace{(E[X])^2}_{0}$$ b/c its Normal with mean 0.

$$\sigma^2 = E[X^2].$$

Similarly, $V[Y] = \sigma^2$.

$$\Rightarrow E[\pi(X^2+Y^2)] = \pi\left(E[X^2] + E[Y^2]\right)$$

$$= \pi(\sigma^2+\sigma^2) = \boxed{2\pi\sigma^2}$$

6. Let consider two pdfs

$$f_1(x) = \mathbb{1}_{[0,1]}(x), \qquad f_2(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}.$$

For some $0 < \alpha < 1$, let define

$$f(x) = \alpha f_1(x) + (1-\alpha) f_2(x).$$

(a) Show that $f(x)$ is a probability density function.

(b) Let X_1 be a random variable whose pdf is $f_1(x)$ and X_2 be a random variable whose pdf is $f_2(x)$ where

$$E[X_1] = \mu_1, \quad \mathrm{Var}[X_1] = \sigma_1^2, \qquad E[X_2] = \mu_2, \quad \mathrm{Var}[X_2] = \sigma_2^2.$$

Let X be a random variable whose pdf is $f(x)$. Find $E[X]$ and $\mathrm{Var}[X]$.

Since $f_1(x)$ and $f_2(x)$ are probability density functions, we know $f_1(x) \geq 0$, $f_2(x) \geq 0$, $\int_{-\infty}^{\infty} f_1(x)dx = 1$, and $\int_{-\infty}^{\infty} f_2(x)dx = 1$. Since $0 < \alpha < 1$ (and so $0 < 1 - \alpha < 1$), we know $f(x) \geq 0$. Also,

$$\begin{aligned}
\int_{-\infty}^{\infty} f(x)dx &= \int_{-\infty}^{\infty} \alpha f_1(x) + (1-\alpha)f_2(x)dx \\
&= \alpha \int_{-\infty}^{\infty} f_1(x)dx + (1-\alpha)\int_{-\infty}^{\infty} f_2(x)dx \text{ by linearity of integrals} \\
&= \alpha + (1-\alpha) \\
&= 1
\end{aligned}$$

$\therefore f(x)$ is a probability density function.

Since $X_1 \sim \text{Uniform}([0,1])$, $\mu_1 = \frac{1}{2}$ and $\sigma_1^2 = \frac{1}{12}$. Similarly, X_2 follows a standard normal distribution, so we know $\mu_2 = 0, \sigma_2^2 = 1$.

Thus,

$$\begin{aligned}
E[X] &= \int_{-\infty}^{\infty} xf(x)dx \\
&= \alpha \int_{-\infty}^{\infty} xf_1(x)dx + (1-\alpha)\int_{-\infty}^{\infty} xf_2(x)dx \\
&= \alpha E[X_1] + (1-\alpha)E[X_2] \\
&= \alpha\left(\frac{1}{2}\right) + (1-\alpha)(0) \\
&= \frac{1}{2}\alpha
\end{aligned}$$

To find the variance, we will use $\text{Var}(X) = E[X^2] - E[X]^2$. First, we compute $E[X^2]$.

$$
\begin{aligned}
E[X^2] &= \int_{-\infty}^{\infty} x^2 f(x)dx \\
&= \int_{-\infty}^{\infty} \alpha x^2 f_1(x) + (1-\alpha)x^2 f_2(x)dx \\
&= \alpha E[X_1^2] + (1-\alpha)E[X_2^2] \\
&= \alpha(\text{Var}(X_1) + E[X_1]^2) + (1-\alpha)(\text{Var}(X_2) + E[X_2]^2) \\
&= \alpha(\frac{1}{12} + \frac{1}{4}) + (1-\alpha)(1) \\
&= 1 - \frac{2}{3}\alpha
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\text{Var}(X) &= E[X^2] - E[X]^2 \\
&= 1 - \frac{2}{3}\alpha - \left(\frac{1}{2}\alpha\right)^2 \\
&= 1 - \frac{2}{3}\alpha - \frac{1}{4}\alpha^2
\end{aligned}
$$

4. Suppose we know that starting from any particular moment in time, the amount of time we must wait for the next person to enter the store is a continuous random variable with probability density function

$$f(x) = \begin{cases} 10e^{-10x}, & \text{if } x \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

Here x is measured in hours.

(a) Find the probability that we must wait more than 6 minutes for the next person to enter the store.

(b) Find the expected time before the next person enters the store.

(c) Find the variance.

X is in hours, so 6 minutes is 1/10 hours.

a) $P(X \geq 1/10) = \int_{1/10}^{\infty} 10e^{-10x}\,dx$

$$= -e^{-10x}\Big|_{1/10}^{\infty} = \boxed{e^{-1}}$$

b) $E[X] = \int_0^{\infty} x\,10e^{-10x}\,dx$ $\quad u = x \quad dv = 10e^{-10x}$ $\quad du = dx \quad v = -e^{-10x}$.

$$= -xe^{-10x}\Big|_0^{\infty} + \int_0^{\infty} e^{-10x}\,dx$$

$$= 0 + -\frac{1}{10}e^{-10x}\Big|_0^{\infty} = \boxed{1/10}$$

Expect to wait 6 minutes before next person enters.

Also, note this is just an exponential ~~distribution~~ density w/ mean 1/10, so can do this without any calculus.

c) $V[X] = E[X^2] - (E[X])^2$.

$E[X^2] = \int_0^{\infty} x^2\,10e^{-10x}\,dx \rightarrow \int$-by-parts twice! $\quad u = x^2 \quad dv = 10e^{-10x}\,dx$ $\quad du = 2x\,dx \quad v = -e^{-10x}$

$$= \underbrace{-x^2e^{-10x}\Big|_0^{\infty}}_{0} + \int_0^{\infty} 2xe^{-10x}\,dx \qquad u = x \quad dv = 2e^{-10x}\,dx \qquad du = dx \quad v = -\frac{1}{5}e^{-10x}$$

$$= \underbrace{-\frac{1}{5}xe^{-10x}\Big|_0^{\infty}}_{0} + \int_0^{\infty} \frac{1}{5}e^{-10x}\,dx$$

$$= -\frac{1}{50}e^{-10x}\Big|_0^{\infty} = 1/50.$$

$$V[X] = \frac{1}{50} - \frac{1}{10^2} = \boxed{\frac{1}{100}}$$