APMA 1650

CH5+6

linghai_liu@brown.edu
janet_qi@brown.edu
jackson_webster@brown.edu
amanda_zhang@brown.edu

Ch. 5.1 + 5.2: Joint Probability, Joint CDF, Marginals, Conditioning and Independence

Ch. 5.3.1: Covariance and Correlation

Ch. 6.1.1 + 6.1.2: Joint Distributions, Sums of Random Variables

Ch. 6.2.0 + 6.2.2 + 6.1.4: Probability Bounds, Markov, Chebyshev, and Cauchy Schwartz

#1: Joint Probabilities

Problem 1

Consider two random variables X and Y with joint PMF given in Table 5.4

Joint PMF of X and Y in Problem 1

	$Y = 1$	$Y = 2$
$X = 1$	$\frac{1}{3}$	$\frac{1}{12}$
$X = 2$	$\frac{1}{6}$	0
$X = 4$	$\frac{1}{12}$	$\frac{1}{3}$

a. Find $P(X \leq 2, Y > 1)$.
b. Find the marginal PMFs of X and Y.
c. Find $P(Y = 2|X = 1)$.
d. Are X and Y independent?

#2: Sum of Random Variables

Problem 4

Let $X_1, X_2, \cdots, X_n$ be i.i.d. random variables, where $X_i \sim Bernoulli(p)$. Define

$$\begin{aligned} Y_1 &= X_1 X_2, \\ Y_2 &= X_2 X_3, \\ &\vdots \\ Y_{n-1} &= X_{n-1} X_n, \\ Y_n &= X_n X_1. \end{aligned}$$

If $Y = Y_1 + Y_2 + \cdots + Y_n$, find

1. $E[Y]$,
2. $\mathrm{Var}(Y)$.

#3: Joint, Marginal, and Conditional Functions

(30 points) Let $f(x, y)$ be the joint density function of X and Y given by

$$f(x,y) = \begin{cases} cx^2y, & -1 \le x \le 1, 0 \le y \le \min\{\sqrt{|1-x|}, \sqrt{|1+x|}\}, \\ 0, & \text{otherwise} \end{cases}$$

(a) (2 points) Determine c which makes $f(x, y)$ a joint probability density function.

#4: Convolutions and Sums of Variables

Let X and Y be independent standard normal RVs.

If $Z = X + Y$, what is the PDF of Z?

#5: Markov and Chebyshev Inequalities

Suppose the average height of a raccoon is 10 inches.

(a) Give an upper bound on the probability that a given raccoon is at least 15 inches tall

(b) If the standard deviation of the height distribution is 2 inches, find a lower bound on the probability that a given raccoon is between 5 and 15 inches tall.

#6: Variances and Correlations

Let X and Y be random variables with standard deviations σ_X, σ_Y.
Prove that:

(a)

$$0 \leq V(\frac{X}{\sigma_X} + \frac{Y}{\sigma_Y}) = 2(1 + \mathrm{COR}(X, Y))$$

(b)

$$0 \leq V(\frac{X}{\sigma_X} - \frac{Y}{\sigma_Y}) = 2(1 - \mathrm{COR}(X, Y))$$

(c)

$$-1 \leq \mathrm{COR}(X, Y) \leq 1$$

#7: More Joint Density

4. **(4 pts).** Let the following joint density function of Y_1, Y_2 be given by

$$f(y_1, y_2) = \begin{cases} e^{-(y_1+y_2)} & y_1 > 0, \quad y_2 > 0, \\ 0, & elsewhere. \end{cases}$$

Find the density function of random variable $\frac{Y_1}{Y_2}$.

#8: More Correlation

3. Let X and Y be random variables whose joint pdf $f(x, y)$ is defined to be

$$f(x,y) = \begin{cases} 1 & 0 \leq y \leq 1, y-1 \leq x \leq 1-y \\ 0 & \text{elsewhere} \end{cases}$$

(a) Find $E[XY]$ and $Cov(X, Y)$.

(b) Are X and Y independent?

(c) Find the correlation coefficient $\rho_{X,Y}$ for X and Y.