APMA 1650

Easy Practice Midterm Exam 2 -Solutions

Problem 1. (Conditional) Let X and Y be random variables with joint density given by:

$$f_{XY}(x,y) = \begin{cases} cx & \text{if} \quad 0 \le x \le 1 \quad \text{and} \quad 0 \le x^2 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

a. Find the value of c for which $f_{XY}(x,y)$ is a valid density.

b. Find $E[X]$.

c. Find the conditional density of X given $Y = y$.

Solution

a. The normality condition is

$$1 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{XY}(x,y) \mathrm{d}x\mathrm{d}y = \int_0^1 \int_{x^2}^1 cx\mathrm{d}y\mathrm{d}x = \int_0^1 cx(1-x^2)\mathrm{d}x = c\left(\frac{x^2}{2} - \frac{x^4}{4}\right)_0^1 = c/4$$

Therefore we need $c = 4$ for f_{XY} to be a valid density.

b. To find EX we integrate

$$\begin{aligned} EX &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x f_{XY}(x,y) \mathrm{d}x\mathrm{d}y = \int_0^1 \int_{x^2}^1 4x^2 \mathrm{d}y\mathrm{d}x \\ &= \int_0^1 4x^2(1-x^2)\mathrm{d}x = 4\left(\frac{x^3}{3} - \frac{x^5}{5}\right)_0^1 = \frac{8}{15} \end{aligned}$$

c. First we find the marginal for Y for each $y \in [0,1]$

$$f_Y(y) = \int_0^{\sqrt{y}} 4x\mathrm{d}x = 2x^2\Big|_0^{\sqrt{y}} = 2y.$$

Therefore we see that for each $y \in (0,1)$

$$f_{X|Y}(x|y) = \frac{f_{XY}(x,y)}{f_Y(y)} = \begin{cases} \frac{2x}{y} & 0 \le x \le \sqrt{y} \\ 0 & \text{otherwise} \end{cases}.$$

Problem 2. (Covariance) Let X be the score of a random student on a final exam and let Y be the number of hours spent studying. Suppose that X and Y are related by $X = 2Y + Z$, where Z is independent of Y. Further suppose $\mathrm{Var}(X) = 40$ and $\mathrm{Var}(Y) = 2$.

a. What is $\text{Var}(Z)$?

b. What is $\text{Cov}(X, Y)$?

Solution:

a. Since Z and Y are independent, we know that $\text{Cov}(Z, Y) = 0$. This means by the properties of covariance

$$\text{Var}(X) = \text{Var}(2Y + Z) = \text{Var}(2Y) + \text{Var}(Z) - 2\text{Cov}(2Y, X) = 4\text{Var}(Y) + \text{Var}(Z).$$

Solving for $\text{Var}(Z)$ gives

$$\text{Var}(Z) = \text{Var}(X) - 4\text{Var}(Y) = 40 - 8 = 32.$$

b. To find $\text{Cov}(X, Y)$, we again use the properties of covariance

$$\text{Cov}(X, Y) = \text{Cov}(2Y + Z, Y) = 2\text{Cov}(Y, Y) + \text{Cov}(Z, Y) = 2\text{Var}(Y) = 4.$$

Problem 3. (Alice and Bob) Alice and Bob each uniformly and independently select a point from the interval $[0, 2]$.

a. What is the joint distribution of the two chosen points?

b. What is the probability that the distance between these two points is no more than 1?

Solution:

a. Let X be the point chosen by Alice and Y be the point chosen by Bob, these are both independent Uniform$(0, 2)$ random variables with PDFs

$$f_X(x) = \frac{1}{2} I_{[0,2]}, \quad f_Y(y) = \frac{1}{2} I_{[0,2]}.$$

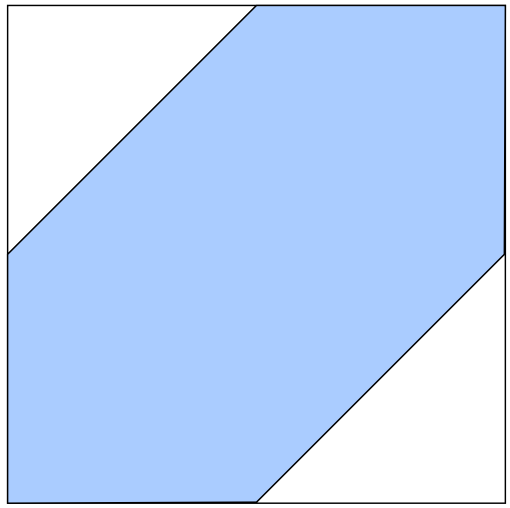
Therefore, since they are independent, the joint PDF is given by

$$f_{XY}(x, y) = f_X(x) f_Y(y) = \begin{cases} \frac{1}{4} & 0 \le x \le 2, 0 \le y \le 2 \\ 0 & \text{otherwise} \end{cases}.$$

b. We are asking what $P(|X - Y| \le 1)$ is this corresponds to finding the probability of the region

$$R = \{(x, y) : |x - y| \le 1,\ 0 \le x \le 2,\ 0 \le y \le 2\}$$

Which is just a stripe down the diagonal of the rectangle $[0, 2]^2$ (see figure). Since the distribution is uniform, we can relate this to the area of this region. There are a number of ways to find the area of this region. For instance you can see that it's area is 3 by subtracting the two triangles of area 1/2 from the rectangle of area 4. This would imply that the probability we seek is just 3/4.



Lets do it by brute force by integration. First we split $R = R_1 \cup R_2$ into two regions separated by when $x \leq 1$ and $x \geq 1$ what we find is

$$R_1 = \{(x, y) \,:\, 0 \leq x \leq 1,\, 0 \leq y \leq 1 + x\}, \quad R_2 = \{(x, y) \,:\, 1 \leq x \leq 2,\, x - 1 \leq y \leq 2\}$$

So

$$\begin{aligned}
P(R) &= \iint_{R_1} f_{X,Y}(x, y) \mathrm{d}A + \iint_{R_2} f_{X,Y}(x, y) \mathrm{d}A \\
&= \frac{1}{4} \int_0^1 \int_0^{1+x} 1 \mathrm{d}y \mathrm{d}x + \frac{1}{4} \int_1^2 \int_{x-1}^2 1 \mathrm{d}y \mathrm{d}x \\
&= \frac{1}{4} \int_0^1 (1 + x) \mathrm{d}x + \frac{1}{4} \int_1^2 (3 - x) \mathrm{d}x \\
&= \frac{1}{4} \frac{3}{2} + \frac{1}{4} \frac{3}{2} = \frac{3}{4}.
\end{aligned}$$

Problem 4. (Stick) Suppose a stick of length 1 is broken in two places. The first break point is chosen uniformly at random along the length of the stick from $[0, 1]$. The second break point is chosen uniformly at random from 0 to the first break point.

a. Find the joint probability distribution of the two break points. (Be careful about your bounds.)

b. What is the covariance of the two break points?

Solution:

a. Let $X_1 \sim \text{Uniform}(0,1)$ be the first break point. It has a PDF

$$f_{X_1}(x_1) = \begin{cases} 1 & 0 \le x_1 \le 1 \\ 0 & \text{otherwise} \end{cases}$$

We see that the second break point X_2 is easily defined by conditioning on X_1, namely

$$(X_2 | X_1 = x_1) \sim \text{Uniform}(0, x_1)$$

Therefore for each $x_1 \in (0,1)$ we have the conditional density

$$f_{X_2|X_1}(x_2|x_1) = \begin{cases} \frac{1}{x_1} & 0 \le x_2 \le x_1 \\ 0 & \text{otherwise} \end{cases}$$

Therefore the joint density is

$$f_{X_1,X_2}(x_1,x_2) = f_{X_2|X_1}(x_2|x_1) f_{X_1}(x_1) = \begin{cases} \frac{1}{x_1} & 0 \le x_2 \le x_1 \le 1 \\ 0 & \text{otherwise} \end{cases}.$$

b. To find the covariance, we compute

$$EX_1 = \int_0^1 x_1 \mathrm{d}x_1 = \frac{1}{2}$$

$$EX_2 = \int_0^1 \int_0^{x_1} \frac{x_2}{x_1} \mathrm{d}x_2 \mathrm{d}x_1 = \frac{1}{2} \int_0^1 x_1 \mathrm{d}x_1 = \frac{1}{4}$$

and

$$E[X_1 X_2] = \int_0^1 \int_0^{x_1} \frac{x_1 x_2}{x_1} \mathrm{d}x_2 \mathrm{d}x_1 = \frac{1}{2} \int_0^1 x_1^2 \mathrm{d}x_1 = \frac{1}{6}.$$

Therefore

$$\text{Cov}(X_1, X_2) = E[X_1 X_2] - (EX_1)(EX_2) = \left(\frac{1}{6}\right) - \left(\frac{1}{2}\right)\left(\frac{1}{4}\right) = \frac{1}{24}$$

Problem 5. (Bounds) Let X be a continuous random variable with PDF,

$$f_X(x) = \begin{cases} e^{x+1} & x \le -1 \\ 0 & \text{otherwise} \end{cases}$$

a. Give a lower bound using Chebyshev for $P(-4 \le X \le 0)$.

b. Use Chebyshev to determine an a such that $P(X \ge a) \ge 0.95$

Solution. First note that by standard calculus

$$EX = \int_{-\infty}^{-1} xe^{x+1}\mathrm{d}x = -2$$

and

$$EX^2 = \int_{-\infty}^{\infty} x^2 e^{x+1}\mathrm{d}x = 5.$$

So

$$\mathrm{Var}(X) = 5 - (-2)^2 = 1$$

a. To apply Chebyshev here, we note that

$$P(-4 \le X \le 0) = P(-2 \le X + 2 \le 2) = P(|X - 2| \le 2)$$

Therefore by Chebyshev

$$P(|X + 2| \le 2) = 1 - P(|X + 2| \ge 2) \ge 1 - \frac{\mathrm{Var}(X)}{4} = \frac{3}{4}$$

b. We will assume that $a < -2$ (at least less than the mean). Using Chebyshev gives

$$P(X \ge a) = P(X + 2 \ge a + 2) \ge P(|X + 2| \le -(a + 2)) \ge 1 - \frac{1}{(a + 2)^2}$$

To ensure the probability is bigger than .95 we need

$$1 - \frac{1}{(a + 2)^2} \ge .95 \quad \Rightarrow \quad |a + 2| \ge 2\sqrt{5}.$$

Since $a + 2 < 0$ we see that this means, we need

$$a + 2 \le -2\sqrt{5} \quad \Rightarrow \quad a \le -2(1 + \sqrt{5}) \approx -6.472$$