APMA 1650

Hard Practice Midterm Exam 2

Problem 1. (Random Rectangle) Suppose that you want to generate a random rectangle with area 1 (pick your favorite unit). You do this by randomly selecting its height H uniformly at random between 1 and 2. The width W is automatically determined by the fact that the area is 1.

a. Compute $E[W]$ and $\text{Var}(W)$.

b. What is the probability density function $f_W(w)$ for W? (Be sure to specify its domain and check that it is valid)

Problem 2. (Joint Conditioning) Let X and Y be continuous random variables with joint probability density

$$f_{XY}(x,y) = \begin{cases} cy & \text{if} \quad 0 \leq x \leq \infty \quad \text{and} \quad 0 \leq y \leq e^{-x} \\ 0 & \text{otherwise} \end{cases}.$$

a. Find the value c for which $f_{XY}(x,y)$ is a valid joint probability density.

b. Find $E[e^X]$.

c. Find the conditional probability density of X given $Y = y$.

Problem 3. (Jobs) Suppose a company has determined that the number of jobs per week, N, varies from week to week and has a Poisson distribution with rate λ. The number of hours to complete each job Y_i is Gamma(α, β). The total time to complete all jobs in a week is $T = \sum_{i=1}^{N} Y_i$. Note that T is the sum of a random number of random variables.

a. Find $E[T|N = n]$.

b. Find ET, the expected time it takes to complete all the jobs.

Problem 4. (Tight)

a. Let X be a Bernoulli(p) random variable. Show that Markov's inequality is actually an equality for $a = 1$, i.e. $P(X \geq 1) = EX$.

b. Let X_1 and X_2 be two independent Bernoulli(p) RVs and let $X = X_1 - X_2$. Show that Chebyshev is an equality for $b = 1$, i.e. $P(|X - EX| \geq 1) = \text{Var}(X)$.

Problem 5. (Boxes) Suppose that m homeworks are distributed randomly into n homework dropboxes (each dropbox can have more than one homework assignment). Let X denote the number of empty dropboxes.

1. Find EX

2. Find $\text{Var}(X)$.