Midterm 2 Practice Problems Solutions

1.

(a) Suppose that X is uniform on $[0,1]$. Compute the pdf and cdf of X.

(b) If $Y = 2X + 5$, compute the pdf and cdf of Y.

Solution:

(a) The PDF and the CDF are

$$f_X(x) = \begin{cases} 1 & x \in [0,1] \\ 0 & \text{otherwise} \end{cases}$$

$$F_X(x) = \begin{cases} 0 & x \leq 0 \\ x & 0 \leq x \leq 1 \\ 1 & x \geq 1 \end{cases}$$

(b) We use the CDF method

$$F_Y(y) = P(2X + 5 \leq y) = P(X \leq (y-5)/2) = F_X((y-5)/2) = \begin{cases} 0 & y \leq 5 \\ \frac{y-5}{2} & 5 \leq y \leq 7 \\ 1 & y \geq 7 \end{cases}$$

The PDF is

$$f_Y(y) = \begin{cases} \frac{1}{2} & 5 \leq y \leq 7 \\ 0 & \text{otherwise} \end{cases}.$$

2.

(a) Suppose that X has probability density function $f_X(x) = \lambda e^{-\lambda x}$ for $x \geq 0$. Compute the cdf, $F_X(x)$.

(b) If $Y = X^2$, compute the pdf and cdf of Y.

Solution:

(a) The CDF is

$$F_X(x) = \int_0^x \lambda e^{-\lambda x} \mathrm{d}x = (1 - e^{-\lambda x})u(x)$$

(b) We use the CDF method

$$F_Y(y) = P(X^2 \leq y) = P(X \leq \sqrt{y}) = F_X(\sqrt{y}) = (1 - e^{-\lambda\sqrt{y}})u(y)$$

and

$$f_Y(y) = \frac{\mathrm{d}}{\mathrm{d}y} F_y(y) = \begin{cases} \frac{\lambda}{2\sqrt{y}} e^{-\lambda\sqrt{y}} & 0 \leq y < \infty \\ 0 & \text{otherwise} \end{cases}$$

3. Let R be the rate at which customers are served in a queue. Suppose that R is exponential with pdf $f_R(r) = 2e^{-2r}$ on $[0, \infty)$. Find the pdf of the waiting time per customer $T = 1/R$.

Solution: We use the CDF method

$$P_T(x) = P(T \leq x) = P(R \geq 1/x) = 1 - P_R(1/x) = e^{-2(1/x)}u(x)$$

therefore

$$f_T(x) = \frac{\mathrm{d}}{\mathrm{d}x} P_T(x) = \begin{cases} \frac{2}{x^2} e^{-2/x} & x > 0 \\ 0 & \text{otherwise} \end{cases}.$$

4. A continuous random variable X has pdf $f_X(x) = x + ax^2$ on $[0, 1]$. Find a, the cdf and $P(.5 < X < 1)$.

Solution: We have

$$1 = \int_0^1 x + ax^2 \mathrm{d}x = \frac{1}{2} + \frac{a}{3} \quad \Rightarrow \quad a = \frac{3}{2},$$

and

$$F_X(x) = \int_0^x u + \frac{3}{2}u^2 \mathrm{d}u = \begin{cases} 0 & x \leq 0 \\ \frac{x^2 + x^3}{2} & 0 \leq x \leq 1 \\ 1 & x \geq 1 \end{cases},$$

$$P(.5 < X < 1) = F_X(1) - F_X(.5) = 1 - \frac{1}{2}\left(.5^2 + .5^3\right) = \frac{13}{16}$$

5. Suppose X has range $[0, 1]$ and has cdf

$$F_X(x) = \begin{cases} 0 & x \leq 0 \\ x^2 & 0 \leq x \leq 1 \\ 1 & \text{otherwise.} \end{cases}$$

Compute $P(1/2 < X < 3/4)$.

Solution:

$$P(1/2 < X < 3/4) = F_X(3/4) - F_X(1/2) = (3/4)^2 - (1/2)^2 = \frac{5}{16}$$

6. Let X be a random variable with range $[0,1]$ and cdf

$$F_X(x) = \begin{cases} 0 & x \le 0 \\ 2x^2 - x^4 & 0 \le x \le 1 \\ 1 & \text{otherwise.} \end{cases}$$

(a) Compute $P(1/4 < X < 3/4)$.

(b) What is the pdf of X?

Solution:

(a)

$$P(1/4 < X < 3/4) = F_X(3/4) - F_X(1/4) = 2(3/4)^2 - (3/4)^4 - 2(1/4)^2 + (1/4)^4 = \frac{11}{16}$$

(b)

$$f_X(x) = \frac{\mathrm{d}}{\mathrm{d}x} F_X(x) = \begin{cases} 4x(1-x^3) & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}.$$

7. Suppose a random circle is formed by randomly choosing a radius uniformly in $[0,1]$.

(a) What is the expected area of the circle?

(b) What is the variance of the area of the circle?

Solution:

(a) The area is $A = \pi R^2$, where $R \sim \text{Uniform}(0,1)$,

$$EA = \int_0^1 \pi r^2 \mathrm{d}r = \frac{\pi}{3}$$

(b)

$$EA^2 = \int_0^1 \pi^2 r^4 \mathrm{d}r = \frac{\pi^2}{5}$$

Therefore

$$\mathrm{Var}(A) = EA^2 - (EA)^2 = \frac{\pi^2}{5} - \left(\frac{\pi}{3}\right)^2 = \frac{4\pi^2}{45}$$

8. Suppose that buses arrive are scheduled to arrive at a bus stop at noon but are always X minutes late, where X is an exponential random variable with probability density function $f_X(x) = \lambda e^{-\lambda x}$. Suppose that you arrive at the bus stop precisely at noon.

(a) Compute the probability that you have to wait for more than five minutes for the bus to arrive.

(b) Suppose that you have already waited for 10 minutes. Compute the probability that you have to wait an additional five minutes or more.

Solution:

(a)

$$P(X \geq 5) = \int_5^\infty \lambda e^{-\lambda x} \mathrm{d}x = e^{-5\lambda}$$

(b) By the memoryless property this is the same as part a $e^{-5\lambda}$.

9. A bus arrives every 15 minutes starting at $7:00$am. You walk to the station and arrive uniformly at a random time between $7:10$am and $7:30$am. What is the pdf for the amount of time spent waiting for the bus?

Solution: During the time window you have, the bus arrives at 7:15 and 7:30. Let T be the time you arrive at the station (in minutes starting from $7:30$), then

$$T \sim \text{Uniform}(0, 20).$$

If $T \leq 5$, then you only have to wait $5 - T$ minutes, if $T > 5$ then miss the first bus and you have to wait $20 - T$ minutes till the next bus. Therefore the amount of minutes you have to wait W is a piecewise function of T

$$W = \begin{cases} 5 - T & T \leq 5 \\ 20 - T & T > 5 \end{cases}$$

To find the PDF, we first calculate the CDF

$$F_W(w) = P(W \leq w) = \begin{cases} 0 & w < 0 \\ P(5 - w \leq T \leq 5) + P(20 - w \leq T \leq 20) & 0 \leq w < 5 \\ P(0 \leq T \leq 5) + P(w - 20 \leq T \leq 20) & 5 \leq w < 15 \\ 1 & w \geq 15 \end{cases}$$

Since

$$P(5 - w \leq T \leq 5) = \frac{w}{20}, \quad P(20 - w \leq T \leq 20) = \frac{w}{20},$$

we see that

$$F_W(w) = \begin{cases} 0 & w \leq 0 \\ \frac{w}{10} & 0 \leq w \leq 5 \\ \frac{5+w}{20} & 5 \leq w \leq 15 \\ 1 & w > 15 \end{cases}.$$

The PDF is given by

$$f_W(w) = \frac{\mathrm{d}}{\mathrm{d}w} F_W(w) = \begin{cases} \frac{1}{10} & 0 \leq w < 5 \\ \frac{1}{20} & 5 \leq 5 < 15 \\ 0 & \text{otherwise} \end{cases}.$$

10. Let $\phi(z)$ and $\Phi(z)$ be the pdf and cdf of the standard normal distribution. Suppose that Z is a standard normal random variable and let $X = 3Z + 1$.

(a) Express $F_X(x) = P(X \leq x)$ in terms of Φ.

(b) Use this to find the pdf of X in terms of ϕ.

(c) Find $P(-1 \leq X \leq 1)$.

(d) Recall that the probability that Z is within one standard deviation of its mean is approximately 68%. What is the probability that X is within one standard deviation of its mean?

Solution:

(a) $F_X(x) = P(3Z + 1 \leq x) = P(Z \leq (x-1)/3) = \Phi\left(\frac{x-1}{3}\right)$.

(b) $f_X(x) = \frac{\mathrm{d}}{\mathrm{d}x} F_X(x) = \phi\left(\frac{x-1}{3}\right)/3$

(c)
$$P(-1 \leq X \leq 1) = \Phi(0) - \Phi(-2/3) = \Phi(2/3) - 1/2 \approx 0.24751$$

(d) It is the same 68%

11. Suppose $Z \sim N(0, 1)$ and $Y = Z^2$. What is the pdf of Y?

Solution: The CDF is for $y \in [0, \infty]$

$$F_Y(y) = P(Z^2 \leq y) = P(-\sqrt{y} \leq Z \leq \sqrt{y}) = \Phi(\sqrt{y}) - \Phi(-\sqrt{y}) = 2\Phi(\sqrt{y}) - 1$$

Therefore

$$f_Y(y) = \frac{\mathrm{d}}{\mathrm{d}y} F_Y(y) = \begin{cases} \frac{\Phi'(\sqrt{y})}{\sqrt{y}} = \frac{1}{\sqrt{2\pi y}} e^{-y/2} & 0 \leq y \leq \infty \\ 0 & \text{otherwise} \end{cases}.$$

12. Let X and Y be independent normal random variables, where $X \sim N(2, 5)$ and $Y \sim N(5, 9)$, let $W = 3X - 2Y + 1$.

(a) Compute $E(W)$ and $\mathrm{Var}(W)$.

(b) It is known that the sum of independent normal random variables is normal (take this as a given). Compute $P(W \le 6)$.

Solution:

(a) By linearity of expectation

$$EW = 3EX - 2EY + 1 = 3(2) - 2(5) + 1 = -3.$$

By independence

$$\mathrm{Var}(W) = 9\mathrm{Var}(X) + 4\mathrm{Var}(Y) = 9(5) + 4(9) = 81.$$

(b) This implies that W is normal with mean -3 and variance 81. Therefore since $(6 - (-3))/\sqrt{81} = 1$

$$P(W \le 6) = P(Z \le 1) = \Phi(1) \approx 0.84134$$

13. Let $X \sim U(a, b)$. Compute $E(X)$ and $\mathrm{Var}(X)$.

Solution:

$$EX = \frac{a+b}{2}, \quad \mathrm{Var}(X) = \frac{(b-a)^2}{12}.$$

14. Compute the median for an exponential distribution with rate λ.

Solution: The median is the value m such that

$$P(X \le m) = P(X \ge m).$$

For the exponential, this gives

$$(1 - e^{-\lambda m}) = e^{-\lambda m} \quad \Rightarrow \quad m = \frac{\ln(2)}{\lambda}.$$

15. Let X and Y be independent Bernoulli(.5) random variables. Define

$$S = X + Y \quad \text{and} \quad T = X - Y.$$

(a) Find the joint and marginal distributions for S and T.

(b) Are S and T independent?

Solution:

(a) Note that $R_S = \{0, 1, 2\}$ and $R_T = \{-1, 0, 1\}$. The joint PMF is

$$P_{ST}(s,t) = P(X+Y=s, X-Y=t)$$

Note that by solving a linear system we have

$$\begin{cases} X+Y=s \\ X-Y=t \end{cases} \quad \Leftrightarrow \quad \begin{cases} X = \frac{s+t}{2} \\ Y = \frac{s-t}{2} \end{cases}$$

Therefore using that X and Y are independent

$$P_{ST}(s,t) = P_X\left(\frac{s+t}{2}\right) P_Y\left(\frac{s-t}{2}\right) = \begin{cases} \frac{1}{4} & (s-t) \in \{0,2\} \text{ and } (s+t) \in \{0,2\} \\ 0 & \text{otherwise} \end{cases}$$

We can write this more succinctly as a table that includes the marginals

S\ T	-1	0	1	P_S
0	0	1/4	0	1/4
1	1/4	0	1/4	1/2
2	0	1/4	0	1/4
P_T	1/4	1/2	1/4	1

(b) They are not independent since (for example)

$$P(S=1, T=0) = 0 \neq \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = P_S(1)P_T(0)$$

16. Data is taken on the height and shoe size of a sample of Brown students. Height is coded by 3 values 1 (short), 2 (medium), 3 (tall) and similarly for shoe size 1 (small), 2 (medium), 3 (large). The total counts are displayed in the following table

Shoe\ Height	1	2	3
1	234	225	84
2	180	453	161
3	39	192	157

Let X be the coded shoe size and Y be the coded shoe height of a random person in the sample.

(a) Find the joint and marginal distributions of X and Y.

(b) Are X and Y independent?

Solution:

(a) The total sum of all the values in the table is

$$234 + 225 + 84 + 180 + 453 + 161 + 39 + 192 + 157 = 1725$$

Dividing each value in the table by this number gives the approximate joint PMF table with marginals

Shoe\ Height	1	2	3	P_{Shoe}
1	0.135652174	0.130434783	0.048695652	0.314782609
2	0.104347826	0.262608696	0.093333333	0.460289855
3	0.022608696	0.111304348	0.091014493	0.224927537
P_{Height}	0.262608696	0.504347827	0.233043478	1

(b) They are not independent, since

$$P(\text{Shoe} = 1, \text{Height} = 1) = 0.135652174 \neq 0.08266465 = P_{\text{Shoe}}(1)P_{\text{Height}}(1)$$

17. Let X and Y be two continuous random variables with joint pdf given by:

$$f(x, y) = \begin{cases} c(x^2y) & 0 \le x \le y,\ x + y \le 2 \\ 0 & \text{otherwise} \end{cases}$$

(a) Find the value of c that makes this a valid density function.

(b) What is the expectation of X?

(c) What is the marginal distribution for Y?

Solution:

(a) We can write the domain as (draw a picture)

$$R = \{(x, y) : 0 \le x \le 1,\ x \le y \le 2 - x\}$$

and so

$$1 = \int_0^1 \int_x^{2-x} cx^2y \,\mathrm{d}y\mathrm{d}x = 2c \int_0^1 x^2(1 - x)\mathrm{d}x = 2c\left(\frac{1}{3} - \frac{1}{4}\right) = \frac{c}{6}$$

Therefore $c = 6$.

(b)

$$E[X] = \int_0^1 \int_x^{2-x} 6x^3y \,\mathrm{d}y\mathrm{d}x = 12 \int_0^1 x^3(1 - x)\mathrm{d}x = 12\left(\frac{1}{4} - \frac{1}{5}\right) = \frac{3}{5}$$

(c) To compute the marginal, we need to split up the domain. When $0 \leq y \leq 1$, we have

$$f_Y(y) = \int_0^y 6x^2 y \mathrm{d}x = 2y^4$$

and when $1 \leq y \leq 2$

$$f_Y(y) = \int_0^{2-y} 6x^2 y \mathrm{d}x = 2y(2-y)^3$$

Therefore

$$f_Y(y) = \begin{cases} 2y^4 & 0 \leq y \leq 1 \\ 2y(2-y)^3 & 1 \leq y \leq 2 \\ 0 & \text{otherwise} \end{cases}.$$

18. Let X_1 and X_2 be two continuous random variables with joint pdf given by:

$$f_{X_1,X_2}(x_1,x_2) = \begin{cases} cx_1x_2 & 0 \leq x_1 \leq 1,\ 0 \leq x_2 \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) Find the value of c that makes this a valid density function.

(b) Find the marginal densities for X_1 and X_2.

(c) Are X_1 and X_2 independent?

(d) What is the probability that $X_1 < 1/4$ given that $X_2 = 1/2$?

(e) Find the expectation of X_1X_2.

Solution:

(a)

$$1 = \int_0^1 \int_0^1 cx_1x_2 \mathrm{d}x_1 \mathrm{d}x_2 = \frac{c}{4}$$

Therefore $c = 4$

(b)

$$f_{X_1}(x_1) = \begin{cases} 2x_1 & 0 \leq x_1 \leq 1 \\ 0 & \text{otherwise} \end{cases}, \quad f_{X_2}(x_2) = \begin{cases} 2x_2 & 0 \leq x_2 \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

(c) Yes, because $f_{X_1,X_2}(x_1,x_2) = f_{X_1}(x_1)f_{X_2}(x_2)$

(d)

$$P(X_1 < 1/4 | X_2 = 1/2) = P(X_1 < 1/4) = \int_0^{1/4} 2x_1 \mathrm{d}x_1 = \frac{1}{16}$$

(e) By independence

$$E[X_1X_2] = (EX_1)(EX_2) = \left(\int_0^1 2x^2 \mathrm{d}x\right)^2 = \left(\frac{2}{3}\right)^2 = \frac{4}{9}$$

19. Let X and Y be two continuous random variables with joint pdf given by:

$$f_{X,Y}(x,y) = \begin{cases} cx^2y(1+y) & 0 \le x \le 3,\ 0 \le y \le 3 \\ 0 & \text{otherwise} \end{cases}$$

(a) Find the value of c that makes this a valid density function.

(b) Find $P(1 \le X \le 2, 0 \le Y \le 1)$.

(c) Determine the joint cdf $F_{X,Y}(x,y)$ of X and Y.

(d) Find the marginal cdf $F_X(x)$.

(e) Find the marginal pdf $f_X(x)$ directly from $f_{X,Y}(x,y)$ and check that it is the derivative of $F_X(x)$.

(f) Are X and Y independent?

Solution:

(a)

$$1 = \int_0^3 \int_0^3 cx^2y(1+y)\mathrm{d}x\mathrm{d}y = 9c \int_0^3 y(1+y)\mathrm{d}y = \frac{3^5 c}{2}$$

Therefore $c = \frac{2}{3^5}$.

(b)

$$P(1 \le X \le 2, 0 \le Y \le 1) = \frac{2}{3^5} \int_0^1 \int_1^2 x^2y(1+y)\mathrm{d}x\mathrm{d}y = \frac{2}{3^5}\left(\frac{7}{3}\right)\left(\frac{5}{6}\right) = \frac{35}{3^7}$$

(c) The joint CDF is for $x, y \in [0,3]$

$$F_{XY}(x,y) = \frac{2}{3^5} \int_0^x \int_0^y u^2v(1+v)\mathrm{d}v\mathrm{d}u = \frac{1}{3^7}x^3(3y^2 + 2y^3)$$

Therefore

$$F_{XY}(x,y) = \begin{cases} 0 & x \le 0 \quad \text{or} \quad y \le 0 \\ \frac{1}{3^7}x^3(3y^2+2y^3) & 0 \le x \le 3,\ 0 \le y \le 3 \\ \frac{x^3}{3^3} & 0 \le x \le 3,\ y \ge 3 \\ \frac{3y^2+2y^3}{3^4} & 0 \le y \le 3,\ x \ge 3 \\ 1 & \text{otherwise} \end{cases}$$

(d) The marginal CDF is for

$$F_{XY}(x,3) = \begin{cases} 0 & x \le 0 \\ \frac{x^3}{3^3} & 0 \le x \le 3 \\ 1 & x \ge 3 \end{cases}$$

(e) For $x \in [0,3]$ we have

$$f_X(x) = \frac{2}{3^5}\int_0^3 x^2 y(1+y)\mathrm{d}y = \frac{2x^2}{3^5}\left(3^2/2 + 3^3/3\right) = \frac{2x^2}{3^3}$$

Similarly

$$f_X(x) = \frac{\mathrm{d}}{\mathrm{d}x}F_X(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x^3}{3^3}\right) = \frac{2x^2}{3^3}.$$

(f) Yes. This is easy to see since

$$F_{XY}(x,y) = \frac{1}{3^7}x^3(3y^2+2y^3) = \left(\frac{x^3}{3^3}\right)\left(\frac{3y^2+2y^3}{3^4}\right) = F_X(x)F_Y(y).$$

20. Suppose Y_1 and Y_2 iid random variables sampled from a Uniform$(0,1)$ distribution.

(a) What is the joint density function $f_{X_1,X_2}(x_1,x_2)$?

(b) What is the probability that $X_1 + X_2$ is less than $\frac{1}{3}$?

(c) Find the expectation of $X_1 + X_2$.

Solution:

(a)

$$f_{X_1,X_2}(x_1,x_2) = \begin{cases} 1 & 0 \le x_1 \le 1,\ 0 \le x_2 \le 1 \\ 0 & \text{otherwise} \end{cases}$$

(b) Draw a picture

$$P(X_1+X_2 \le 1/3) = \int_0^{1/3}\int_0^{1/3-x_1} 1\,\mathrm{d}x_2\mathrm{d}x_1 = \int_0^{1/3}\left(\frac{1}{3}-x_1\right)\mathrm{d}x_1 = \frac{1}{2}\left(\frac{1}{3}\right)^2 = \frac{1}{18}$$

(c)

$$E(X_1+X_2) = EX_1 + EX_2 = \frac{1}{2} + \frac{1}{2} = 1$$

21. Let Y_1 denote the weight (in tons) of a bulk item stocked by a supplier at the beginning of a week and suppose that Y_1 has a uniform distribution over the interval $0 \le y_1 \le 1$. Let Y_2 denote the amount (by weight) of this item sold by the supplier during the week and suppose that Y_2 has as uniform distribution over the interval $0 \le y_2 \le y_1$, where y_1 is specific value of Y_1.

(a) Find the joint pdf for Y_1 and Y_2.

(b) If the supplier stocks a half-ton of the item, what is the probability that she sells more than a quarter-ton?

(c) If it is known that the supplier sold a quarter-ton of the item, what is the probability that she had stocked more than a half-ton.

Solution:

(a) We have from the problem statement

$$f_{Y_1}(y_1) = \begin{cases} 1 & 0 \le y_1 \le 1 \\ 0 & \text{otherwise} \end{cases}$$

and

$$f_{Y_2|Y_2}(y_2|y_1) = \begin{cases} \frac{1}{y_1} & 0 \le y_2 \le y_1 \\ 0 & \text{otherwise} \end{cases}$$

Therefore

$$f_{Y_1,Y_2}(y_1, y_2) = f_{Y_2|Y_1}(y_2|y_1) f_{Y_1}(y_1) = \begin{cases} \frac{1}{y_1} & 0 \le y_2 \le y_1 \le 1 \\ 0 & \text{otherwise} \end{cases}$$

(b) We are computing

$$P(Y_2 \ge 1/4|Y_1 = 1/2) = \int_{1/4}^{\infty} f_{Y_2|Y_1}(y_2|1/2)\mathrm{d}y_2 = \int_{1/4}^{1/2} \frac{1}{1/2}\mathrm{d}y_2 = \frac{1}{2}$$

(c) We are computing $P(Y_1 \ge 1/2|Y_2 = 1/4)$. To do this, we need to compute

$$f_{Y_2}(y_2) = \int_{y_2}^{1} \frac{1}{y_1}\mathrm{d}y_1 = \ln(y_1)|_{y_2}^{1} = -\ln(y_2).$$

Therefore for $y_2 \in (0, 1)$

$$f_{Y_1|Y_2}(y_1|y_2) = \begin{cases} -\frac{1}{y_1 \ln(y_2)} & y_2 \le y_1 \le 1 \\ 0 & \text{otherwise} \end{cases}$$

and so

$$P(Y_1 \ge 1/2|Y_2 = 1/4) = \int_{1/2}^{1} f_{Y_1|Y_2}(y_1|1/4)\mathrm{d}y_1 = \frac{1}{\ln(4)} \int_{1/2}^{1} \frac{1}{y_1}\mathrm{d}y_1 = \frac{\ln(2)}{\ln(4)}.$$

22. If Y_1 is uniformly distributed on the interval $(0,1)$ and, for $0 < y_1 < 1$,

$$f_{Y_2|Y_2}(y_2|y_1) = \begin{cases} 1/y_1 & 0 \le y_2 \le y_1, \\ 0 & \text{otherwise} \end{cases}.$$

(a) Find the joint pdf of Y_1 and Y_2.

(b) Find the marginal pdf for Y_2.

Solution: See the previous problem

23. Let X and Y be two independent exponential random variables with parameters λ_X and λ_Y. Let $Z = X + Y$. Find the covariance of X and Z.

Solution: Using properties of covariance and independence

$$\mathrm{Cov}(X, Z) = \mathrm{Cov}(X, X + Y) = \mathrm{Var}(X) + \underbrace{\mathrm{Cov}(X, Y)}_{=0} = \frac{1}{\lambda_X^2}$$

24. Let X and Y be two random variables and let r, s, t, u be real numbers.

(a) Show that $\mathrm{Cov}(X + s, Y + u) = \mathrm{Cov}(X, Y)$.

(b) Show that $\mathrm{Cov}(rX, tY) = rt\mathrm{Cov}(X, Y)$.

(c) Show that $\mathrm{Cov}(rX + s, tY + u) = rt\mathrm{Cov}(X, Y)$.

Solution:

(a) Using properties of expectation

$$\begin{aligned}
\mathrm{Cov}(X + s, Y + u) &= E[(X + s)(Y + u)] - (E[X + s])(E[Y + u]) \\
&= E\left[XY + sY + uX + us\right] - (EX)(EY) - sEY - uEX - us \\
&= E[XY] + sEY + uEX + us - (EX)(EY) - sEY - uEX - us \\
&= E[XY] - (EX)(EY)
\end{aligned}$$

(b)

$$\mathrm{Cov}(rX, tY) = E[rtXY] - (E[rX]E[tY]) = rt(E[XY] - (EX)(EY))$$

(c) Just combine parts a and b

25. Let X and Y be two discrete random variables whose joint and marginal distributions are partially given in the following table.

$X\backslash Y$	1	2	3	P_X
1	1/6	0	$\cdot$	1/3
2	$\cdot$	1/4	$\cdot$	1/3
3	$\cdot$	$\cdot$	1/4	$\cdot$
P_Y	1/6	1/3	$\cdot$	1

(a) Complete the table.

(b) Are X and Y independent?

Solution:

(a) The table is completed below

$X\backslash Y$	1	2	3	P_X
1	1/6	0	1/6	1/3
2	0	1/4	1/12	1/3
3	0	1/12	1/4	1/3
P_Y	1/6	1/3	1/2	1

(b) No

$$P_{X,Y}(1,2) = 0 \neq P_X(1)P_Y(2) = \left(\frac{1}{3}\right)\left(\frac{1}{3}\right)$$

26. Let X and Y be two discrete random variables which take values $\{1,2,3,4\}$. The following formula gives their joint distribution,

$$P_{X,Y}(i,j) = \frac{i+j}{80},$$

Compute each of the following:

(a) $P(X = Y)$.

(b) $P(XY = 6)$.

(c) $P(1 \leq X \leq 2, 2 < Y \leq 4)$.

Solution:

(a)

$$P(X = Y) = \sum_{i=1}^{4} \frac{2i}{80} = \frac{1}{40}\left(\frac{4(4+1)}{2}\right) = \frac{1}{4}$$

(b) Note $XY = 6$ when $X = 2, Y = 3$ or $X = 3Y = 2$, therefore

$$P(XY = 6) = P_{XY}(2,3) + P_{XY}(3,2) = \frac{2(2+3)}{80} = \frac{1}{8}$$

(c)

$$
\begin{aligned}
P(1 \leq X \leq 2, 2 < Y \leq 4) &= \sum_{i=1}^{2} \sum_{j=3}^{4} \frac{i+j}{80} \\
&= \frac{1}{80} \left(2 \sum_{i=1}^{2} i + 2 \sum_{j=3}^{4} j \right) \\
&= \frac{1}{40} (1+2+3+4) \\
&= \frac{1}{4}
\end{aligned}
$$

27. Toss a fair coin three times. Let X = the number of heads on the first toss, Y = the total number of heads on the last two tosses, and Z = the number of heads on the first two tosses.

(a) Give the joint probability table for X and Y. Compute $\text{Cov}(X, Y)$.

(b) Give the joint probability table for X and Z. Compute $\text{Cov}(X, Z)$.

Solution:

(a) The table is given by

$X \backslash Y$	0	1	2	P_X
0	1/8	1/4	1/8	1/2
1	1/8	1/4	1/8	1/2
P_Y	1/4	1/2	1/4	1

Note that X and Y are independent for $\text{Cov}(X, Y) = 0$. Regardless we can compute explicitly

$$EXY = 1\frac{1}{4} + 2\frac{1}{8} = 1/2$$

$$EX = 1/2, \quad EY = 1\frac{1}{2} + 2\frac{1}{4} = 1$$

and therefore

$$\text{Cov}(X, Y) = 1/2 - 1/2(1) = 0.$$

(b) For this case the table is given as

$X \backslash Z$	0	1	2	P_X
0	1/4	1/4	0	1/2
1	0	1/4	1/4	1/2
P_Z	1/4	1/2	1/4	1

In this case X and Z are clearly no independent. We find that

$$EXZ = 1\left(\frac{1}{4}\right) + 2\left(\frac{1}{4}\right) = \frac{3}{4}$$

Therefore

$$\mathrm{Cov}(X, Z) = \frac{3}{4} - \frac{1}{2}(1) = \frac{1}{4}.$$