APMA 1650 - Spring 2021

Final Exam

Please read the following instructions carefully:

- **This exam is a "take-home" exam**. This means that you have the entire period from Monday 4/19, 11:00AM EDT to Friday 4/23, 11:59PM EDT to work on it.
- **You must show all work and explain/justify your answers**. Answers without sufficient justification may not receive full credit. Partial credit for wrong answers may be given if you show your thought process.
- **The exam is open book**. This means you may use any resources available in the textbook, Canvas or the course webpage. If you use any of these sources, you must indicate when and where you use them. **No other sources are allowed**.
- **No communication with anyone other than course staff** about the exam or course material is allowed during the exam period.
- Questions during the exam week should be posted **privately** to course staff on Campuswire. Do NOT post or answer public questions during the exam week.
- **Calculators are allowed**. You can use any of the combinatorial and CDF calculators located in the online textbook as well as MATLAB if you desire. However calculus must be done by hand. You can also leave answers as basic fractions, elementary functions or known combinatorial quantities unless otherwise indicated.
- Please write each problem on a **separate sheet of paper**. Make sure to use large dark lettering and good light when scanning before uploading to Gradescope, just like you would for homework.
- Technical problems must be communicated **immediately**, via whatever you can get to work. I can upload a submission for you if Gradescope is giving you problems.

Problem 1. (Playing by the rules) (0 pts) You must read and rewrite the following statements for this exam to be graded

- *I have read and understood the above instructions. I understand that failure to follow these instructions may result in loss of points or no credit.*
- *I have not/nor do I plan to communicate or share information about the exam or course material with anyone besides course staff during the exam window.*
- *I have not used/nor plan to use any unapproved sources during the exam window.*
- *I understand that violating these rules may result in referral to the honor committee.*

Problem 2 (Aptitude) (9 pts) Employees in a firm are given an aptitude test when first employed. Experience has shown that of the 60 percent who passed the test, 70 percent of them were rated as good workers, whereas of the 40 percent who failed the test only 50 percent were rated as good workers.

a. (4 pts) What is the probability that an employee, selected at random will be a good worker?

b. (5 pts) What is the probability that a good worker failed the test?

Problem 3 (Pivotal) (17 pts) Let $X_1, X_2, \ldots X_n$ be a random sample from a distribution with PDF

$$f(x;\theta) = \begin{cases} \frac{2x}{\theta^2} & 0 \le x \le \theta \\ 0 & \text{otherwise.} \end{cases}$$

a. (4 pts) Show that $\hat{\Theta} = \max\{X_1, X_2, \ldots, X_n\}$ is a maximum likelihood estimator (MLE) for θ.

b. (4 pts) Find the CDF $F_{\hat{\Theta}}$ for $\hat{\Theta}$.

c. (4 pts) Show that $\hat{\Theta}/\theta$ is a pivotal quantity.

d. (5 pts) Use the pivotal quantity from part (c) to find the general form of a $100(1-\alpha)\%$ confidence interval for θ. State everything as explicitly as possible.

Problem 4 (Vaccine Party) (12 pts) You are celebrating the end of the semester by throwing huge party with all your vaccinated friends (perhaps a bit prematurely...). You have an epic cooler set up with hundreds of your favorite IPA on ice. To make sure they are cold enough you randomly sample 12 beers and measure their temperatures (in Fahrenheit) and obtain a sample mean of 48 with a standard deviation of 3. Assume that the distribution of the temperatures follow a *normal distribution* and the samples are independent.

a. (4 pts) Your friends are very picky about their beer temperature. Give them a 99% confidence interval for the mean temperature of the IPAs.

b. (4 pts) Going one step further, give an 80% confidence interval for variance of the temperature of the IPAs.

c. (4 pts) In general, let $\overline{X}$ and S^2 be the sample mean and sample variance of 12 independent temperature measurements of the IPAs. Find a value a such that

$$P(\overline{X} - \mu \le aS) = 0.01,$$

where μ is the true mean of the temperature in your collection of IPAs. State your answer to three decimal places.

Problem 5 (Study Buddies) (15 pts) Two students, conspicuously named I and II, are studying for an exam. Let X_1 and X_2 be the proportion of time that I and II spend studying during the exam "study week". Assume the joint PDF of X_1 and X_2 is given by

$$f_{X_1,X_2}(x_1,x_2) = \begin{cases} x_1 + x_2, & 0 \le x_1 \le 1, 0 \le x_2 \le 1 \\ 0 & \text{otherwise} \end{cases}.$$

a. (3 pts) Find the marginal densities of X_1 and X_2.

b. (3 pts) These two students are roommates. Does the amount of time one of them studies affect the amount of time the other does? Namely, are X_1 and X_2 independent or dependent?

c. (4 pts) Find $P(X_1 \ge 1/2 | X_2 \ge 1/2)$.

d. (5 pts) Find $\rho(X_1, X_2)$. Interpret your answer. Are the roommates a good or bad influence on each other?

Problem 6 (Light bulbs) (12 pts) The lifetimes of two light bulbs are given by two independent Exponential(λ) random variables X_1 and X_2.

a. (3 pts) What is the joint PDF of X_1 and X_2?

b. (4 pts) What is the $P(X_2 \le X_1)$?

c. (5 pts) Let $Y = X_2/X_1$, what is the CDF and PDF of Y? Does it depend on λ?

Problem 7 (Estimators) (13 pts) Let $Y_1, Y_2, \ldots Y_n$ be a random sample from a Poisson(θ) distribution.

a. (5 pts) Find the maximum likelihood estimator $\hat{\Theta}_{MLE}$ for θ.

b. (3 pts) Show that $\hat{\Theta}_{MLE}$ is an unbiased and consistent estimator for θ.

c. (5 pts) Show that the sample variance S^2 is also an unbiased and consistent estimator for θ.

Problem 8 (Strings) (11 pts) Let X be the number of occurrences of the string 'EXAM' in a random string of length 10 (there are 26 letters in the alphabet). 'EXAM' needs to appear as 4 consecutive letters like in TMDEXAMOPA.

a. (3 pts) How many strings are there with exactly 10 letters?

b. (3 pts) What is the probability of the string 'EXAM' appearing in the first four letters?

c. (5 pts) What is EX, the expected number occurrences of the string 'EXAM'?

Problem 9 (CLTree) (11 pts) You own a struggling tree farm with 100 trees ready to be sold for the year. You found that the heights of the trees (in feet) are independent Exponential(1/6) random variables $X_1, X_2, \ldots X_{100}$. You sell each tree at a price \$10 per foot. Each tree costs you \$57 (to grow and maintain) for the year.

a. (3 pts) Assuming you sell all 100 trees, give an expression I for the total dollar amount earned per tree.

b. (3 pts) Find $E[I]$ and $\mathrm{Var}(I)$. Is the average dollar earned per tree more than your yearly cost per tree?

c. (5 pts) Use the central limit theorem to estimate the probability that you lose money this year.