APMA 1650 - Calculus Preparedness Assignment Solutions

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1. Let $f(x)$ be an integrable continuous function on $\mathbb{R} = (-\infty, \infty)$ and let

$$F(x) = \int_{-\infty}^{x} f(y)\,\mathrm{d}y.$$

What is $F'(x)$? Be sure to name any fundamental theorems you use.

Solution. This is just the fundamental theorem of calculus

$$F'(x) = f(x).$$

2. (Derivatives)

(a) Calculate

$$\frac{\mathrm{d}}{\mathrm{d}x} \ln(x).$$

(b) Calculate

$$\frac{\mathrm{d}}{\mathrm{d}x} \ln(\ln(x)).$$

(c) Let $p \geq 0$. Calculate

$$\lim_{x \to 0} x^{1+p} \ln(x).$$

Solution.

(a) It a standard result from calculus that

$$\frac{\mathrm{d}}{\mathrm{d}x} \ln(x) = \frac{1}{x}.$$

(b) Using the chain rule, we find

$$\frac{\mathrm{d}}{\mathrm{d}x} \ln(\ln(x)) = \frac{1}{\ln(x)} \times \left(\frac{\mathrm{d}}{\mathrm{d}x} \ln(x) \right) = \frac{1}{x \ln(x)}.$$

(c) We write this limit in indeterminate ∞/∞ form and use L'Hopital's rule

$$\lim_{x \to 0} x^{1+p} \ln(x) = \lim_{x \to 0} \frac{\ln(x)}{1/x^{1+p}} = -\lim_{x \to 0} \frac{\frac{1}{x}}{(1+p)/x^{p+2}} = -\lim_{x \to 0} \frac{x^{p+1}}{p+1} = 0.$$

3. (Series) (Updated)

(a) Let $0 \leq \lambda < \infty$. Find a simple formula for the following infinite series

$$\sum_{k=1}^{\infty} \frac{\lambda^k 2^k}{k!}.$$

(b) Give a simple formula for

$$\sum_{k=0}^{n} \lambda^k.$$

when $0 \leq \lambda < 1$

(c) Use the answer to part (b) and the fact that

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\sum_{k=0}^{n} \lambda^k\right) = \sum_{k=1}^{n} k\lambda^{k-1}$$

to calculate

$$\sum_{k=1}^{n} k.$$

Solution.

(a) Using the standard Taylor-series for e^x

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!},$$

we find

$$\sum_{k=1}^{\infty} \frac{\lambda^k 2^k}{k!} = \sum_{k=0}^{\infty} \frac{(2\lambda)^k}{k!} - 1 = e^{2\lambda} - 1.$$

(b) This is a geometric series. An expression can be easily derived by noting that

$$(1-\lambda)\sum_{k=0}^{n} \lambda^k = \sum_{k=0}^{n} \lambda^k - \sum_{k=0}^{n} \lambda^{k+1} = 1 - \lambda^{n+1},$$

where in the last equality we used the fact the the majority of the terms between the two sums cancel. Solving for $\sum_{k=0}^{n} \lambda^k$ above gives the well known formula

$$\sum_{k=0}^{n} \lambda^k = \frac{1-\lambda^{n+1}}{1-\lambda}.$$

(c) Following the suggestion, we find, using the formula from part (b) and the quotient rule

$$
\begin{aligned}
\sum_{k=1}^{n} k\lambda^{k-1} = \frac{\mathrm{d}}{\mathrm{d}\lambda}\sum_{k=1}^{n}\lambda^k &= \frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\frac{1-\lambda^{n+1}}{1-\lambda}\right) \\
&= \frac{-(n+1)\lambda^n(1-\lambda)+(1-\lambda^{n+1})}{(1-\lambda)^2} \\
&= \frac{1-(n+1)\lambda^n - n\lambda^{n+1}}{(1-\lambda)^2}.
\end{aligned}
$$

We would like to send $\lambda = 1$ on both sides of the above identity, however the right-hand side is $0/0$ indeterminate at $\lambda = 1$, meaning we must treat it as a limit,

$$
\sum_{k=1}^{n} k = \lim_{\lambda\to 1}\frac{1-(n+1)\lambda^n - n\lambda^{n+1}}{(1-\lambda)^2}.
$$

Applying L'Hopital's rule once on the right-hand side gives

$$
\begin{aligned}
\lim_{\lambda\to 1}\frac{1-(n+1)\lambda^n - n\lambda^{n+1}}{(1-\lambda)^2} &= \lim_{\lambda\to 1}\frac{\frac{\mathrm{d}}{\mathrm{d}\lambda}(1-(n+1)\lambda^n + n\lambda^{n+1})}{\frac{\mathrm{d}}{\mathrm{d}\lambda}(1-\lambda)^2} \\
&= \lim_{\lambda\to 1}\frac{-n(n+1)\lambda^{n-1}+n(n+1)\lambda^n}{2(\lambda-1)} \\
&= \frac{n(n+1)}{2}\left(\lim_{\lambda\to 1}\frac{\lambda^{n-1}(1-\lambda)}{1-\lambda}\right) \\
&= \frac{n(n+1)}{2}\left(\lim_{\lambda\to 1}\lambda^{n-1}\right) \\
&= \frac{n(n+1)}{2}
\end{aligned}
$$

Putting everything together, we deduce the well known triangular summation formula.

$$
\sum_{k=1}^{n} k = \frac{n(n+1)}{2}.
$$

(c) Original version

$$
\begin{aligned}
\sum_{k=1}^{n} k\lambda^{k-1} = \frac{\mathrm{d}}{\mathrm{d}\lambda}\sum_{k=1}^{n}\lambda^k &= -\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\frac{1-e^{-\lambda(n+1)}}{1-e^{-\lambda}}\right) \\
&= -\frac{(n+1)e^{-\lambda(n+1)}}{1-e^{-\lambda}} + \frac{(1-e^{-\lambda(n+1)})e^{-\lambda}}{(1-e^{-\lambda})^2} \\
&= e^{-\lambda}\left(\frac{1+ne^{-\lambda(n+1)}-(n+1)e^{-\lambda n}}{(1-e^{-\lambda})^2}\right)
\end{aligned}
$$

Sending $\lambda \to 0$, gives

$$\sum_{k=1}^{n} k = \lim_{\lambda\to 0} \frac{1 + ne^{-\lambda(n+1)} - (n+1)e^{-\lambda n}}{(1-e^{-\lambda})^2}$$

which is 0/0 indeterminate. Applying L'Hopitals rule once gives

$$\begin{aligned}
\lim_{\lambda\to 0} \frac{1 + ne^{-\lambda(n+1)} - (n+1)e^{-\lambda n}}{(1-e^{-\lambda})^2} &= \frac{n(n+1)}{2}\left(\lim_{\lambda\to 0} \frac{e^{-\lambda n} - e^{-\lambda(n+1)}}{(1-e^{-\lambda})e^{-\lambda}}\right) \\
&= \frac{n(n+1)}{2}\left(\lim_{\lambda\to 0} e^{-(n-1)\lambda}\frac{1-e^{-\lambda}}{1-e^{-\lambda}}\right) \\
&= \frac{n(n+1)}{2}.
\end{aligned}$$

4. (Integrals)

(a) Calculate

$$\int_0^1 (1-x^2)^2 \mathrm{d}x$$

(b) Let $-\infty < \theta < \infty$, calculate

$$\int_{-\infty}^{\infty} \frac{1}{2} x e^{-|x-\theta|} \mathrm{d}x.$$

(Hint: use symmetry)

(c) Calculate

$$\int \ln(z)\, \mathrm{d}z$$

(d) Let $p \geq 0$. Calculate

$$\int_0^1 x^p \ln(x)\, \mathrm{d}x.$$

Solution.

(a) To solve this, we can expand the square and integrate the polynomials

$$\begin{aligned}
\int_0^1 (1-x^2)^2 \mathrm{d}x &= \int_0^1 (1 - 2x^2 + x^4) \mathrm{d}x \\
&= \left[x - \frac{2}{3}x^3 + \frac{1}{5}x^5\right]_0^1 \\
&= \left(1 - \frac{2}{3} + \frac{1}{5}\right) \\
&= \frac{8}{15}
\end{aligned}$$

(b) In order to properly use symmetry, it is convenient to first change variables $u = x - \theta$, giving

$$\int_{-\infty}^{\infty} \frac{1}{2} x e^{-|x-\theta|} \mathrm{d}x = \int_{-\infty}^{\infty} \frac{1}{2}(u+\theta)e^{-|u|}\mathrm{d}u = \frac{1}{2}\int_{-\infty}^{\infty} ue^{-|u|}\mathrm{d}u + \frac{1}{2}\theta \int_{-\infty}^{\infty} e^{-|u|}\mathrm{d}u$$

Note that $u \mapsto ue^{-|u|}$ is an odd function about 0, hence

$$\frac{1}{2}\int_{-\infty}^{\infty} ue^{-|u|}\mathrm{d}u = 0,$$

while $u \mapsto e^{-|u|}$ is an even function about 0, therefore

$$\frac{1}{2}\theta \int_{-\infty}^{\infty} e^{-|u|}\mathrm{d}u = \theta \int_{0}^{\infty} e^{-u}\mathrm{d}u = \theta(-e^{-u}|_0^{\infty} = \theta.$$

We conclude that

$$\int_{-\infty}^{\infty} \frac{1}{2} x e^{-|x-\theta|} \mathrm{d}x = \theta.$$

(c) To solve this integral, we use a sneaky integration by parts $u = \ln(z)$, $v = z$

$$\int \ln(z)\mathrm{d}z = z\ln(z) - \int z(1/z)\mathrm{d}z = z\ln(z) - z + c$$

(d) Using a similar integraion by parts strategy as the previous problem with $u = \ln(x)$, $v = \frac{1}{1+p}x^{1+p}$,

$$\begin{aligned}\int_0^1 x^p \ln(x)\mathrm{d}x &= \left[\frac{1}{1+p}x^{1+p}\ln(x)\right]_0^1 - \frac{1}{1+p}\int_0^1 x^p \mathrm{d}x \\ &= -\frac{1}{1+p}\left(\lim_{x\to 0} x^{1+p}\ln(x) + \frac{1}{1+p}\right) \\ &= -\frac{1}{(1+p)^2}\end{aligned}$$

Where we used the answer to part 2c to conclude that $\lim_{x\to 0} x^{1+p}\ln(x) = 0$.