APMA 1650

Homework 1 -Solutions

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1. [35 pts] Use DeMorgan's Theorem, the distributive property, and the fact that $(A^c)^c = A$ and $A \cup A^c = S$ to simplify the following expressions (simplified expressions are written using a minimal number of operations $\cup, \cap, {}^c$):

a. $A \cup (A^c \cap B)$

b. $A \cap (A^c \cup B)$

c. $(A \cap B) \cup (A^c \cap B)$

d. $(A^c \cup B^c) \cap (A^c \cup B)$

e. $A \cup B \cup (A^c \cap B^c)$

f. $((A \cup D)^c \cap (B^c \cup C)^c)^c$

g. $(A^c \cup D)^c \cap (B \cup C^c)^c \cap (C^c \cup D)^c$

Solution.

a. $A \cup (A^c \cap B) = (A \cup A^c) \cap (A \cup B) = S \cap (A \cup B) = A \cup B$

b. $A \cap (A^c \cup B) = (A \cap A^c) \cup (A \cap B) = \emptyset \cup (A \cap B) = A \cap B$

c. $(A \cap B) \cup (A^c \cap B) = (A \cup A^c) \cap B = S \cap B = B$

d. $(A^c \cup B^c) \cap (A^c \cup B) = A^c \cup (B \cap B^c) = A^c \cup \emptyset$

e. $A \cup B \cup (A^c \cap B^c) = (A \cup B) \cup (A \cup B)^c = S$

f. $((A \cup D)^c \cap (B^c \cup C)^c)^c = (A^c \cap D^c \cap B \cap C^c)^c = A \cup D \cup B^c \cup C$

g. $(A^c \cup D)^c \cap (B \cup C^c)^c \cap (C^c \cup D)^c = A \cap D^c \cap B^c \cap C$

2. [20 pts] Identify whether the following sets are countable or uncountable

a. $\{x^2 : x \in \mathbb{R}\}$

b. $\bigcup_{i=1}^{\infty} \{1, \dots, i\}$

c. $\{(x, y) \in \mathbb{R}^2 : x = y\}$

d. $[1, 2] - (1, 2)$

e. $\mathbb{Z}^n = \{(i_1, i_2, \dots, i_n) \,:\, i_1 \in \mathbb{Z},\ i_2 \in \mathbb{Z},\ \dots\ i_n \in \mathbb{Z}\}$, for any given $n \in \mathbb{N}$.

Solution.

a. $\{x^2 \,:\, x \in \mathbb{R}\} = [0, \infty)$ - uncountable because it an interval

b. $\bigcup_{i=1}^{\infty}\{1, \dots, i\}$ - countable because it is the countable union of countable sets

c. $\{(x, y) \in \mathbb{R}^2 \,:\, x = y\}$ - uncountable because it the just a rotated version of $\mathbb{R}$ (i.e. can be put into 1-1 correspondence with $\mathbb{R}$)

d. $[1, 2] - (1, 2) = \{1, 2\}$ - countable, it is just a finite set with two elements

e. $\mathbb{Z}^n = \{(i_1, i_2, \dots, i_n) \,:\, i_1 \in \mathbb{Z},\ i_2 \in \mathbb{Z},\ \dots\ i_n \in \mathbb{Z}\}$, for any given $n \in \mathbb{N}$. Countable, because it is a Cartesian product, or we can write it as a countable union of countable sets

$$\mathbb{Z}^n = \bigcup_{i_1 \in \mathbb{Z}} \bigcup_{i_2 \in \mathbb{Z}} \cdots \bigcup_{i_n \in \mathbb{Z}} \{(i_1, i_2, \dots, i_n)\}.$$

3. [12 pts] You roll a fair die once and record the face-up number, if the number is smaller than or equal to 3, you roll again and record that number as well. (To be clear if the first roll is less than or equal to 3 you record two numbers, otherwise you record one number)

a. Write out the sample space for this experiment. How many elements are there?

b. Write out the event that you roll the same number twice. How many elements are there?

c. Write out the event that total sum of all the numbers you roll is a 6.

Solution.

a. One way to write the sample space is

$$\begin{aligned} S = \{&(1,1), (1,2), (1,3), (1,4), (1,5), (1,6), \\ &(2,1), (2,2), (2,3), (2,4), (2,5), (2,6), \\ &(3,1), (3,2), (3,3), (3,4), (3,5), (3,6), \\ &4, 5, 6\} \end{aligned}$$

Hnce there are 21 outcomes.

b. Let A be the event that you roll the same number twice. It is given by

$$A = \{(1,1), (2,2), (3,3)\}.$$

There are 3 elements.

c. Let B be the event that the total sum of all the numbers you roll is a 6.

$$B = \{(1,5),(2,4),(3,3),6\}.$$

There are 4 events.

4.[15 pts] Suppose you are finishing a three-problem homework assignment and have put each problem on a separate page. You are down to the wire, any only have time to randomly scan pages and submit to Gradescope without checking if the pages are assigned to the correct problem. Define the events

$$\begin{aligned} A_1 &: \text{Problem 1 is correctly assigned} \\ A_2 &: \text{Problem 2 is correctly assigned} \\ A_3 &: \text{Problem 3 is correctly assigned} \end{aligned}$$

Using that $P(A_1) = P(A_2) = P(A_3) = 1/3$ and $P(A_1 \cap A_2) = P(A_2 \cap A_3) = P(A_3 \cap A_1) = P(A_1 \cap A_2 \cap A_3) = 1/6$, calculate the probability that

a. none of the problems were assigned correctly

b. exactly one problem was assigned correctly

c. exactly two problems were assigned correctly

Solution. It helps to use Venn diagram here (not included in this solution)

a.

$$\begin{aligned} &P(\text{none of the problems were assigned correctly}) \\ &\quad = 1 - P(A_1 \cup A_2 \cup A_3) \\ &\quad = 1 - (P(A_1) + P(A_2) + P(A_3)) \\ &\qquad\quad + P(A_1 \cap A_2) + P(A_2 \cap A_3) + P(A_1 \cap A_3) \\ &\qquad\quad - P(A_1 \cap A_2 \cap A_3) \\ &\quad = 1 - 3/3 + 3/6 - 1/6 = 1/3. \end{aligned}$$

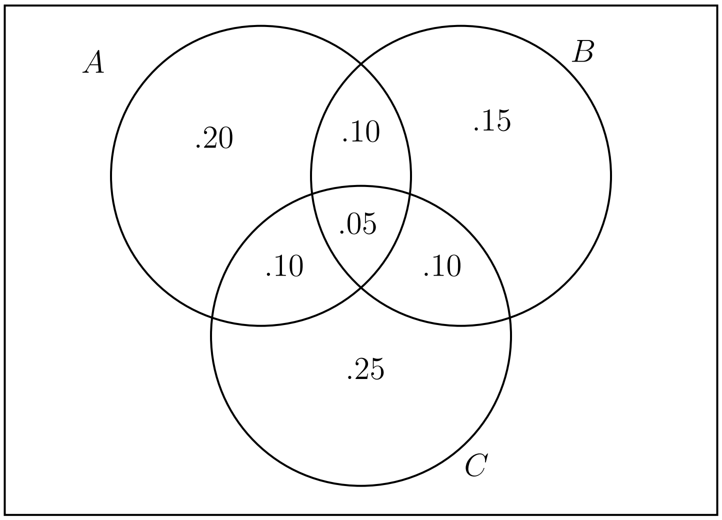
b.

$$\begin{aligned} &P(\text{exactly one problem was assigned correctly}) \\ &\quad = P(A_1) + P(A_2) + P(A_3) \\ &\qquad\quad - 2(P(A_1 \cap A_2) + P(A_2 \cap A_3) + P(A_1 \cap A_3)) \\ &\qquad\quad + 3P(A_1 \cap A_2 \cap A_3) \\ &\quad = 3/3 - 2(3/6) + 3(1/6) = 1/2 \end{aligned}$$

c.

$$\begin{aligned} &P(\text{exactly two problems were assigned correctly}) \\ &\quad = P(A_1 \cap A_2) + P(A_2 \cap A_3) + P(A_1 \cap A_3) \\ &\qquad\quad - 3P(A_1 \cap A_2 \cap A_3) \\ &\quad = 3/6 - 3(1/6) = 0 \end{aligned}$$

5. [18 pts] A group of fictional students on campus are surveyed about whether they posses any of the following three characteristics: (A) they are in a club, (B) they are in a relationship, (C) they watch sports. The percentages of students possessing each of attributes are broken down in the Venn diagram below. For example, we see that 5% of students posses all three characteristics, while 25% of students watch sports, but are not in a club or a relationship.



You randomly choose someone from this group. What is the probability that

a. they are in a club?

b. they are in a club and watch sports?

c. they are not in a relationship, but are in a club and watch sports?

d. they posses exactly one of the characteristics?

e. they posses exactly two of the characteristics?

f. they posses none of the characteristics?

Solution. Using the numbers on the Venn diagram above

a.

$$P(\text{they are in a club}) = .2 + .1 + .1 + .05 = .45$$

b.

$$P(\text{they are in a club and watch sports}) = .1 + .05 = .15$$

c.

$$P(\text{they are not in a relationship, but are in a club and watch sports}) = .1$$

d.

$$P(\text{they posses exactly one of the characteristics}) = .2 + .15 + .25 = .42$$

e.

$$P(\text{they posses exactly two of the characteristics}) = .1 + .1 + .1 = .3$$

f.

$$P(\text{they posses none of the characteristics}) = 1 - (.2 + .25 + .15 + 3(.1) + .05) = .05$$