APMA 1650

Homework 3 - Solutions

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1. (20 pts) Consider two parallel lines, one with n distinct points on it and the other with m distinct points on it.

a. (10 pts) How many triangles (three sided shapes) can you make with these points?

b. (10 pts) How many quadrilaterals (four sided shapes) can you make with these points?

Solution.

a. Triangles are made with three points that do not lie in a line. We can do this by picking one point on one line and two points on the other. The number of ways to choose 1 points on the line with n points and 2 points on the line with m is

$$\binom{n}{1}\binom{m}{2} = n\binom{m}{2}.$$

Similarly we can go the other way around and pick 1 point from the line with m points and 2 from the line with n points, giving

$$m\binom{n}{2}$$

ways to do that. Therefore, overall, there are

$$n\binom{m}{2} + m\binom{n}{2}$$

ways to make a triangle.

b. Quadrilaterals are made from 4 points, 2 points on either line and is therefore the number of ways to choose two points from one line followed by two points from the other line, which is

$$\binom{m}{2}\binom{n}{2} \quad \text{ways.}$$

2. (14 pts) A "hacker" is trying to break into your Brown account. They got their hands on a list of 18 passwords and one of them is yours! They start tying random passwords from this list, never re-trying a password. However, after 5 failed attempts, they will be locked out of your account. What is the probability that they successfully "break-in" to your account before getting locked out?

Solution. To solve this, let A be the event that the correct password is guessed before getting locked out. In this problem, it is easier to consider the complement event A^c, that is, the event that the hacker doesn't get the correct password after 5 attempts. To calculate this probability, remove your password from the list and consider all of the ways to permute 5 passwords out of the list of 17 incorrect passwords. There are P_5^{17} ways to do this. The size of the sample space is just the number of way to permute 5 things out of the original list of 18. Therefore

$$P(A^c) = \frac{P_5^{17}}{P_5^{18}} = \frac{13}{18},$$

and so

$$P(A) = 1 - P(A^c) = \frac{5}{18}.$$

You can also solve this directly without taking the complement by counting the number of ways to get the correct password somewhere in the 5 attempts. To do this, remove your password, and first consider the number of ways to guess 4 wrong attempts, that is P_4^{17} and then multiply it by all the ways to insert the correct guess between these 4 wrong attempts, which is 5. This gives

$$P(A) = \frac{5P_4^{17}}{P_5^{18}} = \frac{5}{18}$$

Alternate Solution Another way to see this problem using material from Chapter 3 is to treat this like a unordered sampling without replacement problem using the hypergeometric distriburtion. If we consider sampling 5 passwords form a list of 18 passwords with 1 correct and 17 incorrect then if X is the number of correct passwords sampled we see that

$$X \sim \text{Hypergeometric}(1, 17, 5)$$

and therefore the probability that you get hacked is

$$P(X = 1) = P_X(1) = \frac{\binom{1}{1}\binom{17}{4}}{\binom{18}{5}} = \frac{5}{18}.$$

3. (20 pts) There are two groups of students at a review session. One group (APMA students) has 8 people and the other group (CS students) has 4 people.

a. (10 pts) If 3 people are chosen from each group to partner up, how many possible pairings are there?

b. (10 pts) Suppose instead, all of the students are randomly divided up into groups of size 3. What is the probability that that each group contains a CS student?

Solution

a. First we choose groups of 3. Note that there are $\binom{8}{3}$ ways to chose the APMA students and $\binom{4}{3}$ ways to choose the CS students. Once we have these groups we can ask how many ways there are to partner them up. This is just $P_1^3 = 3!$, since for a given person, there are 3 ways to pick the first partner, 2 ways to choose a partner for the next person, and so on. This gives a total of

$$\binom{8}{3}\binom{4}{3}3! \quad \text{possible partnerings.}$$

b. First lets calculate the number of ways to divy up the students in groups of size 3, if we view the groups as ordered(i.e. group 1, group 2, ...). This is just the number of partitions of 12 things into 4 groups of size 3, given by the multinomial coefficient

$$\binom{12}{3\,3\,3\,3}.$$

(If you want to view the collection of groups as unordered then you must divide by an additional factor of 4! to account for all the ways we can swap group labels). Since there are exactly 4 CS students, the only way that each group has a CS student is if there is exactly 1 assigned to each group. As we are viewing these groups as ordered, there are 4! ways to assign these CS students to the 4 groups (if they were viewed as unordered then there is only one way). This leaves only 2 spots left in each group for the APMA students. This is the same as partitioning the 8 APMA students into groups of size 2, which is

$$\binom{8}{2\,2\,2\,2}.$$

(this doesn't depend on whether you initially viewed the groups as ordered or not, because once you assign the CS student, the groups are "labeled" by the CS student) Therefore the probability of each group containing a CS student is

$$\frac{4!\binom{8}{2\,2\,2\,2}}{\binom{12}{3\,3\,3\,3}} = \frac{4!(8!)(3^4)}{12!} = \frac{9}{55}$$

4. (18 pts)

a. (8 pts) How many distinct ways can you arrange the letters in the word NARRAGANSETT?

b. (10 pts) If you pick one of these distinct rearrangements at random, what is the probability that the letters G and S are not next to each other?

Solution.

a. NARRAGANSETT has 12 letters with 2 N's, 3 A's, 2 R's, 1 G, 1 S, 1 E, and 2 T's. The number of distinct rearrangements of this word is just the number of ways to partition 12 "slots" into groups of size 2,3, 2, 1, 1, 1, and 2, and is given by

$$\binom{12}{2\,3\,2\,1\,1\,1\,2} = \frac{12!}{(2)^3 3!} = 9979200.$$

b. Let A be the event that G and S are not next to each other. Lets first calculate the ways that G and S *do* remain together. First remove G and S. For the remaining spots, there are

$$\binom{10}{2\,3\,2\,1\,2}$$

ways to distinctly rearrange the remaining letters. G and S can then be placed together as GS or SG. Each choice of SG or GS can be placed back into the word 11 ways, for a total of 22 ways. This means that there are

$$22\binom{10}{2\,3\,2\,1\,2} = \frac{22(10!)}{(2)^3 3!}$$

ways for S and G to end up next to each other. Therefore

$$P(A^c) = \frac{\frac{22(10!)}{(2)^3 3!}}{\frac{12!}{(2)^3 3!}} = \frac{22(10!)}{12!} = \frac{1}{6},$$

and so

$$P(A) = 1 - \frac{1}{6} = \frac{5}{6}.$$

5. (28 pts) 5 cards are dealt from a modified 48 card deck. In this deck, there are still exactly 4 suits ($\heartsuit, \diamondsuit, \spadesuit, \clubsuit$) but only 12 ranks in the following order (2,3,4,5,6,7,8,9,10,J,Q,K) (the Aces are missing).

a. (8 pts) What is the probability that exactly two cards have the same rank?

b. (10 pts) What is the probability that there is exactly one pair of one rank, one pair of a different rank and the remaining card has a third rank? (e.g. $3\heartsuit, 3\spadesuit, 7\heartsuit, 7\diamondsuit, K\clubsuit$)

c. (10 pts) What is the probability that your hand contains 5 cards of sequential rank (e.g. 8,9,10,J,Q)?

Solution.

a. We want to consider the probability that exactly two cards have the same rank. This means that the remaining cards must all have a different rank (otherwise there wouldn't be exactly two cards with the same rank). Lets calculate the number of ways to that exactly two cards can have the same rank in a hand of 5 cards from this modified deck. We will do this be a sequence of actions and use the rule of multiplication to count how many ways it can be done.

- First, pick the rank of the pair out of the 12 ranks: $\binom{12}{1}$
- Next, for the pair we choose 2 out of the four suits: $\binom{4}{2}$
- Since the remaining 3 cards all have to have different rank, we choose 3 ranks out of the remaining 11: $\binom{11}{3}$.
- Finally assign a suit to each of the three remaining cards $\binom{4}{1}^3$.

Therefore, the number of ways to get exactly two cards of the same rank is

$$\binom{12}{1}\binom{4}{2}\binom{11}{3}\binom{4}{1}^3.$$

The total number of 5 card hands that can be dealt is $\binom{48}{5}$ and therefore the probability of getting exactly two cards with the same rank is

$$\frac{\binom{12}{1}\binom{4}{2}\binom{11}{3}\binom{4}{1}^3}{\binom{48}{5}}$$

b. We follow the same procedure for the two pairs.

- First, pick the rank of the 2 pairs out of the 12 ranks: $\binom{12}{2}$
- Next, for the "lower" pair we choose 2 out of the four suits: $\binom{4}{2}$
- Similarly for the "higher" pair we chose 2 suits: $\binom{4}{2}$
- Since the remaining 1 card has to have a different rank we choose its rank out of the 10 remaining ranks: $\binom{10}{1}$.
- Assign a suit to the last card $\binom{4}{1}$.

This gives a probability of

$$\frac{\binom{12}{2}\binom{4}{2}^2\binom{10}{1}\binom{4}{1}}{\binom{48}{5}}$$

c. For the "straight" of sequential cards, we note that the lowest card can range anywhere from 2 to 9 (8 possible ranks). Once the rank of the lowest card is chosen, the ranks of the remaining cards are chosen. We consider the sequence of actions

- Pick the rank of the lowest card: $\binom{8}{1}$
- Next, pick the suit of each of the cards: $\binom{4}{1}^5$

This gives a probability of

$$\frac{\binom{8}{1}\binom{4}{1}^5}{\binom{48}{5}}$$