APMA 1650

Homework 6 - Solutions

Problem 1. (32 pts) You work for an insurance company that covers home flood damage. Suppose that in a flood event, the loss X to a policy holder (measured in tens of thousands of dollars) can modeled by an exponential random variable

$$X \sim \text{Exponential}(\lambda).$$

Suppose the policy has a deductible of $d \geq 0$ and a payout limit of $l \geq 0$ (both measured in tens of thousands of dollars). This means that as long as the loss surpasses the deductible amount, the insurance company will pay the policy holder $X - d$ dollars up to the payout limit l. Let Y be the total payout for the policy holder.

a. (4 pts) Write the payout Y as a piecewise defined function of X. Is Y continuous, discrete, or mixed? What is its range?

b. (10 pts) Find the CDF and generalized PDF of Y.

c. (8 pts) What is the expected payout?

d. (10 pts) The deductible can be used as a simple mechanism to reduce the expected payout without changing the payout limit. Suppose that for this particular flood event that the loss has $\lambda = 0.1$. If the payout limit for this policy is 200 thousand dollars, what is the minimum deductible that ensures that the expected payout is at most 80 thousand dollars. State your answer in numbers of dollars, rounded to the nearest dollar.

Solution:

a. The payout can be written as

$$Y = \begin{cases} 0 & X - d \leq 0 \\ X - d & 0 < X - d \leq l \\ l & X - d > l \end{cases}.$$

Or another way of writing it is

$$Y = \min\{\max\{X - d, 0\}, l\}.$$

b. Note that Y has a range of $[0, l]$. The generalized CDF can be calculated as follows. Suppose $y < 0$, then clearly since Y is always non-negative

$$P(Y \leq y) = 0.$$

Similarly, when $0 \le y < l$ we have

$$\begin{aligned} P(Y \le y) &= P(Y = 0) + P(0 < Y \le y) \\ &= P(X \le d) + P(0 \le X - d < y) \\ &= P(X \le d + y) \\ &= 1 - e^{-\lambda(d+y)} \end{aligned}$$

Finally, when $y \ge l$, since Y is always less then l

$$P(Y \le y) = P(Y \le l) = 1.$$

Therefore we can write the generalized CDF as

$$F_Y(y) = \begin{cases} 0 & y < 0 \\ 1 - e^{-\lambda(d+y)} & 0 \le y < l \\ 1 & l \le y \end{cases}.$$

This CDF has jumps of size $1 - e^{-\lambda d}$ and $e^{-\lambda(d+l)}$ at $y = 0$ and $y = l$ respectively. Therefore the generalized PDF is just given by taking the derivative and using the delta function

$$f_Y(y) = (1 - e^{-\lambda d})\delta(y) + e^{-\lambda(d+l)}\delta(y - l) + \lambda e^{-\lambda(d+y)} I_{[0,l]}(y),$$

where $I_{[0,l]}(y)$ is just the indicator function

$$I_{[0,l]}(y) = \begin{cases} 1 & y \in [0, l] \\ 0 & \text{otherwise} \end{cases}.$$

c. Using the above generalized PDF and the properties of the delta function, we find

$$\begin{aligned} EY &= \int_{-\infty}^{\infty} y f_Y(y) \mathrm{d}y = 0(1 - e^{-\lambda d}) + l e^{-\lambda(d+l)} + \int_0^l y \lambda e^{-\lambda(d+y)} \mathrm{d}y \\ &= l e^{-\lambda(d+l)} - y e^{-\lambda(d+y)} \Big|_0^{\ell} + \int_0^l e^{-\lambda(d+y)} \mathrm{d}y \\ &= e^{-\lambda d} \left(l e^{-\lambda l} - l e^{-\lambda l} + \frac{1}{\lambda}(1 - e^{-\lambda l}) \right) \\ &= \frac{e^{-\lambda d}}{\lambda}(1 - e^{-\lambda l}) \end{aligned}$$

d. In this problem $\lambda = 0.1$ and $l = 20$. Therefore in mathematical language, we want

$$EY = 10 e^{-d/10}(1 - e^{-2}) \le 8$$

solving for d, this gives

$$d \ge 10 \ln\left(\frac{10}{8}(1 - e^{-2})\right) \approx .7773.$$

Therefore the deductible should be at least $7,773$ dollars.

Problem 2. (28 pts) Suppose that X is a *non-negative* continuous random variable.

a. (10 pts) Show the following alternate formula for the expectation,

$$EX = \int_0^\infty x f_X(x)\mathrm{d}x = \int_0^\infty P(X \geq y)\,\mathrm{d}y$$

(Hint: Write $x = \int_0^x 1\mathrm{d}y$ and change the order of integration).

b. (8 pts) Let g be a non-negative, differentiable, strictly increasing function on $[0, \infty)$. Use the formula from part a to show that

$$Eg(X) = \int_0^\infty P(X \geq g^{-1}(y))\mathrm{d}y.$$

c. (10 pts) Use the formulas deduced in parts a and b to calculate the mean and variance of $X \sim \text{Exponential}(\lambda)$.

Solution:

a. Following the hint, we see that

$$EX = \int_0^\infty x f_X(x)\mathrm{d}x = \int_0^\infty \left(\int_0^x f_X(x)\mathrm{d}y\right)\mathrm{d}x$$

Note that this is just an integration over the "triangular" region

$$D = \{(x, y) \,:\, 0 \leq y \leq x \leq 1\}.$$

Changing the order of integration gives

$$\begin{aligned} EX &= \int_0^\infty \left(\int_y^\infty f_X(x)\mathrm{d}x\right)\mathrm{d}y \\ &= \int_0^\infty P(X \geq y)\,\mathrm{d}y. \end{aligned}$$

b. This one is easy. Replace X with $g(X)$ in the above formula to get

$$Eg(X) = \int_0^\infty P(X \geq g^{-1}(y))\,\mathrm{d}y.$$

c. For the exponential random variable, we know that

$$P(X \geq y) = e^{-\lambda y}.$$

Therefore using the formula from part a

$$EX = \int_0^\infty e^{-\lambda y}\mathrm{d}y = \frac{-1}{\lambda}e^{-\lambda y}\Big|_0^\infty = \frac{1}{\lambda}$$

and

$$\begin{aligned}
EX^2 &= \int_0^\infty P(X \geq \sqrt{y})\mathrm{d}y \\
&= \int_0^\infty e^{-\lambda\sqrt{y}}\mathrm{d}y \\
&= \int_0^\infty 2ue^{-\lambda u}\mathrm{d}u, \quad \text{substitution} \quad u = \sqrt{y}, \quad 2u\mathrm{d}u = \mathrm{d}y \\
&= \frac{-2u}{\lambda}e^{-\lambda u}\Big|_0^\infty + \frac{1}{\lambda}\int_0^\infty 2e^{-\lambda u}\mathrm{d}u \quad \text{integration by parts} \\
&= \frac{-2}{\lambda^2}e^{-\lambda u}\Big|_0^\infty = \frac{2}{\lambda^2}
\end{aligned}$$

Therefore

$$\mathrm{Var}(X) = EX^2 - (EX)^2 = \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2 = \frac{1}{\lambda^2}$$

Problem 3. (10 pts) Evaluate the area integral

$$\iint_R 15x^2 - 6y\,\mathrm{d}A,$$

where R is the region bounded by $x = \frac{1}{2}y^2$ and $x = 4\sqrt{y}$.

Solution: First lets find the points where $x = \frac{1}{2}y^2$ and $x = 4\sqrt{y}$ intersect. They are when

$$\frac{1}{2}y^2 = 4\sqrt{y} \quad \Rightarrow \quad y^4 - 64y \quad \Rightarrow \quad y(y^3 - 64) = 0.$$

This means that the two curves intersect between $y = 0$ and $y = 64^{1/3} = 4$. Moreover, when $0 \leq y \leq 4$ the value $4\sqrt{y}$ is bigger than $\frac{1}{2}y^2$. Therefore the region between the two curves is described by

$$R = \left\{(x, y) \,:\, 0 \leq y \leq 4, \quad \frac{1}{2}y^2 \leq x \leq 4\sqrt{y}\right\}.$$

We can then write the above integral as an iterated integral over a region bounded by two vertical curves

$$\iint_R 15x^2 - 6y\,\mathrm{d}A = \int_0^4 \left(\int_{\frac{1}{2}y^2}^{4\sqrt{y}} (15x^2 - 6y)\mathrm{d}x \right) \mathrm{d}y$$
$$= \int_0^4 5x^3 - 6xy\Big|_{\frac{1}{2}y^2}^{4\sqrt{y}} \mathrm{d}y$$
$$= \int_0^4 296y^{3/2} + 3y^3 - \frac{5}{8}y^6 \mathrm{d}y$$
$$= \Big(\frac{592}{5}y^{5/2} + \frac{3}{4}y^4 - \frac{5}{56}y^7\Big|_0^4$$
$$= \frac{18944}{5} + 192 - \frac{10240}{7}$$
$$= \frac{88128}{35}$$
$$\approx 2517.94.$$

Problem 4. (10 pts) Evaluate the area integral

$$\iint_R \sin(3x^2 + 3y^2)\,\mathrm{d}A,$$

where R is the region between $x^2 + y^2 = 1$ and $x^2 + y^2 = 7$.

Solution. The fact that the regions are circular and that the integrand can be expressed as a function of $x^2 + y^2 = r^2$ indicates that we should express this integral in polar coordinates. In polar coordinates the region of integration can be written in polar coordinates as

$$R = \{(r\cos\theta, r\sin\theta) \,:\, 0 \leq \theta \leq 2\pi, \quad 1 \leq r \leq \sqrt{7}\}$$

Therefore we can write the integral in polar coordinates as

$$\iint_R \sin(3x^2 + 3y^2)\,\mathrm{d}A = \int_0^{2\pi} \left(\int_1^{\sqrt{7}} \sin(3r^2) r \mathrm{d}r \right) \mathrm{d}\theta$$
$$= \pi \int_1^7 \sin(3u)\mathrm{d}u$$
$$= \frac{\pi}{3}\Big(- \cos 3u\Big|_1^7$$
$$= \frac{\pi}{3}(\cos 3 - \cos 21)$$
$$\approx -0.4631.$$

Problem 5. (20 pts) Suppose you are playing a game of darts and throwing the darts at the circular board of radius 1. Lets assume that the dart board B is given by a disk centered at the origin

$$B = \{(x, y) \,:\, x^2 + y^2 \leq 1\}.$$

The dart board is mounted to a rectangular mat $M = [-2, 2] \times [-2, 2]$. You aren't very good at darts, but at least you will always hit the mat behind the board. Suppose the probability that you hit a particular region E of the mat is given by an area integral

$$P(E) = c \iint_E (4 - x^2)(4 - y^2)\, \mathrm{d}A$$

a. (10 pts) What is the value of c so that the probability of hitting the mat $P(M) = 1$?

b. (10 pts) What is the probability that you hit the dart board, $P(B)$? State your answer to at least two decimal places. Hint: Use polar coordinates. You may use (without derivation) the fact that

$$\int_0^{2\pi} \cos^2(\theta) \sin^2(\theta) \mathrm{d}\theta = \pi/4.$$

Solution.

a. M is a rectangular region and the integral $(4 - x^2)(4 - y^2)$ is separable. Therefore we can write the integral as

$$\begin{aligned}
P(M) &= c \left(\int_{-2}^{2} (4 - x^2) \mathrm{d}x \right) \left(\int_{-2}^{2} (4 - y^2) \mathrm{d}y \right) \\
&= c \left(\int_{-2}^{2} (4 - x^2) \mathrm{d}x \right)^2 \\
&= c \left(4x - \frac{1}{3} x^3 \Big|_{-2}^{2} \right)^2 \\
&= c \left(16 - \frac{16}{3} \right)^2 \\
&= c \frac{1024}{9}
\end{aligned}$$

Therefore, for $P(M) = 1$, we need $c = 9/1024$.

b. To calculate this, we will find it convenient to switch to polar coordinates, where it is easy to describe the dart board. In this case the integrand transforms to

$$(4 - x^2)(4 - y^2) = 16 - 4(x^2 + y^2) + x^2 y^2 = 16 - 4r^2 + r^4 \cos^2(\theta) \sin^2(\theta).$$

Therefore the integral becomes

$$\begin{aligned}
P(B) &= \left(\frac{9}{1024}\right)\int_0^{2\pi}\left(\int_0^1 (16-4r^2+r^4\cos^2(\theta)\sin^2(\theta))r\mathrm{d}r\right)\mathrm{d}\theta \\
&= \left(\frac{9}{1024}\right)\int_0^{2\pi}\left(8r^2-r^4+\frac{1}{6}r^6\cos^2(\theta)\sin^2(\theta)\right|_0^1 \mathrm{d}\theta \\
&= \left(\frac{9}{1024}\right)\int_0^{2\pi}\left(7+\frac{1}{6}\sin^2(\theta)\cos^2(\theta)\right)\mathrm{d}\theta \\
&= \left(\frac{9}{1024}\right)\left(14\pi+\frac{\pi}{24}\right) \\
&= \pi\left(\frac{1011}{8192}\right) \\
&\approx 0.3877.
\end{aligned}$$

It looks like you are particularly bad at darts...