APMA 1650

Homework 6

Instructions: Homework is due by 11:59pm EST in Gradescope on the day listed on the course webpage. You can use calculators for this assignment. Solutions must be written independently and *cannot* be shared with any other students.

You must *show all work* and *explain your answers thoroughly* to get full credit. **You will be graded partly on how well you explain the answers**.

There will be a 5pt penalty for homework submitted with problems incorrectly assigned to a page. A 10pt penalty will be applied for homework submitted during the late window.

Problem 1. (32 pts) You work for an insurance company that covers home flood damage. Suppose that in a flood event, the loss X to a policy holder (measured in tens of thousands of dollars) can modeled by an exponential random variable

$$X \sim \text{Exponential}(\lambda).$$

Suppose the policy has a deductible of d and a payout limit of l, $0 \leq d < l$ (both measured in tens of thousands of dollars). This means that as long as the loss surpasses the deductible amount, the insurance company will pay the policy holder $X - d$ dollars up to the payout limit l. Let Y be the total payout for the policy holder.

a. (4 pts) Write the payout Y as a piecewise defined function of X. Is Y continuous, discrete, or mixed? What is its range?

b. (10 pts) Find the CDF and generalized PDF of Y.

c. (8 pts) What is the expected payout?

d. (10 pts) The deductible can be used as a simple mechanism to reduce the expected payout without changing the payout limit. Suppose that for this particular flood event that the loss has $\lambda = 0.1$. If the payout limit for this policy is 200 thousand dollars, what is the minimum deductible that ensures that the expected payout is at most 80 thousand dollars. State your answer in numbers of dollars, rounded to the nearest dollar.

Problem 2. (28 pts) Suppose that X is a *non-negative* continuous random variable.

a. (10 pts) Show the following alternate formula for the expectation,

$$EX = \int_0^\infty x f_X(x) \mathrm{d}x = \int_0^\infty P(X \geq y) \, \mathrm{d}y$$

(Hint: Write $x = \int_0^y 1 \mathrm{d}y$ and change the order of integration).

b. (8 pts) Let g be a non-negative, strictly increasing function on $[0, \infty)$. Use the formula from part a to show that

$$Eg(X) = \int_0^\infty P(X \geq g^{-1}(y))\mathrm{d}y.$$

c. (10 pts) Use the formulas deduced in parts a and b to calculate the mean and variance of $X \sim \text{Exponential}(\lambda)$.

Problem 3. (10 pts) Evaluate the area integral

$$\iint_R 15x^2 - 6y\,\mathrm{d}A,$$

where R is the region bounded by $x = \frac{1}{2}y^2$ and $x = 4\sqrt{y}$.

Problem 4. (10 pts) Evaluate the area integral

$$\iint_R \sin(3x^2 + 3y^2)\,\mathrm{d}A,$$

where R is the region between $x^2 + y^2 = 1$ and $x^2 + y^2 = 7$.

Problem 5. (20 pts) Suppose you are playing a game of darts and throwing the darts at the circular board of radius 1. Lets assume that the dart board B is given by a disk centered at the origin

$$B = \{(x, y) \,:\, x^2 + y^2 \leq 1\}.$$

The dart board is mounted to a rectangular mat $M = [-2, 2] \times [-2, 2]$. You aren't very good at darts, but at least you will always hit the mat behind the board. Suppose the probability that you hit a particular region E of the mat is given by an area integral

$$P(E) = c \iint_E (4 - x^2)(4 - y^2)\,\mathrm{d}A$$

a. (10 pts) What is the value of c so that the probability of hitting the mat $P(M) = 1$?

b. (10 pts) What is the probability that you hit the dart board, $P(B)$? State your answer to at least two decimal places. Hint: Use polar coordinates. You may use (without derivation) the fact that

$$\int_0^{2\pi} \cos^2(\theta)\sin^2(\theta)\mathrm{d}\theta = \pi/4.$$