APMA 1650

Homework 7

Instructions: Homework is due by 11:59pm EST in Gradescope on the day listed on the course webpage. You can use calculators for this assignment. Solutions must be written independently and *cannot* be shared with any other students.

You must *show all work* and *explain your answers thoroughly* to get full credit. **You will be graded partly on how well you explain the answers**.

There will be a 5pt penalty for homework submitted with problems incorrectly assigned to a page. A 10pt penalty will be applied for homework submitted during the late window.

Problem 1. (22 pts) Suppose that you are taking only two classes this semester (sounds nice). The professor for each class has three different dates to choose for their final exam (Day 1,Day 2 or Day 3). Each professor chooses one of the three days at random for their final. Let Y_1 be the number of exams you have on day 1, and Y_2 be the number of exams you have on day 2.

a. (5 pts) Find the joint PMF for Y_1 and Y_2. Write it as a table.

b. (5 pts) What is $F(1, 0)$?

c. (5 pts) Find the marginal PMFs for Y_1 and Y_2 from the joint PMF.

d. (2 pts) Identify the name of the distribution for Y_1 and give it's appropriate parameters. Explain why this is the case.

e. (5 pts) Are Y_1 and Y_2 independent? Explain.

Problem 2. (20 pts) You have a bag with 9 balls in it 4 red, 3 green and 2 blue. You reach in an pull out 3 balls.Out of the three balls you pulled out, let X_1 be the number of red balls, and X_2 be the number of green balls.

a. (5 pts) What is the joint PMF of X_1 and X_2? State it as a function of x_1, x_2. Clearly state the range R_{X_1,X_2}. You may leave your answer in terms of combinatorial quantities.

b. (5 pts) Are X_1 and X_2 independent?

c. (5 pts) What is the conditional PMF of X_1 given $X_2 = x_2$? State it as a function of x_1 and x_2.

d. (5 pts) What is the probability $P(X_1 = 1 | X_2 \geq 1)$?

Problem 3. (25 pts) Let X and Y be two jointly continuous random variables with joint PDF

$$f_{XY}(x, y) = \begin{cases} cx & 0 \leq x \leq 1, \quad 0 \leq y \leq x^2 \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

a. (5 pts) Find the constant c that makes f_{XY} a valid joint PDF.

b. (5 pts) Find the marginal PDFs of X and Y. Be sure to state their range.

c. (5 pts) Find the conditional densities $f_{X|Y}(x|y)$, $f_{Y|X}(y|x)$. Be sure to state them as PDFs for each fixed value of the conditioned variable (e.g. state $f_{X|Y}(x|y)$ as a PDF in x for each fixed value $y \in R_Y$.) What is the "name" of the conditional distribution of Y given $X = x$?

d. (5 pts) Find the conditional expectations $E[X|Y = y]$ and $E[Y|X = x]$.

e. Let $Z = E[Y|X]$. What is the PDF of Z?

Problem 4. (15 pts) Let (Y_1, Y_2) be uniformly distributed in the triangle $T \subset \mathbb{R}^2$ with corners

$$(0, 0), (3, 1), (3, 3).$$

a. (5 pts) What is the joint PDF of Y_1, Y_2? Be sure to specify it's domain and make sure that it is a valid density function.

b. (5 pts) What is $P(Y_1 \geq 1, Y_2 \leq 1)$? (It helps to draw a picture!)

c. (5 pts) What is $P(3Y_2 - 2Y_1 \geq 0)$? (Again, draw a picture!)

Problem 5. (18 pts) Suppose that the number of defects Y in a given chip fabrication process is known to follow a Poisson distribution with rate Λ. However, the rate Λ is itself an Exponential(2) random variable.

a. (5 pts) Use the law of iterated expectation find the expected number of defects per chip by first finding the expected number of defects for a given $\Lambda = \lambda$.

b. (5 pts) Use the law of total variance to find the variance of Y.

c. (8 pts) Show that Y is just a Geometric random variable $Y \sim \text{Geometric}(p)$. What is the value of p? (Hint: Use the law of total probability and the definition of the Gamma function $\Gamma(k + 1) = \int_0^\infty x^k e^{-x} \mathrm{d}x = k!$ to find the PMF of Y).