APMA 1650

Homework 8

Instructions: Homework is due by 11:59pm EST in Gradescope on the day listed on the course webpage. You can use calculators for this assignment. Solutions must be written independently and *cannot* be shared with any other students.

You must *show all work* and *explain your answers thoroughly* to get full credit. **You will be graded partly on how well you explain the answers**.

There will be a 5pt penalty for homework submitted with problems incorrectly assigned to a page. A 10pt penalty will be applied for homework submitted during the late window.

Problem 1. (20 pts)

a. (10 pts) Let $X \sim \text{Gamma}(n, \lambda)$ and $Y \sim \text{Exponential}(\lambda)$ be independent. Use the convolution to show that

$$Z = X + Y \sim \text{Gamma}(n+1, \lambda).$$

b. (10 pts) Let $X \sim \text{Pascal}(n, p)$ and $Y \sim \text{Geometric}(p)$ be independent. Use the convolution to show that

$$Z = X + Y \sim \text{Pascal}(n+1, p).$$

Hint: You may find useful the following "hockey stick" identity

$$\sum_{k=0}^{m} \binom{n+k}{k} = \binom{n+m+1}{m}$$

Problem 2. (50 pts) A certain hospital is stocking COVID vaccines at the beginning of the week. The hospital has only a certain amount of freezer space to store the vaccine and the supply varies from week to week. Let Y_1 denote the supply of vaccines at the beginning of the week, measured in terms of the proportion of the capacity of the freezer (assumed to be a number from 0 to 1). Assume the PDF of Y_1 is given by

$$f_{Y_1}(y_1) = \begin{cases} 2y_1 & 0 \le y_1 \le 1 \\ 0 & \text{otherwise} \end{cases}.$$

Let Y_2 be the amount of vaccines actually administered during the week (measured in terms of proportion of freezer capacity). Assume that the conditional PDF for Y_2 given Y_1 is given for each $0 < y_1 \le 1$ by

$$f_{Y_2|Y_1}(y_2|y_1) = \begin{cases} \frac{3y_2^2}{y_1^3} & 0 \le y_2 \le y_1 \\ 0 & \text{otherwise} \end{cases}.$$

a. (10 pts) Find EY_1 and EY_2

b. (10 pts) Find $\text{Var}(Y_1)$ and $\text{Var}(Y_2)$.

c. (20 pts) Find the covariance and correlation of Y_1 and Y_2. Explain in words what the sign your correlation means here.

d. (10 pts) The amount of surplus vaccines at the end of the week is $Y_2 - Y_1$. Find $E(Y_1 - Y_2)$ and $\text{Var}(Y_1 - Y_2)$ using your answers to parts a-c.

Problem 3. (30 pts) The average weight of a golden retriever is 70 pounds.

a. (10 pts) Give an upper bound on the probability that a certain golden retriever is at least 80 pounds.

b. (10 pts) Suppose you know the standard deviation for the weight distribution is 5 pounds. Find a lower bound on the probability that a certain golden retriever is between 82 and 58 pounds.

c. (10 pts) Suppose you know that the weight distribution is normal (with the same mean and standard deviation given above). Repeat part b using the CDF calculator for the normal distribution. How close was your estimate in part b?