SCORE: ....OVER TOTAL: **90 points** ....

MATH 122 - CALCULUS II

SOLUTION TO MIDTERM EXAM III

Problem 1 (15 points)**.** *Solve the differential equation* $xy' - 2y = x^2$, $x > 0$.

Solution 1. Divide both sides by x to get: $y' - \frac{2}{x}y = x$. This is a **linear** differential equation. The *integrating factor* is:

$$I(x) = e^{\int -\frac{2}{x}dx} = e^{-2\ln x} = \left(e^{\ln x}\right)^{-2} = x^{-2} = \frac{1}{x^2}.$$

Multiply both sides of our equation with $I(x)$:

$$\frac{1}{x^2}y' - \frac{2}{x^3}y = \frac{1}{x}$$
$$\left(\frac{1}{x^2}y\right)' = \frac{1}{x}.$$

Integrate both sides:

$$\frac{1}{x^2}y = \int \frac{1}{x}dx$$
$$= \ln x + C.$$

Thus, $y = x^2(\ln x + C)$.

Problem 2 (45 points)**.** *Let C be the curve with parametric equations:*

$$\begin{cases} x = 3t - t^3 \\ y = 3t^2. \end{cases}$$

(1) *Find the Cartesean equation for the tangent line of C at the point $(\frac{11}{8}, \frac{3}{4})$.*
(2) *Find the points on C where the tangent line is horizontal or vertical.*
(3) *Find the arc length along C from $t = 0$ to $t = 1$.*

Solution 2. (1) We shall use the formula $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$, where

$$\frac{dy}{dt} = 6t$$
$$\frac{dx}{dt} = 3 - 3t^2.$$

At $(\frac{11}{8}, \frac{3}{4})$, the value of t is $\frac{1}{2}$. Thus,

$$\left.\frac{dy}{dx}\right|_{(\frac{11}{8},\frac{3}{4})} = \frac{3}{9/4} = \frac{4}{3}.$$

This is the slope of the tangent line at $(\frac{11}{8}, \frac{3}{4})$, so the equation of the tangent line at this point is:

$$y - \frac{3}{4} = \frac{4}{3}\Big(x - \frac{11}{8}\Big).$$

(2) Solve $dy/dt = 0$ we get $t = 0$. At $t = 0$, $dx/dt = 3 \neq 0$. Thus, at $t = 0$, i.e. at the point $(0, 0)$ the tangent line is horizontal.

Solve $dx/dt = 0$ we get $t = \pm 1$. At $t = \pm 1$, $dy/dt = \pm 6 \neq 0$. Thus, at $t = \pm 1$, i.e. at the points $(2, 3)$ and $(-2, 3)$, the tangent lines are vertical.

(3) The arc length L along C from $t = 0$ to $t = 1$ is:

$$\begin{aligned} L &= \int_0^1 \sqrt{\Big(\frac{dx}{dt}\Big)^2 + \Big(\frac{dy}{dt}\Big)^2} dt \\ &= \int_0^1 \sqrt{(3 - 3t^2)^2 + (6t)^2} dt \\ &= \int_0^1 \sqrt{9 - 18t^2 + 9t^4 + 36t^2} dt \\ &= \int_0^1 \sqrt{(3 + 3t^2)^2} dt = \int_0^1 (3 + 3t^2) dt \\ &= (3t + t^3)\Big|_0^1 = 4. \end{aligned}$$

Problem 3 (30 points)**.** *Determine if the following sequences are convergent or divergent. Explain your answers in detail.*

(1) $\left\{\frac{(-1)^n \ln n}{n}\right\}_{n=1}^{\infty}$.

(2) $\{a_n\}_{n=1}^{\infty}$ *given by* $a_1 = 2$ *and* $a_{n+1} = \sqrt{2a_n - 1}$ *for* $n \geq 1$.

Solution 3. (1) For $a_n = \frac{(-1)^n \ln n}{n}$, $|a_n| = \frac{\ln n}{n}$. Consider the function

$$f(x) = \frac{\ln x}{x}.$$

As $x \to \infty$, by applying l'Hospital's rule we get:

$$\lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{1/x}{1} = 0.$$

Thus, $|a_n| \to 0$ as $n \to \infty$. By the absolute value test, this implies that $\{a_n\}$ converges to 0 as $n \to \infty$.

(2) From the definition, $a_n \geq 0$ for any $n \geq 1$.

Observe that $a_n^2 - a_{n+1}^2 = a_n^2 - (2a_n - 1) = a_n^2 - 2a_n + 1 = (a_n - 1)^2 \geq 0$. Thus, $a_n^2 \geq a_{n+1}^2$. That is, $a_n \geq a_{n+1}$.

We have shown that $\{a_n\}$ is a decreasing sequence which is bounded below by 0 (and clearly, bound above by $a_1 = 2$). Therefore, $\{a_n\}$ is convergent.