Math 122 - Exam 1

Name: ______________________

TA (Lab time): ______________________

OFFICIAL USE ONLY
P1 ______
P2 ______
P3 ______
P4 ______
Total: ______

INSTRUCTIONS — READ THIS NOW

- Print your name in CAPITAL letters and your lab time **right now**.
- Show your work step by step. To receive full credit, your answers must be neatly written, logically organized, and clearly explained.
- If you need more space, write on the back of the previous sheet.
- Please turn off all cellphones, audio recording/playing devices and any other electronic device that could disrupt others taking the test around you.
- This is a 50 minutes test. Symbolic calculators are not permitted. **You DO NOT need to SIMPLIFY your answers UNLESS SPECIFICALLY ASKED to do so.**

Formulas:

- $\int x^n dx = \frac{x^{n+1}}{n+1} + C \ (n \neq -1)$
- $\int \frac{1}{x} dx = \ln|x| + C$
- $\int e^x dx = e^x + C$
- $\int a^x dx = \frac{a^x}{\ln a} + C$
- $\int \sin x \ dx = -\cos x + C$
- $\int \cos x \ dx = \sin x + C$
- $\int \sec^2 x \ dx = \tan x + C$
- $\int \csc^2 x \ dx = -\cot x + C$
- $\int \sec x \tan x \ dx = \sec x + C$
- $\int \csc x \cot x \ dx = -\csc x + C$
- $\int \tan x \ dx = \ln|\sec x| + C$
- $\int \cot x \ dx = \ln|\sin x| + C$
- $\int \frac{1}{x^2+a^2} dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$
- $\int \frac{1}{\sqrt{a^2-x^2}} dx = \sin^{-1}\left(\frac{x}{a}\right) + C$

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P1 (10 points). Let A be the region enclosed by the curves $y = \sqrt{x}$ and $y = x^2$. Let S be the solid obtained by rotating A about the y-axis.

(a) Find the area of A.

(b) Use the method of cylindrical shells to compute the volume of S.

Solution: (Students should sketch the graph!)

(a) The two curves $y = \sqrt{x}$ and $y = x^2$ intersect at $(0,0)$ and $(1,1)$. On $[0,1]$, the graph of $y = \sqrt{x}$ lies above the graph of $y = x^2$. Thus, the area of A is

$$\begin{aligned}\int_0^1 (\sqrt{x} - x^2)dx &= \Big[\frac{2}{3}x^{\frac{3}{2}} - \frac{1}{3}x^3\Big]_0^1 \\ &= (\frac{2}{3} - \frac{1}{3}) - 0 = \frac{1}{3}.\end{aligned}$$

(b) We use the formula

$$\text{Volume of } S = \int_a^b [\text{circumference}][\text{height}]dx$$

to get

$$\begin{aligned}\text{Volume of } S &= \int_0^1 2\pi x(\sqrt{x} - x^2)dx = 2\pi \int_0^1 (x^{\frac{3}{2}} - x^3)dx \\ &= 2\pi \Big[\frac{2}{5}x^{\frac{5}{2}} - \frac{1}{4}x^4\Big]_0^1 \\ &= 2\pi \Big[\Big(\frac{2}{5} - \frac{1}{4}\Big) - 0\Big] = 2\pi\frac{3}{20} \\ &= \frac{3\pi}{10}.\end{aligned}$$

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P2 (10 points).

(a) When a particle is located a distance x feet from the origin, a force of $x^3 + 2x^2 - 1$ pounds acts on it. How much work is done in moving the particle from $x = 1$ to $x = 2$?

(b) A force of 30 N is required to hold a spring that has been stretched from its natural length of 15 cm to a length of 20 cm. How much work will be done in stretching the spring from 20 cm to 24 cm?

Solution: (a) The work done is:

$$\begin{aligned}\int_1^2 (x^3 + 2x^2 - 1)dx &= \Big[\frac{1}{4}x^4 + \frac{2}{3}x^3 - x\Big]_1^2 \\ &= \Big(\frac{16}{4} + \frac{16}{3} - 2\Big) - \Big(\frac{1}{4} + \frac{2}{3} - 1\Big) \\ &= \frac{88}{12} - \Big(-\frac{1}{12}\Big) = \frac{89}{12}(\text{ft-lb}).\end{aligned}$$

(b) Let $f(x)$ be the force to stretch the spring x meters beyond its natural length. By Hooke's Law,

$$f(x) = kx.$$

As given, $f(0.05) = 30$ (since $20 - 15 = 5\text{cm} = 0.05\text{m}$), i.e. $0.05k = 30$. Thus, $k = 600$. Therefore, the work done in stretching the spring from 20 cm to 24 cm is:

$$\begin{aligned}\int_{0.05}^{0.09} 600x dx &= 300x^2\Big|_{0.05}^{0.09} \\ &= 2.43 - 0.75 = 1.68(\text{Joule}).\end{aligned}$$

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P3 (15 points). Evaluate the following integrals:

$$(a) \int x^2 e^x \; dx \qquad (b) \int \cos^2 x \sin^5 x \; dx \qquad (c) \int \tan^3 x \sec^5 x \; dx.$$

Solution: (a) Use integration by parts twice to get:

$$\begin{aligned}\int x^2 e^x dx &= x^2 e^x - \int e^x d(x^2) = x^2 e^x - \int 2x e^x dx \\ &= x^2 e^x - 2\big(x e^x - \int e^x dx\big) = x^2 e^x - 2(x e^x - e^x) + C \\ &= x^2 e^x - 2x e^x + 2e^x + C.\end{aligned}$$

(b) Use $d(-\cos x) = \sin x dx$ to write

$$\begin{aligned}\int \cos^2 x \sin^5 x dx &= \int \cos^2 x \sin^4 x d(-\cos x) = -\int \cos^2 x (1 - \cos^2 x)^2 d(\cos x) \\ &\overset{u=\cos x}{=} -\int u^2 (1 - u^2)^2 du = -\int u^2 (1 - 2u^2 + u^4) du \\ &= -\int (u^2 - 2u^4 + u^6) du = -\Big(\frac{1}{3}u^3 - \frac{2}{5}u^5 + \frac{1}{7}u^7\Big) + C \\ &= -\frac{1}{3}\cos^3 x + \frac{2}{5}\cos^5 x - \frac{1}{7}\cos^7 x + C.\end{aligned}$$

(c) Use $d(\sec x) = \sec x \tan x dx$ to write

$$\begin{aligned}\int \tan^3 x \sec^5 x dx &= \int \tan^2 x \sec^4 x d(\sec x) = \int (\sec^2 x - 1) \sec^4 x d(\sec x) \\ &\overset{u=\sec x}{=} \int (u^2 - 1) u^4 du = \int (u^6 - u^4) du \\ &= \frac{1}{7}u^7 - \frac{1}{5}u^5 + C = \frac{1}{7}\sec^7 x - \frac{1}{5}\sec^5 x + C.\end{aligned}$$

P4 (15 points).

(a) Use the Trapezoidal Rule with $n = 6$ to approximate the integral $\int_0^3 e^{x^2}\,dx$ (round off at 6 decimal places).

(b) Determine whether the following improper integrals are convergent/divergent:

$$\text{(a)} \int_{-\infty}^{0} \frac{1}{2x-5}dx \qquad \text{(b)} \int_0^{\infty} \frac{x}{x^3+1}dx.$$

Solution: (a) $a = 0, b = 3$ and $n = 6$, so $\Delta x = 0.5$. The Trapezoidal Rule gives

$$\begin{aligned}\int_0^3 e^{x^2}dx &\simeq \frac{0.5}{2}\left[e^{0^2} + 2e^{0.5^2} + 2e^{1^2} + 2e^{1.5^2} + 2e^{2^2} + 2e^{2.5^2} + e^{3^2}\right]\\ &= \frac{0.5}{2}\left[1 + 2e^{0.25} + 2e + 2e^{2.25} + 2e^4 + 2e^{6.25} + e^9\right]\\ &= \frac{0.5}{2}.(9276.285963) = 2319.071491.\end{aligned}$$

(b) (a) We have

$$\begin{aligned}\int_{-\infty}^{0} \frac{1}{2x-5}dx &= \lim_{t\to-\infty}\int_t^0 \frac{1}{2x-5}dx\\ &= \lim_{t\to-\infty}\left(\frac{1}{2}\ln(5-2x)\Big|_t^0\right)\\ &= \lim_{t\to-\infty}\left(\frac{1}{2}\ln 5 - \frac{1}{2}\ln(5-2t)\right) = -\infty.\end{aligned}$$

Thus, $\int_{-\infty}^{0} \frac{1}{2x-5}dx$ is divergent.

(b) We have $\frac{x}{x^3+1} \le \frac{x}{x^3} = \frac{1}{x^2}$ for $x \ge 1$. On the other hand,

$$\begin{aligned}\int_1^{\infty} \frac{1}{x^2}dx &= \lim_{t\to\infty}\int_1^t \frac{1}{x^2}dx\\ &= \lim_{t\to\infty}\left(-\frac{1}{x}\Big|_1^t\right)\\ &= \lim_{t\to\infty}\left(-\frac{1}{t}+1\right) = 1.\end{aligned}$$

Thus, by Comparison Test, $\int_0^{\infty} \frac{x}{x^3+1}dx = \int_0^1 \frac{x}{x^3+1}dx + \int_1^{\infty} \frac{x}{x^3+1}dx$ is convergent.

STOP. THIS IS THE LAST PAGE.