SCORE: ....OVER TOTAL: **50 points** ....

MATH 122 - CALCULUS II

SOLUTION TO MIDTERM EXAM II

Problem 1 (15 points). *Let C be the arc of the curve $y = \sqrt{4-x^2}$ from $(-1, \sqrt{3})$ to $(1, \sqrt{3})$.*

(1) *Find the arc length of C.*

(2) *Find the area of the surface obtained by rotating C about the x-axis.*

Solution 1. (1) We have $y' = \frac{1}{2}(4-x^2)^{-\frac{1}{2}}(-2x) = \frac{-x}{\sqrt{4-x^2}}$. Thus,

$$\text{Arc length of } C = \int_{-1}^{1} \sqrt{1+\left(\frac{dy}{dx}\right)^2}dx$$
$$= \int_{-1}^{1} \sqrt{1+\frac{x^2}{4-x^2}}dx = \int_{-1}^{1} \frac{2}{\sqrt{4-x^2}}dx.$$

Use the substitution $x = 2\cos\theta$ for $0 \leq \theta \leq \pi$, we get $dx = -2\sin\theta d\theta$, and $\sqrt{4-x^2} = \sqrt{4-4\cos^2\theta} = 2\sin\theta$. Therefore,

$$\text{Arc length of } C = \int_{\arccos(-0.5)}^{\arccos(0.5)} \frac{2}{2\sin\theta}(-2\sin\theta)d\theta$$
$$= \int_{\frac{2\pi}{3}}^{\frac{\pi}{3}} -2d\theta = -2\theta\Big|_{\frac{2\pi}{3}}^{\frac{\pi}{3}}$$
$$= 2\frac{2\pi}{3} - 2\frac{\pi}{3} = \frac{2\pi}{3}.$$

(2) We have

$$\text{Surface area } = \int_{-1}^{1} 2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}dx$$
$$= 2\pi \int_{-1}^{1} \sqrt{4-x^2}\sqrt{1+\frac{x^2}{4-x^2}}dx$$
$$= 2\pi \int_{-1}^{1} \sqrt{4-x^2}\frac{2}{\sqrt{4-x^2}}dx$$
$$= 2\pi \int_{-1}^{1} 2dx = 8\pi.$$

Problem 2 (10 points)**.** *Solve the differential equation* $xy' - 2y = x^2$, $x > 0$.

Solution 2. Divide both sides by x to get: $y' - \frac{2}{x}y = x$. This is a **linear** differential equation. The *integrating factor* is:

$$I(x) = e^{\int -\frac{2}{x}dx} = e^{-2\ln x} = \left(e^{\ln x}\right)^{-2} = x^{-2} = \frac{1}{x^2}.$$

Multiply both sides of our equation with $I(x)$:

$$\begin{aligned} \frac{1}{x^2}y' - \frac{2}{x^3}y &= \frac{1}{x} \\ \left(\frac{1}{x^2}y\right)' &= \frac{1}{x}. \end{aligned}$$

Integrate both sides:

$$\begin{aligned} \frac{1}{x^2}y &= \int \frac{1}{x}dx \\ &= \ln x + C. \end{aligned}$$

Thus, $y = x^2(\ln x + C)$.

Problem 3 (10 points). *Consider the initial-value problem* $\begin{cases} xy' = y^2 - y \\ y(1) = -1. \end{cases}$

(1) *In the direction field of the given differential equation, what is the equation of the short line segment drawn at* $(1, -1)$*?*

(2) *Use Euler's method with step size* $h = 0.5$ *to approximate* $y(3)$*.*

Solution 3. (1) We rewrite the differential equation as $y' = \dfrac{y^2 - y}{x} = F(x, y)$. Then the slope of the line segment at $(1, -1)$ is 2. Thus, the equations for this line segment is:

$$y + 1 = 2(x - 1).$$

(2) Using Euler's method we get $x_0 = 1, x_1 = 1.5, x_2 = 2, x_3 = 2.5, x_4 = 3$, and

$$\begin{aligned}
y_0 &= y(1) = -1 \\
y_1 &= y_0 + hF(x_0, y_0) = -1 + (0.5).2 = 0 \\
y_2 &= y_1 + hF(x_1, y_1) = 0 + (0.5).0 = 0 \\
y_3 &= y_2 + hF(x_2, y_2) = 0 + (0.5).0 = 0 \\
y_4 &= y_3 + hF(x_3, y_3) = 0 + (0.5).0 = 0
\end{aligned}$$

Thus, $y(3)$ is approximated by $y_4 = 0$.

Problem 4 (15 points). *Let C be the curve with parametric equations:*

$$\begin{cases} x = 3t - t^3 \\ y = 3t^2. \end{cases}$$

(1) *Find the Cartesean equation for the tangent line of C at the point $(\frac{11}{8}, \frac{3}{4})$.*

(2) *Find the points on C where the tangent line is horizontal or vertical.*

(3) *Find the arc length along C from $t = 0$ to $t = 1$.*

Solution 4. (1) We shall use the formula $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt}$, where

$$\frac{dy}{dt} = 6t$$
$$\frac{dx}{dt} = 3 - 3t^2.$$

At $(\frac{11}{8}, \frac{3}{4})$, the value of t is $\frac{1}{2}$. Thus,

$$\left.\frac{dy}{dx}\right|_{(\frac{11}{8}, \frac{3}{4})} = \frac{3}{9/4} = \frac{4}{3}.$$

This is the slope of the tangent line at $(\frac{11}{8}, \frac{3}{4})$, so the equation of the tangent line at this point is:

$$y - \frac{3}{4} = \frac{4}{3}\left(x - \frac{11}{8}\right).$$

(2) Solve $dy/dt = 0$ we get $t = 0$. At $t = 0$, $dx/dt = 3 \neq 0$. Thus, at $t = 0$, i.e. at the point $(0, 0)$ the tangent line is horizontal.

Solve $dx/dt = 0$ we get $t = \pm 1$. At $t = \pm 1$, $dy/dt = \pm 6 \neq 0$. Thus, at $t = \pm 1$, i.e. at the points $(2, 3)$ and $(-2, 3)$, the tangent lines are vertical.

(3) The arc length L along C from $t = 0$ to $t = 1$ is:

$$\begin{aligned} L &= \int_0^1 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt \\ &= \int_0^1 \sqrt{(3 - 3t^2)^2 + (6t)^2} dt \\ &= \int_0^1 \sqrt{9 - 18t^2 + 9t^4 + 36t^2} dt \\ &= \int_0^1 \sqrt{(3 + 3t^2)^2} dt = \int_0^1 (3 + 3t^2) dt \\ &= (3t + t^3)\Big|_0^1 = 4. \end{aligned}$$