ARITHMETIC MACAULAYFICATION OF PROJECTIVE SCHEMES

STEVEN DALE CUTKOSKY AND HUY TÀI HÀ

Abstract. In this paper, we study arithmetic Macaulayfication of projective schemes and Rees algebras of ideals. We discuss the existence of an arithmetic Macaulayfication for projective schemes. We give a simple neccesary and sufficient condition for nonsingular projective varieties to possess an arithmetic Macaulayfication (Theorem 1.5). We also show that this condition is sufficient in general, but give examples to show that it is not in general necessary. We further consider Rees algebras $\mathcal{R}_\lambda(I) = R[(I_\lambda)t]$ (truncated Rees algebras) associated to a homogeneous ideal I and show that they are Cohen-Macaulay for large λ in some important cases (Theorem 2.1 and Corollary 2.2.1).

Dedicated to Wolmer Vasconcelos on the occation of his sixty fifth birthday.

0. Introduction

In this paper, we study arithmetic Macaulayfication of projective schemes and Rees algebras of ideals.

In the first part of the paper, we discuss the problem of arithmetic Macaulayfication of projective schemes. This is a globalization of the problem of arithmetic Macaulayfication of local rings, which was first considered by Barshay in [3], and then studied extensively by many authors, such as Goto and Shimoda [12], Goto and Yamagishi [13], Brodmann [4], Schenzel [25], Lipman [23], Aberbach [1], Kurano [22], Aberbach, Huneke and Smith [2], and finally solved by Kawasaki [21]. We give a neccesary and sufficient condition for a nonsingular projective scheme over a field k of characteristic 0 to have an arithmetic Macaulayfication.

Theorem 0.1. *(Theorem 1.3) Suppose X is a nonsingular projective scheme over a field k of characteristic 0. Then, X has an arithmetic Macaulayfication if and only if $H^0(X, \mathcal{O}_X) = k$ and $H^i(X, \mathcal{O}_X) = 0$ for all $i = 1, \ldots, \dim X - 1$.*

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We show that the cohomological conditions of Theorem 1.3 are sufficient conditions for an unmixed projective scheme to have an arithmetic Macaulayfication (Theorem 1.5). This result follows from the work of Kawasaki ([20], [21]). However, we show that the cohomological conditions of Theorem 1.3 are not necessary in general (Example 1.6, Example 1.7).

In the second part of this paper, we consider a natural class of Rees algebras associated to a homogeneous ideal, the *truncated Rees algebras.* This class of Rees algebras was first considered by the second author in [14] and [15] for the defining ideal of a set of points in $\mathbb{P}^2$. It gave a new tool to completely answer the question on defining equations of projective embeddings of certain rational surfaces (see [14, Section 4.3]).

Our main result of this section is Theorem 2.1, from which we can conclude results such as the following.

Corollary 0.1.1. *(Corollary 2.2.1) Suppose that* X *is a projective Cohen-Macaulay scheme over a field* k *such that* $H^0(X,\mathcal{O}_X) = k$, $H^i(X,\mathcal{O}_X) = 0$ *for* $i > 0$ *and* $\mathcal{I} \subset \mathcal{O}_X$ *is an ideal sheaf which is locally a complete intersection. Then*

1. *There exists a Cohen-Macaulay standard graded* k*-algebra* R *with* $a(R) < 0$ *such that* $X \cong \mathit{Proj}\ R$.
2. *If* $I \subset R$ *is a homogeneous ideal such that* $\tilde{I} \cong \mathcal{I}$, *then there exists* $\lambda_0 \geq \delta(I) + 1$ *(where* $\delta(I)$ *is the maximum degree of a minimal set of homogeneous generators of* I*) such that the truncated Rees algebra*

$$\mathcal{R}_\lambda(I) = R[(I_\lambda)t]$$

is Cohen-Macaulay for all $\lambda \geq \lambda_0$.

To prove Theorem 2.1, we combine the method of [19] for studying the local cohomology of multigraded algebras with the results of [6].

Throughout this paper, let k be a field. We follow the notations of [8] and [16].

1. Arithmetic Macaulayfication

Suppose that X is a projective scheme over a field k. We will say that X is *arithmetically Cohen-Macaulay* if there exists a Cohen-Macaulay standard graded k algebra S such that $X \cong \text{Proj } S$.

Definition. Suppose that X is a projective scheme over a field k. An *arithmetic Macaulayfication* of X is a proper birational morphism $\pi : Y \to X$ such that Y is arithmetically Cohen-Macaulay.

We shall first prove a very basic result on arithmetically Cohen-Macaulay schemes.

Lemma 1.1. (1) *Suppose that* $Y = \text{Proj } S$ *is an arithmetically Cohen-Macaulay scheme. Then,* Y *is a Cohen-Macaulay scheme,* $H^i(Y, \mathcal{O}_Y) = 0$ *for* $i = 1, \dots, \dim Y - 1$, *and* $H^0(Y, \mathcal{O}_Y) = k$.

(2) *Suppose that* $Y = \text{Proj } S$ *is a Cohen-Macaulay scheme,* $H^i(Y, \mathcal{O}_Y) = 0$ *for* $i = 1, \dots, \dim Y - 1$ *and* $H^0(Y, \mathcal{O}_Y) = k$. *Then, there exists an integer* n_0 *such that for all* $n \geq n_0$, *the Veronese embedding of* Y *by* $H^0(Y, \mathcal{O}_Y(n))$ *is arithmetically Cohen-Macaulay.*

Proof. Let $\mathfrak{m}$ be the maximal homogeneous ideal of S. We have isomorphisms

$$\oplus_{n\in\mathbb{Z}} H^i(Y, \mathcal{O}_Y(n)) \cong H^{i+1}_{\mathfrak{m}}(S), \forall\ i \geq 1,$$

and an exact sequence

$$0 \to H^0_{\mathfrak{m}}(S) \to S \to \oplus_{n\in\mathbb{Z}} H^0(Y, \mathcal{O}_Y(n)) \to H^1_{\mathfrak{m}}(S) \to 0.$$

(1) is immediate since S is Cohen-Macaulay if and only if $H^i_{\mathfrak{m}}(S) = 0$ for $i \leq d = \dim \text{Proj } S$.

To prove (2) we first observe that $H^i(Y, \mathcal{O}_Y(n)) = 0$ for $i > 0$ and $n \gg 0$ by Serre vanishing. Since Y is Cohen-Macaulay, we also have $H^i(Y, \mathcal{O}_Y(n)) = 0$ for $i < d$ and $n \ll 0$. ☐

Theorem 1.2. *(Hironaka [18]) Suppose that* $\pi : Y \to X$ *is a birational morphism of projective nonsingular varieties over a field* k *of characteristic 0. Then*

$$H^i(Y, \mathcal{O}_Y) \simeq H^i(X, \mathcal{O}_X)\ \forall\ i.$$

Proof. By resolution of indeterminancy ([18]), there exists a commutative diagram of projective morphisms

$$\begin{tikzcd} Z \arrow[r, "f"] \arrow[dr, "g"'] & Y \arrow[d, "\pi"] \\ & X \end{tikzcd}$$

such that g is a product of blowups of nonsingular subvarieties,

$$g : Z = Z_n \xrightarrow{g_n} Z_{n-1} \xrightarrow{g_{n-1}} \dots \xrightarrow{g_2} Z_1 \xrightarrow{g_1} Z_0 = X.$$

We have ([24] or Lemma 2.1 [6])

$$R^i {g_j}_* \mathcal{O}_{Z_j} = \begin{cases} 0, & i > 0 \\ \mathcal{O}_{Z_{j-1}}, & i = 0 \end{cases}$$

Thus,

$$R^i g_* \mathcal{O}_Z \quad = \quad \begin{cases} 0, & \text{if } i > 0 \\ \mathcal{O}_X, & \text{if } i = 0 \end{cases} \tag{1.1}$$

and

$$g^* : H^i(X, \mathcal{O}_X) \cong H^i(Z, \mathcal{O}_Z)$$

for all i. Now, by considering the commutative diagram

$$\begin{tikzcd} H^i(Z, \mathcal{O}_Z) & H^i(Y, \mathcal{O}_Y) \arrow[l, "f^*"'] \\ & H^i(X, \mathcal{O}_X) \arrow[ul, "g^*"] \arrow[u, "\pi^*"'] \end{tikzcd}$$

we conclude that π^* is one-to-one. To show that π^* is an isomorphism we now only need to show that f^* is also one-to-one.

By resolution of indeterminancy it also gives a new diagram

$$\begin{tikzcd} W \arrow[r, "\gamma"] \arrow[dr, "\beta"'] & Z \arrow[d, "f"] \\ & Y \end{tikzcd}$$

where β is a product of blowups of nonsingular subvarieties, so similarly we have

$$\beta^* : H^i(Y, \mathcal{O}_Y) \cong H^i(W, \mathcal{O}_W)$$

for all i. This implies that f^* is one-to-one, and the theorem is proved. □

Suppose that $f : Y \to X$ is a morphism of schemes, and $\mathcal{F}$ is a sheaf of Abelian groups on Y. From the Leray spectral sequence $H^i(X, R^j f_* \mathcal{F}) \Rightarrow H^{i+j}(Y, \mathcal{F})$, we deduce the following exact sequence

$$0 \to H^1(X, f_*\mathcal{F}) \to H^1(Y, \mathcal{F}) \to H^0(X, R^1 f_* \mathcal{F}) \to H^2(X, f_*\mathcal{F}) \to H^2(Y, \mathcal{F}). \tag{1.2}$$

In the case of nonsingular varieties over a field of characteristic 0, we have a good necessary and sufficient condition for the existence of an arithmetic Macaulayfication.

Theorem 1.3. *Suppose X is a nonsingular projective scheme over a field k of characteristic 0. Then, X has an arithmetic Macaulayfication if and only if $H^0(X, \mathcal{O}_X) = k$ and $H^i(X, \mathcal{O}_X) = 0$ for all $i = 1, \ldots, \dim X - 1$.*

Proof. Suppose that X is nonsingular and there exists an arithmetic Macaulayfication $f : Y = \operatorname{Proj} S \to X$. By Lemma 1.1, we have $H^0(Y, \mathcal{O}_Y) = k$ and $H^i(Y, \mathcal{O}_Y) = 0$ for $0 < i < \dim Y$.

Let $g : Z \to Y$ be a resolution of singularites. Set $h = f \circ g$. Then, $H^i(Z, \mathcal{O}_Z) \cong H^i(X, \mathcal{O}_X)$ for all i by Theorem 1.2.

We have sequences

$$H^i(X, \mathcal{O}_X) \xrightarrow{f^*} H^i(Y, \mathcal{O}_Y) \xrightarrow{g^*} H^i(Z, \mathcal{O}_Z).$$

$h^* = g^* \circ f^*$ is an isomorphism, so we have $H^0(X, \mathcal{O}_X) = k$ and $H^i(X, \mathcal{O}_X) = 0$ for $0 < i < \dim X = \dim Y$. The necessary condition is proved.

Now suppose that $H^0(X, \mathcal{O}_X) = k$ and $H^i(X, \mathcal{O}_X) = 0$ for $i = 1, \ldots, \dim X - 1$. X is a Cohen-Macaulay scheme since X is nonsingular. Now using Lemma 1.1, we can embed X as an arithmetically Cohen-Macaulay scheme Y. The sufficient condition is proved. ☐

Remark 1.4. *The same proof shows that the conclusions of Theorem 1.3 hold if X has rational singularities, over a field of characteristic zero.*

From Kawasaki's work we easily deduce a very strong criterion for the existence of an arithmetic Macaulayfication over a field of arbitrary characteristic.

Theorem 1.5. *Suppose that X is an unmixed projective scheme of dimension ≥ 1 over a field k, $H^i(X, \mathcal{O}_X) = 0$ for $1 \leq i \leq$ dim $X - 1$ and $H^0(X, \mathcal{O}_X) = k$. Then there exists an arithmetic Macaulayfication of X.*

Proof. $X = \operatorname{Proj} R$ where $R = \oplus_{i \geq 0} R_i$ is an unmixed, standard graded k-algebra. Let V be the (reduced) closed subscheme of X of non Cohen-Macaulay points, $s =$ dim V, $d =$ dim X, $z_1, \ldots, z_d$ be homogeneous elements of R satisfying the conclusions of Lemma

5.3 [20]. Since R is unmixed $s < d-1$ (as follows from Corollary 2.4 [21]). Let $Q_i = (z_i, \ldots, z_d) \subset R$ for $1 \leq i \leq s+1$, and $I = Q_1 \cdots Q_s Q_{s+1}^{d-s-1} \subset R$.

Suppose that $\alpha \in X$ is a closed point, $y \in R_1 - \alpha$, $x_i = \frac{z_i}{y^{\deg z_i}}$ for $1 \leq i \leq d$. Let $q_i = (x_i, \ldots, x_d)$, $\beta = q_1 \cdots q_s q_{s+1}^{d-s-1} \subset R_{(\alpha)}$. We have $q_i = (Q_i)_{(\alpha)}$ and $\beta = I_{(\alpha)}$. If $\beta = R_{(\alpha)}$, then $R_{(\alpha)}$ is Cohen-Macaulay and $R_{(\alpha)}[I_{(\alpha)}t]$ is Cohen-Macaulay. If $\beta \neq R_{(\alpha)}$, then there exists l such that $x_l, \ldots, x_d \in \alpha_{(\alpha)}$ and $x_{l-1} \not\in \alpha_{(\alpha)}$. As in the proof of Theorem 5.1 [20], $x_l, \ldots, x_d$ is a subsystem of a p-standard system of parameters for $R_{(\alpha)}$ and $R_{(\alpha)}/(x_l, \ldots, x_d)R_{(\alpha)}$ is a Cohen-Macaulay ring if $l > 1$. $R_{(\alpha)}[\beta t] = R_{(\alpha)}[q_t \cdots q_s q_{s+1}^{d-s-1} t]$ is Cohen-Macaulay by Corollary 4.5 [21], since $s < d-1$ and $((0) : x_d) = (0)$ as $R_{(\alpha)}$ is unmixed.

Let $\mathcal{I}$ be the sheafication of I, $Y = \text{Proj } (\oplus_{n \geq 0} \mathcal{I}^n)$, with projection $\pi : Y \to \text{Proj } R = X$. For $\alpha \in X$ a closed point, $R^i \pi_* \mathcal{O}_{Y,\alpha} = H^i(Y_\alpha, \mathcal{O}_{Y_\alpha})$ where $Y_\alpha = \text{Proj } R_{(\alpha)}[I_{(\alpha)}t] = Y \times_X \text{Spec } \mathcal{O}_{X,\alpha}$. Since $R_{(\alpha)}[I_{(\alpha)}t]$ is Cohen-Macaulay, we have $H^i(Y_\alpha, \mathcal{O}_{Y_\alpha}) = 0$ for $i > 0$ and $(\pi_* \mathcal{O}_Y)_\alpha = R_{(\alpha)} = \mathcal{O}_{X,\alpha}$ by Theorem 4.1 [23]. Thus $R^i \pi_* \mathcal{O}_Y = 0$ for $i > 0$ and $\pi_* \mathcal{O}_Y = \mathcal{O}_X$. From the Leray spectral sequence we deduce that

$$H^i(Y, \mathcal{O}_Y) = H^i(X, \pi_* \mathcal{O}_Y) = H^i(X, \mathcal{O}_X) = 0$$

for $1 \leq i \leq d-1$ and

$$H^0(Y, \mathcal{O}_Y) = H^0(X, \pi_* \mathcal{O}_Y) = H^0(X, \mathcal{O}_X) = k.$$

It also follows from what was shown above that if $\gamma \in Y$ is a closed point, $\mathcal{O}_{Y,\gamma}$ is Cohen-Macaulay, so that Y is a Cohen-Macaulay scheme. Lemma 1.1 now implies that $Y = \text{Proj } S$ for some Cohen-Macaulay ring S. □

Example 1.6. *The converse of Theorem 1.5 is not true, as can be seen from the following simple example. Suppose that k is an algebraically closed field, $X = Proj\ S$ is the cuspidal plane curve with coordinate ring $S = k[y_0, y_1, y_2]/(y_0 y_2^2 - y_1^3)$. $H^1(X, \mathcal{O}_X) \cong k$. Let $Y = X \times \mathbb{P}^1_k$. Note that Y is a Cohen-Macaulay scheme. $H^1(Y, \mathcal{O}_Y) \cong k \neq 0$, by the Künneth formula. There is a natural resolution of singularites $\mathbb{P}^1_k \times \mathbb{P}^1_k \to Y$, which is an arithmetic Macaulayfication, as $\mathbb{P}^1_k \times \mathbb{P}^1_k \cong Proj\ R$, with $R = k[x_0, x_1, x_2, x_3]/(x_0 x_2 - x_1 x_3)$.*

We observe that the converse of Theorem 1.5 is true for normal projective surfaces. For if X is a projective normal surface and $f : Y \to X$ is an arithmetic Macaulayfication, then

$f_*\mathcal{O}_Y = \mathcal{O}_X$, so that $k = H^0(Y, \mathcal{O}_Y) = H^0(X, \mathcal{O}_X)$, and $H^1(X, \mathcal{O}_X) = H^1(Y, \mathcal{O}_Y) = 0$ by (1.2) and Lemma 1.1.

The following example is of a normal 3-fold X such that the converse of Theorem 1.5 is false.

Example 1.7. *There exists a normal projective 3-fold B such that $H^2(B, \mathcal{O}_B) \neq 0$ and B has an arithmetic Macaulayfication.*

Proof. In section III of [5] an example is given of an m-primary ideal I in the power series ring $\mathbb{C}[[x, y, z]]$ such that $\oplus_{n \geq 0} I^n$ is normal but not Cohen-Macaulay. The construction there yields an example of the desired type.

Let $\beta : A \to \mathbb{P}^3_{\mathbb{C}}$ be the morphism obtained by blowing up a point in $\mathbb{P}^3_{\mathbb{C}}$, and then blowing up the 12 points which are the intersection points of a general hypersurface on the exceptional $\mathbb{P}^2$ with a general cubic curve C'' on the exceptional $\mathbb{P}^2$. Let C' be the strict transform of C'' on A. In section III of [5], it is shown that there exists a projective morphism $\alpha : A \to B$ such that B is normal, $\alpha(C')$ is a point Q, $A - C' \to B - Q$ is an isomorphism, $\alpha_*\mathcal{O}_A \cong \mathcal{O}_B$ and $R^1\alpha_*\mathcal{O}_A \neq 0$. Since $R^1\alpha_*\mathcal{O}_A$ is supported at the single point Q, we have $H^0(B, R^1\alpha_*\mathcal{O}_A) \neq 0$. Since $H^i(A, \mathcal{O}_A) \cong H^i(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3})$ for all i (by Theorem 1.2), we have $H^i(A, \mathcal{O}_A) = 0$ for $i = 1, 2$ and $H^0(A, \mathcal{O}_A) = \mathbb{C}$. Hence (by Lemma 1.1) $A \cong \operatorname{Proj} S$ where S is a Cohen-Macaulay standard graded $\mathbb{C}$-algebra. By (1.2), we have an isomorphism $H^2(B, \mathcal{O}_B) \cong H^0(B, R^1\alpha_*\mathcal{O}_A) \neq 0$. □

2. Truncated Rees algebras

Let R be a standard graded k-algebra, $I \subseteq R$ a homogeneous ideal. The truncated Rees algebras associated to I are defined as follows.

Definition. Suppose that $I = \oplus_{t \geq \alpha} I_t$ is the homogeneous decomposition of I, where $\alpha = \alpha(I)$ is the minimum degree in I. For each $\lambda \geq \alpha$, we define the *truncated Rees algebra of I at degree λ* to be the Rees algebra

$$\mathcal{R}_\lambda(I) = R[(I_\lambda)t] \subseteq R[t]$$

of the ideal generated by I_λ.

Define $\delta = \delta(I)$, the maximum degree of a minimal system of homogeneous generators of I.

We will assume that $\lambda \geq \delta$. The truncated Rees algebra $\mathcal{R}_\lambda(I)$ has a bi-gradation determined by $\deg F = (d,0)$ if $F \in R$ is homogeneous of degree d, and $\deg t = (-\lambda, 1)$, i.e.

$$\mathcal{R}_\lambda(I)_{(p,q)} = I^q_{p+q\lambda}t^q.$$

It can be seen that

$$R = \oplus_{n\geq 0} R_\lambda(I)_{(n,0)}$$

as a graded subring of $\mathcal{R}_\lambda(I)$, and

$$S_\lambda = \oplus_{n\geq 0} R_\lambda(I)_{(0,n)}$$

is another subring of $R_\lambda(I)$ which we will consider. There is a natural isomorphism $S_\lambda \cong k[I_\lambda]$.

Set $X = \text{Proj } R$, $V_\lambda = \text{Proj } R_\lambda(I)$ (with respect to the above bi-grading), $\overline{V}_\lambda = \text{Proj } S_\lambda$. We have canonical projections $\pi_1 : V_\lambda \to X$ and $\pi_2 : V_\lambda \to \overline{V}_\lambda$.

V_λ can be identified with the graph of the rational map $X \dashrightarrow \overline{V}_\lambda$ induced by the natural inclusion $k[I_\lambda] \to R$, and we have an isomorphism $V_\lambda \cong \text{Proj } (\oplus_{n\geq 0}\mathcal{I}^n)$, the blowup of the sheafification $\mathcal{I}$ of I (c.f. [10]). From now on we will assume that $\lambda \geq \delta + 1$. We then also have that $\overline{V}_\lambda \cong \text{Proj } (\oplus_{n\geq 0}\mathcal{I}^n)$ ([6, Lemma 1.1]) so that $V_\lambda \to \overline{V}_\lambda$ is an isomorphism, and we have a natural diagram of morphisms (where π_2 is an isomorphism):

$$\begin{tikzcd} & V_\lambda \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ X & & \overline{V}_\lambda \arrow[ll, "\pi"'] \end{tikzcd}$$

Let $\mathcal{L} = \mathcal{I}\mathcal{O}_{\overline{V}_\lambda}$. The respective gradings on R, S_λ and $R_\lambda(I)$ are related by isomorphisms

$$\mathcal{L}^q \otimes \pi^*\mathcal{O}_X(q\lambda) \cong \mathcal{O}_{\overline{V}_\lambda}(q),$$

$$\mathcal{O}_{V_\lambda}(p,q) \cong \pi_1^*\mathcal{O}_X(p) \otimes \pi_2^*\mathcal{O}_{\overline{V}_\lambda}(q). \tag{2.1}$$

For $q \in \mathbb{Z}$, let $\mathcal{M}_q$ be the sheafification on X of the graded R-module

$$M_q = \oplus_{i\geq 0} R_\lambda(I)_{(i,q)}$$

so that (since $\lambda \geq \delta$)

$$\mathcal{M}_q = \begin{cases} \mathcal{O}_X & \text{if } q = 0 \\ \mathcal{I}^q(\lambda q) & q > 0 \\ 0 & q < 0 \end{cases}$$

Thus for $p \in \mathbb{Z}$,

$$\mathcal{M}_q(p) = \begin{cases} \mathcal{O}_X(p) & \text{if } q = 0 \\ \mathcal{I}^q(\lambda q + p) & q > 0 \\ 0 & q < 0 \end{cases}$$

For $p \in \mathbb{Z}$, let $\mathcal{N}_p$ be the sheafification on $\overline{V}_\lambda$ of the graded S_λ module

$$N_p = \oplus_{i \geq 0} R_\lambda(I)_{(p,i)}.$$

Observe that $N_0 = k[I_\lambda] = S_\lambda$,

$$\mathcal{N}_p = \begin{cases} \mathcal{O}_{\overline{V}_\lambda} & \text{if } p = 0 \\ \pi^*\mathcal{O}_X(p) & p > 0 \\ 0 & p < 0 \end{cases}$$

Thus for $q \in \mathbb{Z}$,

$$\mathcal{N}_p(q) = \begin{cases} \pi^*\mathcal{O}_X(p) \otimes \mathcal{O}_{\overline{V}_\lambda}(q) & \text{if } p \geq 0 \\ 0 & p < 0 \end{cases}$$

Our main result in this section is to show that for a certain class of standard graded k-algebras R and homogeneous ideals I, the truncated Rees algebras $\mathcal{R}_\lambda(I)$ of I are Cohen-Macaulay for large λ.

Theorem 2.1. *Suppose that R is a Cohen-Macaulay standard graded k-algebra of positive dimension $d+1$ that has negative a-invariance, $a(R) < 0$ (Since R is Cohen-Macaulay this is equivalent to $H^{d+1}_{\mathfrak{m}_1}(R)_p = 0\ \forall p \geq 0$, where $\mathfrak{m}_1$ is the maximal ideal of R). Let $I \subseteq R$ be a homogeneous ideal, and suppose that $\lambda \geq \delta(I) + 1$.*

Let $\mathcal{I}$ be the ideal sheaf associated to I on $X = Proj\ R$,

$$E \cong Proj\ \oplus_{n \geq 0} \mathcal{I}^n/\mathcal{I}^{n+1}$$

be the exceptional divisor of $\pi_1 : V_\lambda \to X$, with dualizing sheaf ω_E on E. Suppose that

$$\begin{cases} \pi_{1*}\mathcal{O}_E(-\lambda m, m) &=& \mathcal{I}^m/\mathcal{I}^{m+1}, \forall\ m \geq 0, \\ R^i\pi_{1*}\mathcal{O}_E(-\lambda m, m) &=& 0, \forall\ i > 0, m \geq 0, \\ R^i\pi_{1*}\omega_E(-\lambda m, m) &=& 0, \forall\ i > 0, m \geq 2 \end{cases} \tag{2.2}$$

Then, there exists an integer λ_0 such that for all

$$\lambda \geq \lambda_0 \geq \delta + 1$$

the truncated Rees algebra $\mathcal{R}_\lambda(I)$ is Cohen-Macaulay.

To prove Theorem 2.1, we shall combine the method of [19] for studying the local cohomology of multi-graded algebras with the results of [6]. Suppose that $\lambda \geq \delta + 1$. For convenience, denote $S_{V_\lambda} = \mathcal{R}_\lambda(I)$. We need to show that S_{V_λ} is a Cohen-Macaulay ring for $\lambda \gg 0$.

Let

$$\mathfrak{m}_1 = \oplus_{i>0} R_\lambda(I)_{(i,0)}$$

be the irrelevant ideal of R,

$$\mathfrak{n}_1 = \mathfrak{m}_1 R_\lambda(I) = \oplus_{i>0, j\geq 0} R_\lambda(I)_{(i,j)}.$$

Let

$$\mathfrak{m}_2 = \oplus_{j>0} R_\lambda(I)_{(0,j)}$$

be the irrelevant ideal of S_λ,

$$\mathfrak{n}_2 = \mathfrak{m}_2 R_\lambda(I) = \oplus_{i\geq 0, j>0} R_\lambda(I)_{(i,j)}.$$

Let

$$\mathfrak{m} = \oplus_{i+j>0} R_\lambda(I)_{(i,j)}$$

and

$$\mathfrak{n} = \oplus_{i,j>0} R_\lambda(I)_{(i,j)}.$$

Then $\mathfrak{n}_1 + \mathfrak{n}_2 = \mathfrak{m}$ and $\mathfrak{n}_1 \cap \mathfrak{n}_2 = \mathfrak{n}$.

S_{V_λ} is Cohen-Macaulay if and only if

$$H^i_{\mathfrak{m}}(S_{V_\lambda}) = 0, \ \forall\, i = 0, \ldots, d+1,$$

where $d = \dim X = \dim S_{V_\lambda} - 2$, so that from the Mayer-Vietoris sequence of cohomologies,

$$\ldots \to H^i_{\mathfrak{m}}(S_{V_\lambda}) \to H^i_{\mathfrak{n}_1}(S_{V_\lambda}) \oplus H^i_{\mathfrak{n}_2}(S_{V_\lambda}) \to H^i_{\mathfrak{n}}(S_{V_\lambda}) \to H^{i+1}_{\mathfrak{m}}(S_{V_\lambda}) \to \ldots.$$

we see that S_{V_λ} is Cohen-Macaulay if and only if

$$\left\{ \begin{array}{lcccl} H^i_{\mathfrak{n}_1}(S_{V_\lambda}) & \oplus & H^i_{\mathfrak{n}_2}(S_{V_\lambda}) & \overset{\sim}{\to} & H^i_{\mathfrak{n}}(S_{V_\lambda}), \ \forall\, i = 0, \ldots, d \\ H^{d+1}_{\mathfrak{n}_1}(S_{V_\lambda}) & \oplus & H^{d+1}_{\mathfrak{n}_2}(S_{V_\lambda}) & \hookrightarrow & H^{d+1}_{\mathfrak{n}}(S_{V_\lambda}) \end{array} \right. \tag{2.3}$$

We have isomorphisms

$$H^i_{\mathfrak{m}_1}(M_q)_p \cong H^i_{\mathfrak{n}_1}(S_{V_\lambda})_{(p,q)}$$

and

$$H^i_{\mathfrak{m}_2}(N_p)_q \cong H^i_{\mathfrak{n}_2}(S_{V_\lambda})_{(p,q)}$$

for all $p, q \in \mathbb{Z}$ (c.f. Lemma 2.1 [7]).

For $p, q \in \mathbb{Z}$, we have commutative diagrams with exact rows [19, Theorem 1.4].

$$\begin{tikzcd}
0 \arrow[r] & H^0_{\mathfrak{n}}(S_{V_\lambda})_{(p,q)} \arrow[r] & (S_{V_\lambda})_{(p,q)} \arrow[r] & H^0(V_\lambda, \mathcal{O}_{V_\lambda}(p,q)) \arrow[r] & H^1_{\mathfrak{n}}(S_{V_\lambda})_{(p,q)} \arrow[r] & 0 \\
0 \arrow[r] & H^0_{\mathfrak{n}_1}(S_{V_\lambda})_{(p,q)} \arrow[r] \arrow[u] & (S_{V_\lambda})_{(p,q)} \arrow[r] \arrow[u, "\wr"'] & H^0(X, \mathcal{M}_q(p)) \arrow[r] \arrow[u] & H^1_{\mathfrak{n}_1}(S_{V_\lambda})_{(p,q)} \arrow[r] \arrow[u] & 0
\end{tikzcd} \tag{2.4}$$

and isomorphisms

$$H^i_{\mathfrak{n}}(S_{V_\lambda})_{(p,q)} \cong H^{i-1}(V_\lambda, \mathcal{O}_{V_\lambda}(p,q)) \tag{2.5}$$

and

$$H^i_{\mathfrak{n}_1}(S_{V_\lambda})_{(p,q)} \cong H^{i-1}(X, \mathcal{M}_q(p)) \tag{2.6}$$

for all $i \geq 2$.

For $p, q \in \mathbb{Z}$, we have commutative diagrams with exact rows ([19, Theorem 1.4]).

$$\begin{tikzcd}
0 \arrow[r] & H^0_{\mathfrak{n}}(S_{V_\lambda})_{(p,q)} \arrow[r] & (S_{V_\lambda})_{(p,q)} \arrow[r] & H^0(V_\lambda, \mathcal{O}_{V_\lambda}(p,q)) \arrow[r] & H^1_{\mathfrak{n}}(S_{V_\lambda})_{(p,q)} \arrow[r] & 0 \\
0 \arrow[r] & H^0_{\mathfrak{n}_2}(S_{V_\lambda})_{(p,q)} \arrow[r] \arrow[u] & (S_{V_\lambda})_{(p,q)} \arrow[r] \arrow[u, "\wr"'] & H^0(\overline{V}_\lambda, \mathcal{N}_p(q)) \arrow[r] \arrow[u] & H^1_{\mathfrak{n}_1}(S_{V_\lambda})_{(p,q)} \arrow[r] \arrow[u] & 0
\end{tikzcd} \tag{2.7}$$

and isomorphisms

$$H^i_{\mathfrak{n}_2}(S_{V_\lambda})_{(p,q)} \cong H^{i-1}(\overline{V}_\lambda, \mathcal{N}_p(q)) \tag{2.8}$$

for all $i \geq 2$.

By Lemma 2.1 [6]

$$R^i \pi_{1*} \mathcal{O}_{V_\lambda}(p,q) = 0 \text{ for } i > 0, q \geq 0, p \in \mathbb{Z}$$

and

$$\pi_{1*} \mathcal{O}_{V_\lambda}(p,q) \cong \mathcal{I}^q(p + q\lambda) \text{ for } q \geq 0, p \in \mathbb{Z}.$$

Thus by the Leray spectral sequence,

$$H^i(\overline{V}_\lambda, \mathcal{O}_{\overline{V}_\lambda}(q) \otimes \pi^* \mathcal{O}_X(p)) \cong H^i(V_\lambda, \mathcal{O}_{V_\lambda}(p,q)) \cong H^i(X, \mathcal{I}^q(p + q\lambda)) \tag{2.9}$$

for $i \geq 0, q \geq 0, p \in \mathbb{Z}$.

Proposition 2.2. *There exists $\lambda_0 \geq \delta + 1$ such that for $\lambda \geq \lambda_0$,*

(1) $H^0(V_\lambda, \mathcal{O}_{V_\lambda}(p,q)) = (S_{V_\lambda})_{(p,q)}$ *for* $p, q \geq 0$.

(2) $H^0(V_\lambda, \mathcal{O}_{V_\lambda}(p,q)) = 0$ *for* $p, q < 0$

(3) $H^i(V_\lambda, \mathcal{O}_{V_\lambda}(p,q)) = 0$ *for* $i > 0, p \geq 0, q > 0$.

(4) $H^i(V_\lambda, \mathcal{O}_{V_\lambda}(p,q)) = 0$ *for* $i < d, p, q < 0$

Proof. (1) follows from (2.1) and Lemma 1.3 [6].

We now prove (2). After possibly tensoring with an extension field of k, we may suppose that k is an infinite field. Suppose that $\lambda > \delta+1$. Then $\mathcal{O}_{V_\lambda}(-r(\lambda-(\delta+1)), r)$ is very ample for $r \geq 1$, by (2.1) and Lemma 1.1 [6]. For $r \geq 1$, there exists $\sigma_r \in H^0(V_\lambda, \mathcal{O}_{V_\lambda}(-r(\lambda-(\delta+1)), r))$ such that (σ_r) contains no associated primes of V_λ. Thus

$$\sigma_r : \mathcal{O}_{V_\lambda}(r(\lambda-(\delta+1)), -r) \rightarrow \mathcal{O}_{V_\lambda}$$

are inclusions. For $s > 0$, we have inclusions

$$\mathcal{O}_{V_\lambda}(r(\lambda-(\delta+1)) - s, -r) \rightarrow \mathcal{O}_{V_\lambda}(-s, 0)$$

which induce inclusions

$$\pi_{1*}\mathcal{O}_{V_\lambda}(r(\lambda-(d+1)) - s, -r) \rightarrow \mathcal{O}_X(-s)$$

and thus inclusions

$$H^0(V_\lambda, \mathcal{O}_{V_\lambda}(r(\lambda-(\delta+1)) - s, -r)) \rightarrow H^0(X, \mathcal{O}_X(-s))$$

$H^0(X, \mathcal{O}_X(-s)) = 0$ for $s < 0$ since R is Cohen-Macaulay. Since $\lambda > \delta+1$, we have

$$H^0(V_\lambda, \mathcal{O}_{V_\lambda}(p, q)) = 0$$

if $p, q < 0$.

(3) follows from (2.1) and Proposition 3.1 [6].

(4) follows from (2.1), Lemma 2.2 [6] and Proposition 3.2 [6]. □

We also have

$$H^i(X, \mathcal{O}_X(p)) = 0 \text{ for } i > 0, p \geq 0 \tag{2.10}$$

since $a(R) < 0$, and

$$H^i_{\mathfrak{m}_1}(R) = 0 \text{ for } i < d+1, \tag{2.11}$$

since R is Cohen-Macaulay, so that

$$H^0(X, \mathcal{O}_X(p)) = \begin{cases} 0 & p < 0 \\ (S_{V_\lambda})_{(p,0)} & p \geq 0. \end{cases}$$

The conditions of (2.3) now follow for $\lambda \geq \lambda_0$ by (2.9), (2.1), Proposition 2.2, (2.10), (2.11) and (2.4) - (2.8). We have thus finished the proof Theorem 2.1.

Corollary 2.2.1. *Suppose that X is a projective Cohen-Macaulay scheme over a field k such that $H^0(X, \mathcal{O}_X) = k$, $H^i(X, \mathcal{O}_X) = 0$ for $i > 0$ and $\mathcal{I} \subset \mathcal{O}_X$ is an ideal sheaf which is locally a complete intersection. Then*

(1) *There exists a Cohen-Macaulay standard graded k-algebra R xith $a(\alpha) < 0$ such that $X \cong$ Proj R.*

(2) *If $I \subset R$ is a homogeneous ideal such that $\tilde{I} = \mathcal{I}$, then there exists λ*

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Department of Mathematics, University of Missouri, Columbia MO 65201, USA

E-mail address: `dale@math.missouri.edu`

E-mail address: `tai@math.missouri.edu`