ALGEBRAIC METHODS FOR DETECTING ODD HOLES IN A GRAPH

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ABSTRACT. Let G denote a finite simple graph with edge ideal $I(G)$. Letting $I(G)^\vee$ denote the Alexander dual of $I(G)$, we show that a description of the induced cycles of G of odd length is encoded in the associated primes of $(I(G)^\vee)^2$. This result forms the basis for an algorithm to detect all the odd induced cycles of a graph via ideal operations, e.g., intersections, products, and colon operations. Moreover, we get simple algebraic algorithms for determining whether a graph is perfect. We also show how to determine the existence of odd induced cycles in a graph from the value of the arithmetic degree or the regularity of $(I(G)^\vee)^2$.

1. INTRODUCTION

One of the most important properties of a graph is the type of odd cycles that appear as induced subgraphs. The notion of a Berge graph is fundamental; a graph is Berge if neither it nor its complementary graph has an odd induced cycle of length at least five; these cycles are often called odd holes. Understanding this cycle structure is vital for results on colorings, cliques, and in particular, determining whether a graph is perfect. A simple graph G (a graph with no loops or multiple edges) is said to be perfect if for all induced subgraphs H of G, the chromatic number of H equals the clique number of H. (We define all terms formally in the next section.) The Strong Perfect Graph Theorem, one of the major breakthroughs in graph theory in recent years, was proven by Chudnovsky, Robertson, Seymour and Thomas [4]; it characterizes perfect graphs as exactly the Berge graphs. The study of perfect graphs is especially important since not only are both the Berge and coloring/clique definitions of great interest theoretically, but the notion of perfection is vital for efficiency reasons in applied problems. An example from a *Science* article [27] on the proof of the Strong Perfect Graph Theorem illustrates this well; think of cell-phone transmitters as the vertices of a graph with two transmitters connected by an edge if and only if their ranges overlap. One colors the graph so that no two adjacent vertices have the same color, and this corresponds to using different channels when ranges overlap. Perfect graphs minimize the number of colors used and thus maximize efficiency.

Consequently, the ability to detect odd induced cycles in a graph in systematic ways is significant. Chudnovsky, Cornuéjols, Liu, Seymour, and Vušković [3] proved the existence of a polynomial time algorithm to determine if a graph is perfect; in particular, their algorithm determines whether G is a Berge graph. If G is not perfect, however, this

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algorithm does not tell whether it is G or G^c that contains an odd hole. More recently, Conforti, Cornuéjols, Liu, Vušković, and Zambelli [6] showed that one can determine if a graph has an odd hole in polynomial time provided the graph has a bounded clique number. In general, there is no known effective algorithm for detecting the existence of odd holes.

Our goal in this paper is to expand the dictionary between graph theory and commutative algebra by providing simple, explicit ways to detect all odd induced cycles in graphs, allowing us to determine whether a graph is Berge (and thus perfect) and, if not, where the offending odd hole lies. The novelty of our work is the surprising connection between odd holes on the graph-theoretic side and associated primes on the commutative algebra side.

More precisely, suppose that $G = (V_G, E_G)$ is a finite simple graph on the vertex set $V_G = \{x_1, \ldots, x_n\}$ with edge set E_G. By identifying the vertices with the variables in the polynomial ring $R = k[x_1, \ldots, x_n]$ over the field k, one can associate to G a square-free quadratic monomial ideal

$$I(G) = (\{x_ix_j \mid \{x_i, x_j\} \in E_G\}).$$

The ideal $I(G)$ is called the **edge ideal** of G. The edge ideal $I(G)$, which was first introduced by Villarreal [35], is an algebraic object whose invariants can be related to the properties of G, and vice-versa. Simple graphs and hypergraphs can also be viewed as clutters, and so, edge ideals of clutters can be defined in the same way (cf. [8, 15, 16]). Many researchers have been interested in using the edge ideal construction to build a dictionary between the two fields, graph theory and commutative algebra. For general references, see [30, 34, 35]; for invariants encoded in the resolution, see [7, 10, 17, 18, 19, 25, 28]; for classes of (sequentially) Cohen-Macaulay graphs, see [12, 13, 21].

The first main result of this paper is to show that every odd induced cycle in a graph can be detected from the associated primes of $R/(I(G)^\vee)^2$, where $I(G)^\vee$ is the Alexander dual of $I(G)$, thus giving us an algorithm for determining perfection. In fact, we can show a much stronger result about the associated primes. Not only do they tell us if an odd hole exists, the associated primes actually indicate which vertices make up the odd hole. In particular, we show:

Theorem 1.1 (Corollary 3.3). *Let $J = I(G)^\vee$. A prime ideal $P = (x_{i_1}, \ldots, x_{i_r})$ is in $\mathrm{Ass}(R/J^2)$, the set of associated prime ideals of R/J^2, if and only if:*

1. *$P = (x_{i_1}, x_{i_2})$, and $\{x_{i_1}, x_{i_2}\}$ is an edge of G, or*
2. *r is odd, and the induced graph on $\{x_{i_1}, x_{i_2}, \ldots, x_{i_r}\}$ is an induced cycle of G.*

Theorem 1.1 is not the first time the induced odd cycles of a graph have been found using commutative algebra. Simis and Ulrich [29] showed that $I(G)^{\{2\}}$, the join of $I(G)$ with itself, is generated by the square-free monomials $x_{i_1}x_{i_2}\cdots x_{i_r}$, where r is odd, and the induced graph on $\{x_{i_1}, x_{i_2}, \ldots, x_{i_r}\}$ is an induced cycle of G. We find (see Theorem 3.2) an irreducible decomposition for the ideal $(I(G)^\vee)^2$, and then pair this decomposition with a result of Sturmfels and Sullivant [31] to recover the result of Simis and Ulrich (see Corollary 3.7).

Our proof of Theorem 3.2, and subsequently, Theorem 1.1, is based upon the notion of a 2-cover of a graph. If G is simple graph on n vertices, then $a = (a_1, \ldots, a_n) \in \mathbb{N}^n$ is a **2-cover** if $a_i + a_j \geq 2$ for all edges $\{x_i, x_j\} \in E_G$. A 2-cover a is **reducible** if there exist $b, c \in \mathbb{N}^n$ such that $a = b + c$, where b and c are both 1-covers, or one is a 2-cover, and the other is a 0-cover. (A 0-cover is simply any nonzero vector $a \in \mathbb{N}^n$.) Otherwise, we say a is **irreducible**. If we let $J = I(G)^\vee$, then each generator of $J^{(2)}$, the second symbolic power of J, corresponds to some 2-cover of G, while the generators of J^2 correspond only to the reducible 2-covers that are the sum of two 1-covers. The key ingredient in our proof is Dupont and Villarreal's classification of irreducible 2-covers [8]. This classification allows us to describe the irreducible decomposition of J^2.

The algebra of vertex covers of a graph, or more generally, a hypergraph, was first studied by Herzog, Hibi and Trung [20]. In [20], the authors use the terminology of *decomposable* and *indecomposable* covers in place of our reducible and irreducible ones. We choose to use reducible and irreducible covers to be consistent with the result of [8] that we use.

Theorem 1.1 forms the basis for an algorithm to detect the existence of odd holes in a graph using only the operations of commutative algebra. In particular, the existence of odd holes can be characterized algebraically as follows:

Theorem 1.2 (Theorems 4.5 and 4.12). *Let G be a simple graph with edge ideal $I(G)$. Set $J = I(G)^\vee$, and let*

$$L = \prod_{1 \leq i_1 < i_2 < i_3 < i_4 \leq n} (x_{i_1} + x_{i_2} + x_{i_3} + x_{i_4}).$$

Then the following are equivalent:

(a) *G has no odd hole.*
(b) *$J^2 : (L) = J^2$.*
(c) $\operatorname{adeg}(J^2) = 3|E_G| + t(G)$*, where* $\operatorname{adeg}(J^2)$ *denotes the arithmetic degree of J^2, and $t(G)$ is the number of triangles of G.*

The proof of the equivalence of (a) and (b) follows from a well-known lemma (see Lemma 4.1) that the saturation of an ideal J by an ideal K results in an ideal whose associated primes do not contain K. For the equivalence of (a) and (c), we use Sturmfels, Trung, and Vogel's notion of standard pairs to compute the arithmetic degree [32].

The main bottleneck in using Theorem 1.2.(b) to determine the existence of odd holes occurs in computing J. The generators of J are the vertex covers of G, and determining the vertex covers of a graph is an NP-complete problem [24]. However, the algorithm is simple to code and generally runs reasonably quickly in Macaulay 2.

The structure of our paper is as follows. In Section 2 we collect together the needed graph theoretic and algebraic results. In particular, we introduce Dupont and Villarreal's classification of irreducible 2-covers. Section 3 is devoted to the proof of our main theorem (Theorem 1.1). Section 4 contains two algebraic characterizations of graphs with odd holes. Finally, in Section 5, we make some remarks about associated primes of higher powers of the Alexander dual of the edge ideal.

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2. Graph Theory and Irreducible Covers

In this section we recall the needed terms and results from graph theory, and furthermore, we introduce our key ingredient for the proof of Theorem 3.2 and Corollary 3.3, Dupont and Villarreal's classification of irreducible of 2-covers [8]. We continue to use the definitions and terms from the introduction.

Let $G = (V_G, E_G)$ denote a finite simple graph (no loops or multiple edges) on the vertex set $V_G = \{x_1, \ldots, x_n\}$ and edge set E_G. We shall abuse notation and write x_ix_j for an edge $\{x_i, x_j\} \in E_G$. If $S \subseteq V_G$, the **induced subgraph** of G on S, denoted by G_S, is the graph with vertex set S and edge set $E_{G_S} = \{x_ix_j \in E_G \mid \{x_i, x_j\} \subseteq S\}$.

Definition 2.1. A **cycle** in a simple graph G is an alternating sequence of distinct vertices and edges $C = x_{i_1}e_1x_{i_2}e_2 \cdots x_{i_{n-1}}e_{n-1}x_{i_n}e_nx_{i_1}$ in which the edge e_j connects the vertices x_{i_j} and $x_{i_{j+1}}$ ($x_{i_{n+1}} = x_{i_1}$) for all j. In this case, we say C has **length** n and call C an n-cycle. We shall often write a cycle simply as $x_{i_1}x_{i_2} \cdots x_{i_n}x_{i_1}$ or $x_{i_1} \cdots x_{i_n}$, omitting the edges. A **chord** is an edge that joins two nonadjacent vertices in the cycle. We shall use C_n to denote an n-cycle without any chords. We usually refer to C_n as an **induced cycle** since the induced graph on $\{x_{i_1}, x_{i_2}, \ldots, x_{i_n}\}$ contains only the edges and vertices in the cycle. If an induced cycle has an odd (resp. even) number of vertices, we shall call it an **odd** (resp., **even**) **cycle**. An odd induced cycle of length at least five is called an **odd hole**.

A subset W of V_G is a **vertex cover** of G if every edge is incident to at least one vertex of W. A vertex cover W is a **minimal vertex cover** if no proper subset of W is a vertex cover. More generally, we can define vertex covers of any order.

Definition 2.2. Let $\mathbb{N}$ denotes the set of nonnegative integers. If G is simple graph on n vertices, then a nonzero vector $a = (a_1, \ldots, a_n) \in \mathbb{N}^n$ is a **vertex cover of order** k (or a k-**cover**) if $a_i + a_j \geq k$ for all edges $x_ix_j \in E_G$. A k-cover a is **reducible** if there exists an i-cover $b \in \mathbb{N}^n$ and a j-cover $c \in \mathbb{N}^n$ such that $a = b + c$ and $k = i + j$. Otherwise, we say a is **irreducible**.

Remark 2.3. When $(a_1, \ldots, a_n)$ is a $(0, 1)$-tuple, then a vertex cover of order 1 corresponds to the standard notion of a vertex cover. At times, we shall write vertex covers of order k as monomials, using the usual correspondence between monomials and vectors of nonnegative integers. Thus the monomial $x_1^{a_1} \cdots x_n^{a_n}$ corresponds to the cover $(a_1, \ldots, a_n)$.

We shall use a result of Dupont and Villarreal [8] that classifies irreducible 2-covers. In fact, we shall state a slightly more general version than their result. The proof of the theorem was already embedded in the proof of [20, Theorem 5.1]. In the statement below, the set $N(A)$ denotes the **neighbors** of the set $A \subseteq V_G$, that is,

$$N(A) = \{y \in V_G \setminus A \mid \text{there exists } x \in A \text{ such that } xy \in E_G\}.$$

A set of vertices $A \subseteq V_G$ is **independent** if the induced graph G_A contains no edges, that is, there are no edges among the vertices of A. As well, we say that G is **bipartite** if we can partition $V_G = V_1 \cup V_2$ so that every $xy \in E_G$ has the property that $x \in V_1$ and $y \in V_2$.

Theorem 2.4 (see [8, Theorem 2.6]). *Let G be a simple graph.*

(i) *If G is bipartite, then G has no irreducible 2-covers.*
(ii) *If G is not bipartite and a is a 2-cover that cannot be written as the sum of two 1-covers, then (up to some permutation of the vertices)*

$$a = (\underbrace{0, \ldots, 0}_{|A|}, \underbrace{b_1, \ldots, b_{|B|}}_{|B|}, 1, \ldots, 1)$$

for some (possibly empty) independent set A and a set $B \supseteq N(A)$ such that
(1) *$b_j \geq 2$ for all $j = 1, \ldots, |B|$,*
(2) *B is not a vertex cover of G and $V \neq A \cup B$, and*
(3) *the induced subgraph on $C = V \setminus (A \cup B)$ is not bipartite.*
Moreover, if a is irreducible, then

$$a = (\underbrace{0, \ldots, 0}_{|A|}, \underbrace{2, \ldots, 2}_{|B|}, 1, \ldots, 1),$$

$B = N(A)$, and the induced subgraph on C has no isolated vertices.

Proof. Part (i) follows from part (b) of [20, Theorem 5.1]. To prove part (ii), we let A be the set of vertices x_i such that $a_i = 0$. Since a is a 2-cover, for any $x_i \in N(A)$, we must have $a_i \geq 2$. We may also include in B all other vertices x_j not in $N(A)$ such that $a_j \geq 2$. Clearly, $B \supseteq N(A)$, and (1) is satisfied.

If B is a vertex cover, then $(0, \ldots, 0, c_1, \ldots, c_{|B|}, d_1, \ldots, d_{|C|})$, where $c_i \geq 1$ and $d_j \geq 0$ for all i and j, is a 1-cover of G. Thus, a can be written as the sum of two 1-covers, a contradiction. Therefore, B is not a vertex cover of G. This also implies that C is not empty, and (2) is satisfied.

It follows from part (b) of [20, Theorem 5.1] that if the induced subgraph on C is bipartite, then it admits $(1, \ldots, 1)$ as a 2-cover that can be written as the sum of two 1-covers. Moreover, $(\underbrace{0, \ldots, 0}_{|A|}, \underbrace{c_1, \ldots, c_{|B|}}_{|B|})$, where $c_i \geq 1$ for all i, is a 1-cover of the induced subgraph on $A \cup B$. Thus, a can be written as the sum of two 1-covers, a contradiction. Thus (3) is satisfied.

To prove the last statement we observe that if $b_j > 2$ for some j, then a can be written as the sum of a 0-cover $(\underbrace{0,\ldots,0}_{|A|},\underbrace{0,\ldots,0,1,0\ldots,0}_{1 \text{ at the } j\text{-th place}},\underbrace{0,\ldots,0}_{|C|})$ and another 2-cover. This contradicts the irreducibility of a. Also, if there exists some $x_j \in B \setminus N(A)$, then a can be written in a similar fashion as the sum of a 0-cover and a 2-cover. This contradiction thus implies that $B = N(A)$. Similarly, if the induced subgraph on C has an isolated vertex, say the last one, then a can be written as the sum of the 0-cover $(\underbrace{0,\ldots,0}_{|A|},\underbrace{0,\ldots,0}_{|B|},\underbrace{0,\ldots,0,1}_{|C|})$ and another 2-cover, again a contradiction. □

We round out this section by explaining how information about vertex covers and 2-covers of a graph G are related to the edge ideal $I(G)$. We begin by recalling the notion of the Alexander dual of a monomial ideal.

Definition 2.5. Suppose I is a square-free monomial ideal. The **Alexander dual** of I, denoted by $I^\vee$, is the ideal obtained by mapping the generators of I to primary components. That is, if $I = (x_{1,1}\cdots x_{1,t_1},\ldots,x_{r,1}\cdots x_{r,t_r})$, then

$$I^\vee = (x_{1,1},\ldots,x_{1,t_1}) \cap \cdots \cap (x_{r,1},\ldots,x_{r,t_r}).$$

The ideal $I(G)^\vee$ is sometimes referred to as the **cover ideal** because of the well-known fact that the generators of $I(G)^\vee$ correspond to vertex covers (see, e.g., [13]):

Lemma 2.6. *Let G be a simple graph with edge ideal $I(G)$. Then*

$$I(G)^\vee = (\{x_{i_1}\cdots x_{i_s} \mid \{x_{i_1},\ldots,x_{i_s}\} \text{ is a vertex cover of } G\}).$$

Finally, we mention the notion of the symbolic power of an ideal, restricting to the case in which $I \subset R$ is a square-free monomial ideal. Suppose I has primary decomposition

$$I = P_1 \cap \cdots \cap P_r,$$

where each P_i is an ideal generated by a subset of the variables of R. The ***j*-th symbolic power** of I is the ideal

$$I^{(j)} = P_1^j \cap \cdots \cap P_r^j.$$

Set $J = I(G)^\vee$. Graph-theoretically, we can interpret the minimal generators of $J^{(2)}$ and J^2 in terms of 2-covers. For convenience, we denote covers by their corresponding monomials instead of the vectors themselves. Note that

$$J^{(2)} = \bigcap_{x_ix_j \in E_G} (x_i,x_j)^2,$$

so $J^{(2)}$ is the ideal whose minimal generators yield a 2-cover of G. On the other hand, J^2 is more restrictive. Its minimal generators are still 2-covers, but they must be able to be partitioned into two ordinary vertex covers. That is, if $m \in J^2$, then $m = m'm''$, where m' and m'' are 1-covers of G. A main part of the proof of Theorem 3.2 is to understand, via Theorem 2.4, the difference between monomials in $J^{(2)}$ and those in J^2.

3. Odd cycles and associated primes

In this section we prove the main result of our paper, that is, the odd cycle structure of a graph G appears in the associated primes of R/J^2, where $J = I(G)^\vee$.

The following definition is classical:

Definition 3.1. Let M be an R-module. A prime ideal P is called an **associated prime** of M if $P = \mathrm{Ann}(m)$, the annihilator of m, for some $m \in M$. The set of all associated primes of M is denoted by $\mathrm{Ass}(M)$.

We begin with some observations. Because we will only be dealing with the case that I is a monomial ideal, all $P \in \mathrm{Ass}(R/I)$ will have the form $P = (x_{i_1}, \ldots, x_{i_t})$ for some subset $\{x_{i_1}, \ldots, x_{i_t}\} \subset \{x_1, \ldots, x_n\}$. Since $J = I(G)^\vee = \bigcap_{x_ix_j \in E_G}(x_i, x_j)$, the associated primes of R/J are exactly the primes corresponding to the edges of G, that is, the prime ideals (x_i, x_j) where x_ix_j is an edge of G. Moreover, $\mathrm{Ass}(R/J^{(2)}) = \mathrm{Ass}(R/J)$. However, R/J^2 can have additional associated primes, and it is these primes we seek to identify.

We proceed by computing something stronger, namely, an irreducible decomposition for J^2. An **irreducible** monomial ideal in n variables is an ideal of the form $(x_1^{a_1}, \ldots, x_n^{a_n})$ with $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{N}^n$. This ideal is usually denoted as $\mathbf{m}^{\mathbf{a}}$, so, for example, the maximal homogeneous ideal would be $\mathbf{m}^{(1,\ldots,1)}$. If $a_i = 0$, then we adopt the convention that $\mathbf{m}^{\mathbf{a}} = (x_1^{a_1}, \ldots, \widehat{x_i^{a_i}}, \ldots, x_n^{a_n})$; that is, no power of x_i is in the ideal. Every monomial ideal I can be decomposed into the intersection of finitely many irreducible ideals, i.e., $I = \mathbf{m}^{\mathbf{a}_1} \cap \cdots \cap \mathbf{m}^{\mathbf{a}_s}$ (see, for example, [26, Lemma 5.18]).

Theorem 3.2. *Let G be a finite simple graph. If $J = I(G)^\vee$, then J^2 has the following irreducible decomposition:*

$$J^2 = \bigcap_{x_ix_j \in E_G} [(x_i^2, x_j) \cap (x_i, x_j^2)] \cap \bigcap_{\substack{\text{induced graph on } \{x_{i_1}, \ldots, x_{i_s}\} \\ \text{is an odd cycle}}} (x_{i_1}^2, \ldots, x_{i_s}^2).$$

Proof. Let L denote the ideal on the right-hand side in the statement of the theorem.

Consider a minimal generator $M \in J^2$, and thus M is the product of two 1-covers of G. Since $I(G)^\vee \subseteq (x_i, x_j)$ for every $x_ix_j \in E_G$, it follows that $J^2 \subseteq (x_i, x_j)^2 \subseteq (x_i^2, x_j)$, and similarly, $J^2 \subseteq (x_i, x_j^2)$. Hence

$$M \in \bigcap_{x_ix_j \in E_G} [(x_i^2, x_j) \cap (x_i, x_j^2)].$$

Suppose that the graph G has an odd induced cycle on the vertices $\{x_{i_1}, \ldots, x_{i_s}\}$. We claim that there exists some $x_{i_j} \in \{x_{i_1}, \ldots, x_{i_s}\}$ such that $x_{i_j}^2 \mid M$. Suppose not. Then $M = x_{i_1}^{a_{i_1}} \cdots x_{i_s}^{a_{i_s}} M'$, where $0 \leq a_{i_j} \leq 1$ for all $j = 1, \ldots, s$, and no x_{i_j} divides M'. Since $M = M_1M_2$, where $M_1, M_2 \in J$, both M_1 and M_2 must contain at least $(s+1)/2$ vertices of $\{x_{i_1}, \ldots, x_{i_s}\}$ in order to cover the odd induced cycle on these vertices. So, in the variables $\{x_{i_1}, \ldots, x_{i_s}\}$, M must have degree at least $s+1$. But we have assumed that M has degree at most s in these variables, a contradiction. So, there exists some x_{i_j} such that $x_{i_j}^2 \mid M$. Hence $M \in (x_{i_1}^2, \dots, x_{i_s}^2)$. Because this is true for each odd induced cycle, M is also in the second set of intersections. Thus, $J^2 \subseteq L$.

We now prove the converse. Consider a minimal generator N of L. Since $N \in \bigcap_{x_ix_j \in E_G}[(x_i^2, x_j) \cap (x_i, x_j^2)]$, it is clear that N is a 2-cover of G. It suffices to show that N can be written as the sum of two 1-covers. Suppose that this is not the case. Then by Theorem 2.4, for some independent set A, $B \supseteq N(A)$ and $C = V \setminus (A \cup B) \neq \emptyset$, we have

$$N = \prod_{x_j \in B} x_j^{b_j} \prod_{x_j \in C} x_j$$

where $b_j \geq 2$ for all j, and the induced subgraph on C is not a bipartite graph. This implies that the induced subgraph on C contains an odd cycle, say on the vertices $\{x_{i_1}, \dots, x_{i_s}\}$. From the expression of L, we have $N \in (x_{i_1}^2, \dots, x_{i_s}^2)$. However, as we have seen, the power of any vertices of C in N is exactly one. This is a contradiction. Hence, N can be written as the sum of two 1-covers. That is, $N \in J^2$. This is true for any minimal generator of L, so $L \subseteq J^2$. □

Our main result is now an immediate corollary.

Corollary 3.3. *Let G be a finite simple graph, and set $J = I(G)^\vee$. A prime $P = (x_{i_1}, \dots, x_{i_r})$ is in* $\operatorname{Ass}(R/J^2)$ *if and only if:*

1. *$P = (x_{i_1}, x_{i_2})$, and $x_{i_1}x_{i_2}$ is an edge of G, or*
2. *r is odd, and after re-indexing, $x_{i_1}x_{i_2}\cdots x_{i_r}x_{i_1}$ is an induced cycle of G.*

Moreover, we get an algorithm for detecting perfect graphs from the following corollary:

Corollary 3.4. *Let G be a finite simple graph with $J = I(G)^\vee$ and $J_c = I(G^c)^\vee$, where G^c is the complementary graph of G. Then G is perfect if and only if neither* $\operatorname{Ass}(R/J^2)$ *nor* $\operatorname{Ass}(R/J_c^2)$ *contains a prime of height greater than three.*

While our algorithm is unlikely to run in polynomial time, it has the advantage that it tells us exactly where any odd holes occur, and whether they are in G or G^c, which the polynomial time algorithm from [3] does not.

Remark 3.5. J. Mermin pointed out to us that one can use these techniques also to detect even induced cycles; most simply (though inefficiently), one can break up each edge of the graph xy one at a time, add a vertex to make two new edges xz and zy, and test the new graph for odd induced cycles. We restrict our attention to odd induced cycles since they seem to be most interesting for graph-theoretic purposes.

Corollary 3.3 also provides some crude bounds on the depth and projective dimension of R/J^2 in terms of the size of the largest induced odd cycle.

Corollary 3.6. *Let G be a finite simple graph on n vertices, and let t denote the size of the largest induced odd cycle of G. If $J = I(G)^\vee$, then*

(a) $\operatorname{depth}(R/J^2) \leq n - t$,
(b) $\operatorname{projdim}(R/J^2) \geq t$.

Proof. By the Auslander-Buchsbaum Formula, it suffices to prove (a). It is well-known that for any ideal I of R, $\operatorname{depth}(R/I) \leq \dim R/P$ for any $P \in \operatorname{Ass}(R/I)$. The conclusion now follows from Corollary 3.3. $\square$

We round out this section by using our methods to give an alternate proof of a result of Simis and Ulrich [29] and Sturmfels and Sullivant [31] about the second-secant of the ideal $I(G)$.

The join of an ideal was studied in [29] and [31]. We recall a special case of this definition. If I and J are ideals of $k[x_1,\ldots,x_n]$, then their **join**, denoted $I * J$, is a new ideal of $k[x_1,\ldots,x_n]$ which is computed as follows: Introduce n new variables $y_1,\ldots,y_n$, and let $I(y)$ (resp. $J(y)$) denote the image of the ideal I (resp. J) under the map $x_i \mapsto y_i$ in the ring $k[x_1,\ldots,x_n,y_1,\ldots,y_n]$. Then

$$I * J = (I(y) + J(y) + (y_1 - x_1, y_2 - x_2, \ldots, y_n - x_n)) \cap k[x_1,\ldots,x_n].$$

When $I = J$, we call $I * I$ the **second-secant ideal** of I and denote it by $I^{\{2\}}$. In the proof of the following theorem, we use the notation of $I^{[\mathbf{a}]}$ found in Section 5.2 of [26] for the generalized Alexander dual of I.

Corollary 3.7 ([29, Proposition 5.1],[31, Corollary 3.3])**.** *Let G be a finite simple graph. Then*

$$I(G)^{\{2\}} = (\{x_{i_1}\cdots x_{i_s} \mid G_{\{x_{i_1},\ldots,x_{i_s}\}} \text{ is an odd induced cycle}\});$$

that is, the generators correspond to the vertices of the induced odd cycles.

Proof. Since $J = I(G)^\vee$ is a square-free monomial ideal, every monomial generator of J divides $x_1\cdots x_n$, and so every generator of J^2 divides $x_1^2\cdots x_n^2$. Set $\mathbf{1} = (1,\ldots,1) \in \mathbb{N}^n$. By applying [31, Corollary 2.7] we have

$$I(G)^{\{2\}} = \left((I(G)^{[\mathbf{1}]})^2\right)^{[2\cdot\mathbf{1}]} \text{ modulo } \mathbf{m}^{\mathbf{1}+\mathbf{1}},$$

where modulo $\mathbf{m}^{\mathbf{1}+\mathbf{1}} = \mathbf{m}^{\mathbf{2}} = (x_1^2,\ldots,x_n^2)$ refers to removing all the monomial generators divisible by x_i^2 for some i. Now $I(G)^{[\mathbf{1}]} = I(G)^\vee$, so $I(G)^{\{2\}} = (J^2)^{[\mathbf{2}]}$ modulo $\mathbf{m}^{\mathbf{2}}$, where $\mathbf{2} := 2\cdot\mathbf{1} = (2,\ldots,2)$. By [26, Theorem 5.27], the generators of $(J^2)^{[\mathbf{2}]}$ are in one-to-one correspondence with the irreducible components of J^2; in particular, by Theorem 3.2, combined with [26, Theorem 5.27], we have

$$(J^2)^{[\mathbf{2}]} = (\{x_ix_j^2, x_i^2x_j \mid x_ix_j \in E_G\}) + (\{x_{i_1}\cdots x_{i_s} \mid G_{\{x_{i_1},\ldots,x_{i_s}\}} \text{ is an odd induced cycle}\}).$$

When we remove the monomial generators of $(J^2)^{[\mathbf{2}]}$ divisible by x_i^2 for some i, we are removing the first ideal, while the second remains, and hence the conclusion follows. $\square$

4. Algebraic classification of odd cycles

In this section we describe two algebraic approaches to detecting the existence of odd induced cycles (and, in particular, odd holes) in a graph. The first method is based upon taking quotients of ideals and is well-suited for constructing an algorithm to detect odd cycles using the ideal operations of commutative algebra. The second method is based upon the arithmetic degree of an ideal, which, although hard to compute, is interesting from a theoretical point of view. Of course, one could use Corollary 3.3 to determine if a graph has an odd cycle; however, Corollary 3.3 not only tells us if an odd cycle exists, it tells us which vertices make up the cycle. If one is simply interested in the question of existence, the results of this section may be more relevant and computationally more effective in situations in which computing $\mathrm{Ass}(R/J^2)$ is difficult.

4.1. **Method 1: Colon ideals.**

Using the technique of ideal saturation, we can describe an algebraic approach to detecting odd cycles. Recall that if I and K are ideals of R, then the **saturation** of I with respect to K, denoted $(I : K^\infty)$, is defined by:

$$(I : K^\infty) = (\cdots(((I : K) : K) : K)\cdots).$$

The ideal $I : K^\infty$ is then related to the primary decomposition of I as in Lemma 4.1. We omit the proof; see, for example [11, Lemma 2.4].

Lemma 4.1. *Let I be an ideal of $R = k[x_1, \ldots, x_n]$ with primary decomposition*

$$I = Q_1 \cap Q_2 \cap \cdots \cap Q_r.$$

If K is an ideal of R, then

$$(I : K^\infty) = \bigcap_{K \not\subseteq \sqrt{Q_i}} Q_i.$$

We use this to give a saturation method for detecting odd induced cycles.

Theorem 4.2. *Let G be a simple graph, and set $J = I(G)^\vee$. Fix an integer $t > 1$, and set*

$$L_t = \prod_{1 \leq i_1 < i_2 < \cdots < i_t \leq n} (x_{i_1} + x_{i_2} + \cdots + x_{i_t}).$$

Then G has no odd induced cycle of length $\geq t$ if and only $J^2 : (L_t)^\infty = J^2$.

Proof. Let $J^2 = Q_1 \cap \cdots \cap Q_r$ be the primary decomposition of J^2. By Corollary 3.3, we know that $\sqrt{Q_i} = (x_{i_1}, x_{i_2})$ where $\{x_{i_1}, x_{i_2}\}$ is an edge of our graph, or $\sqrt{Q_i} = (x_{i_1}, \ldots, x_{i_s})$ with s odd, and the induced graph on the vertices in $\sqrt{Q_i}$ is a cycle of odd length.

Suppose that G has no odd induced cycle of length $\geq t$, i.e., if $P_i = (x_{i_1}, \ldots, x_{i_s})$ is an associated prime, then $s = 2$ or s is odd and $s < t$. In both cases $(L_t) \not\subseteq \sqrt{Q_i} = P_i$ for all i. Hence, by Lemma 4.1

$$J^2 : (L_t)^\infty = \bigcap_{(L_t) \not\subseteq \sqrt{Q_i}} Q_i = \bigcap_{i=1}^{r} Q_i = J^2.$$

On the other hand, suppose that G has an odd induced cycle of length $\geq t$, i.e., there exists some Q_i such that $\sqrt{Q_i} = (x_{i_1}, \ldots, x_{i_s})$ with $s \geq t$ odd. Now, $x_{i_1} + x_{i_2} + \cdots + x_{i_t} \in \sqrt{Q_i}$, so $L_t \in \sqrt{Q_i}$, and hence

$$J^2 : (L_t)^\infty = \bigcap_{(L_t) \not\subseteq \sqrt{Q_i}} Q_i \supsetneq \bigcap_{i=1}^{r} Q_i = J^2.$$

The result now follows. $\square$

By specializing Theorem 4.2 to the case $t = 4$, we can detect graphs with odd holes:

Corollary 4.3. *Let G be a simple graph, and set $J = I(G)^\vee$. Set*

$$L = \prod_{1 \le i_1 < i_2 < i_3 < i_4 \le n} (x_{i_1} + x_{i_2} + x_{i_3} + x_{i_4}).$$

Then G has no odd hole if and only $J^2 : (L)^\infty = J^2$.

Because R is a Noetherian ring, there exists some integer $N \gg 0$ such that $I : K^\infty = I : K^N$. Since N may be quite large, computing the saturation of an ideal can be an expensive operation. Although Corollary 4.3 uses saturation to detect odd holes, one can in fact get away with only using one colon operation. We now prove this fact, beginning with the following lemma.

Lemma 4.4. *Suppose the induced subgraph on the vertices $\{x_{i_1}, \ldots, x_{i_s}\}$ of G is an odd hole. Let $U = \dfrac{\prod_{\ell=1}^n x_\ell}{\prod_{i=1}^s x_{i_j}}$. Set $J = I(G)^\vee$ and*

$$L = \prod_{1 \le i_1 < i_2 < i_3 < i_4 \le n} (x_{i_1} + x_{i_2} + x_{i_3} + x_{i_4}).$$

Then

$$M = x_{i_1} \cdots x_{i_s} U^2 \in J^2 : (L) \setminus J^2.$$

Proof. For simplicity, suppose $\{i_1, \ldots, i_s\} = \{1, \ldots, s\}$. So

$$M = x_1 \cdots x_s x_{s+1}^2 \cdots x_n^2.$$

First we show that $M \notin J^2$. If M did belong to J^2, then there would exist $M_1, M_2 \in J$ (not necessarily distinct) such that $M_1 M_2 \mid M$. Now M_1 and M_2 correspond to vertex covers of G, and thus must cover the odd cycle on the vertices $\{x_1, \ldots, x_s\}$. So, M_1 and M_2 must be divisible by at least $\frac{s+1}{2}$ of the variables of $\{x_1, \ldots, x_s\}$, and thus they must have at least one variable in common, i.e., there exists $i \in \{1, \ldots, s\}$ such that $x_i^2 \mid M_1 M_2$. But this contradicts the fact that $M_1 M_2 \mid M$.

We now show that $M \in J^2 : (L)$. The form L is divisible by $x_1 + x_2 + x_3 + x_4$, i.e., $L = (x_1 + x_2 + x_3 + x_4)L'$. Now $M x_1 = x_1^2 x_2 \cdots x_s x_{s+1}^2 \cdots x_n^2$. Then

$$x_1 x_3 x_5 \cdots x_s U \text{ and } x_1 x_2 x_4 x_6 \cdots x_{s-1} U$$

correspond to two vertex covers of G, and thus both monomials are elements of J. So, $Mx_1 \in J^2$. A similar argument will show that Mx_2, Mx_3 and Mx_4 are in J^2. Hence, $M(x_1+x_2+x_3+x_4) \in J^2$, whence $M(x_1+x_2+x_3+x_4)L' = ML \in J^2$. Thus $M \in J^2 : (L)$, as desired. □

Theorem 4.5. *Let G be a simple graph, and let $J = I(G)^\vee$. Set*

$$L = \prod_{1 \le i_1 < i_2 < i_3 < i_4 \le n} (x_{i_1} + x_{i_2} + x_{i_3} + x_{i_4}).$$

Then G has an odd hole if and only if $J^2 : (L) \supsetneq J^2$.

Proof. Because $J^2 : (L)^\infty \supseteq J^2 : (L)$, Theorem 4.3 implies that if $J^2 : (L) \supsetneq J^2$, then G has an odd hole. Conversely, the above lemma showed that if G has an odd hole, then $J^2 : (L) \supsetneq J^2$. □

Remark 4.6. Although we have only stated the above result for detecting odd holes, similar results could be stated for detecting odd induced cycles of length $\geq t$ for any integer $t > 1$.

Example 4.7. The converse of Lemma 4.4 does not hold. For example, consider the graph with edge set

$$\{x_1, x_2\}, \{x_2, x_3\}, \{x_3, x_4\}, \{x_4, x_5\}, \{x_5, x_1\}, \{x_5, x_6\}, \{x_6, x_7\}, \{x_7, x_8\}, \{x_8, x_1\}.$$

This graph is two five-cycles sharing an edge. Then $x_1x_2x_3x_4x_5x_6x_7x_8^2 \in J^2 : (L)$ (because $x_1 \cdots x_8$ is a generator of $J^2 : (L)$), but the graph does not have an odd induced cycle on $\{x_1, \ldots, x_7\}$.

Theorem 4.5 enables us to use ideal operations to detect odd holes in graphs. Since many of these operations have been implemented into computer algebra packages such as CoCoA [5] and Macaulay 2 [14], we can write simple procedures to determine the existence of odd holes. We provide the needed pseudo-code below:

Algorithm 4.8 (Existence of odd holes)**.**

```
 Input:  Edge ideal $I = I(G)$ of $G$
Output:  TRUE if $G$ contains an odd hole; FALSE otherwise

    • Compute $J :=$ dual $I$ by taking the intersection $\bigcap_{x_ix_j \in E_G}(x_i, x_j)$.
    • Compute all subsets of size four of $\{x_1, \ldots, x_n\}$.
    • Sum the elements inside each subset, and set $L$ to be the product of all the subsets.
    • If $J^2 \neq J^2 : (L)$, return TRUE. Else return FALSE.
```

4.2. **Method 2: Arithmetic degree and regularity.** The second main result of this section is to show that one can identify graphs with odd holes via the arithmetic degree and regularity.

Definition 4.9. Let I be a homogeneous ideal of $R = k[x_1, \ldots, x_n]$. The *arithmetic degree* of I is

$$\operatorname{adeg}(I) = \sum_{\text{homogeneous prime ideals } P \subseteq R} \operatorname{mult}_I(P) \deg(P).$$

In the above definition, $\operatorname{mult}_I(P)$ is the length of the largest ideal of finite length in the ring R_P/IR_P. It can be shown that $\operatorname{mult}_I(P) > 0$ if and only if P is an associated prime of I. So, the above formula gives us information about the existence of certain associated primes. Note that when I is a monomial ideal, all the associated primes have the form $P = (x_{i_1}, \ldots, x_{i_s})$, and $\deg(P) = 1$ for all of these ideals. So, when I is a monomial ideal, the above formula reduces to

$$\operatorname{adeg}(I) = \sum_{P \in \operatorname{Ass}(R/I)} \operatorname{mult}_I(P).$$

In the paper of Sturmfels, Trung, and Vogel [32], a combinatorial formula for $\operatorname{mult}_I(P)$ is given when I is a monomial ideal. Let $X = \{x_1, \ldots, x_n\}$. Any prime monomial ideal of R is generated by some subset of the variables. In particular, any monomial prime is determined by the variables not in the ideal; that is, for each monomial prime ideal P, there is a subset $Z \subseteq X$ such that $P = P_Z := (\{x_i \mid x_i \in X \setminus Z\})$. For a monomial $M \in R$, we let $\operatorname{supp}(M)$ denote the **support** of M, i.e., the set of variables appearing in M. By [32, Lemma 3.3], $\operatorname{mult}_I(P_Z)$ equals the number of standard pairs of the form $(\cdot, Z)$. If M is a monomial, and $Z \subseteq X$, a pair (M, Z) is **standard** if

(a) (M, Z) is **admissible**, i.e., $\operatorname{supp}(M) \cap Z = \emptyset$,
(b) $(M \cdot k[Z]) \cap I = \emptyset$, and
(c) (M, Z) is minimal with respect to the partial order

$$(M, Z) \leq (M', Z') \Leftrightarrow \ M \text{ divides } M' \text{ and } \operatorname{supp}(M'/M) \cup Z' \subseteq Z$$

for all pairs (M, Z) that satisfy (b).

We now specialize to the case of the monomial ideal $J^2 = (I(G)^\vee)^2$. By Corollary 3.3 we know that P_Z is an associated prime of J^2 if the induced graph on $X \setminus Z$ is either an edge of G or an odd cycle of G. We will now calculate $\operatorname{mult}_{J^2}(P_Z)$ for some special Z.

Lemma 4.10. *Let $x_ix_j \in E_G$ and set $Z = X \setminus \{x_i, x_j\}$. Then* $\operatorname{mult}_{J^2}(P_Z) = 3$.

Proof. We begin with some observations about the generators of $J = I(G)^\vee$ and J^2. Because the minimal generators of $I(G)^\vee$ correspond to the minimal vertex covers of G, and a vertex cover must contain either x_i or x_j (to cover the edge x_ix_j), every minimal generator of $I(G)^\vee$ is divisible by either x_i or x_j. Hence every minimal generator of J^2 is divisible by either x_i^2, x_ix_j, or x_j^2.

Since $Z = X \setminus \{x_i, x_j\}$, there are only four possible types of admissible pairs $(\cdot, Z)$:

$$(1, Z), (x_i^a, Z), (x_j^b, Z), \text{ and } (x_i^cx_j^d, Z),$$

where a, b, c, and $d \geq 1$. We consider these cases separately.

We claim first that $x_i^cx_j^dk[Z] \cap J^2 \neq \emptyset$, proving that $(x_i^cx_j^d, Z)$ is not standard. To see this, consider first (x_ix_j, Z) Any minimal vertex cover of G will contain x_i or all the neighbors of x_i. Let M_1 be a minimal vertex cover (written as a monomial) that does not contain x_i (and hence is divisible by x_j), and let M_2 be a minimal vertex cover that does not contain x_j (and hence is divisible by x_i). So $M_1M_2 = x_ix_jN$ for some N not divisible by either x_i or x_j, and thus $M_1M_2 \in x_ix_jk[Z] \cap J^2$, whence (x_ix_j, Z) fails to be a standard pair. Moreover, $x_i^{c-1}x_j^{d-1}M_1M_2 = x_i^cx_j^dN \in x_i^cx_j^dk[Z] \cap J^2$, and thus $x_i^cx_j^dk[Z] \cap J^2$ is nonempty, so $(x_i^cx_j^d, Z)$ is not a standard pair when $c, d \geq 1$.

Next, we show that if $a \geq 2$, then (x_i^a, Z) is not a standard pair. First consider (x_i^2, Z). Let M_1 be the monomial corresponding to a minimal vertex cover of G not containing x_j; then $x_i | M$ since x_ix_j is an edge of G. Hence $x_i^2 | M_1^2$ and $x_j \nmid M_1^2$, and we can write $M_1^2 = x_i^2N_1^2$, where $N_1 \in k[Z]$. Thus, because $M_1 \in J$, $M_1^2 \in x_i^2k[Z] \cap J^2$, so (x_i^2, Z) is not standard because it fails property (b). Moreover, $x_i^{a-2}M_1^2 = x_i^aN_1^2 \in x_i^ak[Z] \cap J^2$. Therefore when $a \geq 2$, (x_i^a, Z) is not standard (again failing property (b)), and by symmetry, if $b \geq 2$, (x_j^b, Z) is not standard either.

We move now to finding the pairs that are standard, beginning with $(1, Z)$. Since $k[Z] \cap J^2 = \emptyset$ (every monomial of J^2 is divisible by either x_i or x_j), we just need to show that $(1, Z)$ is a minimal pair with respect to this property. So, suppose there is $(M, Y) < (1, Z)$. By the definition of the ordering, we must have $M|1$, i.e., $M = 1$. So, $Z \subsetneq Y$. Thus, $Z \cup \{x_i\} \subseteq Y$ or $Z \cup \{x_j\} \subseteq Y$. But in both cases, $k[Y] \cap J^2 \neq \emptyset$. For example, if $Z \cup \{x_i\} \subseteq Y$, let M be any vertex cover that does not contain x_j and thus contains x_i. Then $M^2 \in k[Y] \cap J^2$. So $(1, Z)$ is a standard pair.

We now show that (x_i, Z) is a standard pair and omit the similar proof that (x_j, Z) is a standard pair. Now $x_i k[Z] \cap J^2 = \emptyset$ since every monomial of J^2 is divisible by $x_i^2, x_i x_j$, or x_j^2, but none of the monomials of $x_i k[Z]$ have this property. It suffices to show that (x_i, Z) is minimal. So, suppose $(M, Y) < (x_i, Z)$ and $Mk[Y] \cap J = \emptyset$. Because $M|x_i$, we have $M = 1$ or $M = x_i$. If $M = x_i$, then we have $(x_i, Y) < (x_i, Z)$, so to get the strict inequality, we must have $Z \subsetneq Y$. Therefore $Z \cup \{x_i\} \subset Y$ or $Z \cup \{x_j\} \subset Y$. But then we will fail to have $x_i k[Y] \cap J^2 = \emptyset$ in both cases. Indeed, if $Z \cup \{x_i\} \subseteq Y$, then let M be any vertex cover that does not contain x_j, and so it contains x_i. But then $M^2 \in x_i k[Y] \cap J^2$. On the other hand, if $Z \cup \{x_j\} \subseteq Y$, let M_1 by any vertex cover that does not contain x_i, i.e., it contains x_j, and let M_2 be any vertex cover that does not contain x_j, i.e., it contains x_i. Then $M_1 M_2 \in x_i k[Y] \cap J^2$. If $M = 1$, then again we must have $Z \cup \{x_i\} \subseteq Y$, but again, we fail to have $k[Y] \cap J^2 \neq \emptyset$.

So, $(1, Z), (x_i, Z), (x_j, Z)$ are the only standard pairs of the form $(\cdot, Z)$, and hence $\mathrm{mult}_{J^2}(P_Z) = 3$. □

Lemma 4.11. *Let the induced graph on* $\{x_i, x_j, x_k\}$ *be a three-cycle, and set* $Z = X \setminus \{x_i, x_j, x_k\}$. *Then* $\mathrm{mult}_{J^2}(P_Z) = 1$.

Proof. We once again begin with some observations about the generators of $J = I(G)^\vee$ and J^2. If M is a minimal generator of $I(G)^\vee$, then M must be divisible by one of $x_i x_j$, $x_i x_k$ or $x_j x_k$ because M corresponds to a minimal vertex cover, and we need at least two of the three vertices to cover the edges of the triangle formed by $\{x_i, x_j, x_k\}$. Hence, any monomial of J^2 must be divisible by one of $x_i^2 x_j^2, x_i^2 x_j x_k, x_i x_j^2 x_k, x_i^2 x_k^2, x_i x_j x_k^2, x_j^2 x_k^2$. In particular, every monomial of J must be divisible by at least one of x_i^2, x_j^2, x_k^2.

All the admissible pairs of the form $(\cdot, Z)$ are:

$$(1, Z), (x_i^a, Z), (x_j^b, Z), (x_k^c, Z), (x_i^a x_j^b, Z), (x_i^a x_k^c, Z), (x_j^b x_k^c, Z), (x_i^a x_j^b x_k^c, Z).$$

We claim that all but the last pair fail to be a standard pair, and the last is standard only when $a = b = c = 1$. The conclusion of the lemma will then follow once we prove this fact.

Note that $(1, Z \cup \{x_i\}) < (1, Z)$, and $(1, Z \cup \{x_i\})$ has the property that $k[Z \cup \{x_i\}] \cap J^2 = \emptyset$ since every monomial of J^2 is divisible by either x_j or x_k (because $x_j x_k$ is an edge of G), but no such monomial belongs to $k[Z \cup \{x_i\}]$. So, $(1, Z)$ is not a standard pair since it is not minimal with respect to the partial order.

For (x_i^a, Z), we have $(1, Z \cup \{x_i\}) < (x_i^a, Z)$ since $1|x_i^a$ and $\mathrm{supp}(x_i^a/1) \cup Z \subseteq Z \cup \{x_i\}$. But as noted above, $k[Z \cup \{x_i\}] \cap J^2 = \emptyset$. Thus (x_i^a, Z) is not minimal, so it cannot be a standard pair. A similar argument eliminates (x_j^b, Z) and (x_k^c, Z).

To rule out $(x_i^a x_j^b, Z)$, we first note that $(x_j, Z \cup \{x_i\}) < (x_i^a x_j, Z)$. But we also have $x_j k[Z \cup \{x_i\}] \cap J^2 = \emptyset$ since for every monomial M in J^2 such that $x_j | M$ but

$x_j^2 \nmid M$, we must have $x_k | M$. But $x_k \notin k[Z \cup \{x_i\}]$, and hence the intersection is empty. Therefore $(x_i^a x_j, Z)$ is not standard, and by a symmetric argument, neither is $(x_i x_j^b, Z)$. Suppose now that $a, b > 1$, and consider $(x_i^a x_j^b, Z)$. Pick a minimal vertex cover M_1 of G containing x_i and x_j but not x_k. Then $M_1^2 = x_i^2 x_j^2 N_1^2$, where $N_1 \in k[Z]$. Thus $x_i^{a-2} x_j^{b-2} M_1^2 = x_i^a x_j^b N_1^2 \in x_i^a x_j^b k[Z] \cap J^2$, and $(x_i^a x_j^b, Z)$ is not standard. The same argument with the variables permuted eliminates $(x_i^a x_k^c, Z)$ and $(x_j^b x_k^c, Z)$.

Suppose now that $a > 1$, and consider the pair $(x_i^a x_j^b x_k^c, Z)$. Note that $x_i x_j$ and $x_i x_k$ are each covers of the three-cycle, and $x_i^2 x_j x_k$ divides $x_i^a x_j^b x_k^c$. Let $M_1 = x_i x_j N_1$ be any minimal vertex cover of G divisible by $x_i x_j$ but not x_k, and let $M_2 = x_i x_k N_2$ be any minimal vertex cover of G divisible by $x_i x_k$ but not x_j. Then $x_i^{a-2} x_j^{b-1} x_k^{c-1} M_1 M_2 = x_i^a x_j^b x_k^c N_1 N_2 \in x_i^a x_j^b x_k^c k[Z] \cap J^2$, and therefore $(x_i^a x_j^b x_k^c, Z)$ fails property (b) when $a > 1$. A similar argument shows that $(x_i^a x_j^b x_k^c, Z)$ fails property (b) if $b > 1$ or $c > 1$.

Hence $\operatorname{mult}_{J^2}(P_Z) \leq 1$ since $(x_i x_j x_k, Z)$ is the only remaining candidate for a standard pair. Because P_Z is an associated prime, $\operatorname{mult}_{J^2}(P_Z) \geq 1$, and thus the multiplicity is equal to 1. □

Theorem 4.12. *Let G be a simple graph with $|E_G|$ the number of edges of G and $t(G)$ the number of triangles. Set $J = I(G)^\vee$. Then G has no odd holes if and only if*

$$\operatorname{adeg}(J^2) = 3|E_G| + t(G).$$

Proof. By definition

$$\operatorname{adeg}(J^2) = \sum_{P \in \operatorname{Ass}(R/J^2)} \operatorname{mult}_{J^2}(P)$$

The associated primes P of J^2 are either $P = (x_i, x_j)$ where $x_i x_j \in E_G$, $P = (x_i, x_j, x_k)$, where the induced graph on $\{x_i, x_j, x_k\}$ is a triangle (i.e., a 3-cycle), or $P = (x_{i_1}, \ldots, x_{i_s})$ where the induced graph on $\{(x_{i_1}, \ldots, x_{i_s}\}$ is an odd hole. Thus

$$\begin{aligned}\operatorname{adeg}(J^2) &= \sum_{x_i x_j \in E_G} \operatorname{mult}_{J^2}((x_i, x_j)) + \sum_{\{x_i, x_j, x_k\} \text{ is a triangle}} \operatorname{mult}_{J^2}((x_i, x_j, x_k)) \\ &\quad + \sum_{\{x_{i_1}, \ldots, x_{i_s}\} \text{ is an odd hole}} \operatorname{mult}_J((x_{i_1}, \ldots, x_{i_s})).\end{aligned}$$

Note that $\operatorname{mult}_J(P) > 0$ if and only if P is an associated prime. So G has no odd hole if and only if

$$\begin{aligned}\operatorname{adeg}(J^2) &= \sum_{x_i x_j \in E_G} \operatorname{mult}_{J^2}((x_i, x_j)) + \sum_{\{x_i, x_j, x_k\} \text{ is a triangle}} \operatorname{mult}_{J^2}((x_i, x_j, x_k)) \\ &= 3|E(G)| + t(G)\end{aligned}$$

where the final equality follows from Lemmas 4.10 and 4.11. □

Corollary 4.13. *Let G be a simple graph, and $J = I(G)^\vee$. Then*

$$\deg(J^2) = 3|E(G)|.$$

Proof. The formula for the degree of an ideal I is similar to the formula of the arithmetic degree of I, except one sums over all minimal associated primes, instead of all associated primes. Since the minimal associated primes of J^2 are precisely those that correspond to edges of G, the result now follows. $\square$

Theorem 4.12 is interesting more from a theoretical point of view rather than a computational point of view. Unlike Theorem 4.5, using the arithmetic degree to detect odd holes does not lend itself well to computations since there is no known fast method for computing this invariant. Moreover, to use Theorem 4.12, one would first need to count all the triangles in the graph.

We end this section by pointing out that the Castelnuovo-Mumford regularity of $(I(G)^\vee)^2$ can sometimes be used to detect graphs with odd holes. Here, the **Castelnuovo-Mumford regularity** of a homogeneous ideal I is defined by

$$\operatorname{reg}(I) = \min\{t \mid H^i_{(x_1,\ldots,x_n)}(I)_{n-i} = 0 \text{ for all } n > t,\ i \geq 0\}$$

where $H^i_{(x_1,\ldots,x_n)}(I)_{n-i}$ denotes degree $n-i$ piece of the i-th local cohomology module of I with respect to the maximal ideal $(x_1,\ldots,x_n)$.

Theorem 4.14. *Let G be a simple graph with $J = I(G)^\vee$. If*

$$\operatorname{reg}(J^2) > 3|E_G| + t(G),$$

then G has an odd hole.

Proof. For any homogeneous ideal I of R, Hoa and Trung [23, Theorem 1.1] proved that $\operatorname{reg}(I) \leq \operatorname{adeg}(I)$. Thus, if $3|E_G| + t(G) < \operatorname{reg}(J^2) \leq \operatorname{adeg}(J^2)$, the conclusion follows from Theorem 4.12. $\square$

5. Higher powers of the Alexander dual

Let $J = I(G)^\vee$. Having characterized $\operatorname{Ass}(R/J^2)$, in this section, we discuss $\operatorname{Ass}(R/J^s)$ when $s > 2$. In [1], Brodmann showed that if I is any ideal in R, then the sequence of sets of associated primes $\operatorname{Ass}(R/I^r)$ eventually stabilizes, though the sequence need not be monotone.(Brodmann's result is actually more general than this.) There have been a number of papers investigating when this chain stabilizes. Hoa [22] gave a very general bound when I is a monomial ideal, though the bound may be much larger than what is required in particular cases. Chen, Morey, and Sung [2] computed explicit bounds for when $\operatorname{Ass}(R/I^r)$ stabilizes when I is the edge ideal of a graph; the bounds encode the size of the smallest odd cycle in the graph.

In this paper, we are investigating the Alexander duals of edge ideals, and the situation is considerably more delicate when we consider powers higher than two. The primes that appear in $\operatorname{Ass}(R/J^s)$ are related to vertex colorings of induced subgraphs of G, and we will devote a future paper to this topic. Here, we limit our discussion to two cases more directly related to the topic of this paper.

The first result is a new proof of a result characterizing bipartite graphs in terms of symbolic and ordinary powers. Since a bipartite graph is a graph with no odd cycles, we get the following corollary of Corollary 3.3:

Corollary 5.1 (see [20, Theorem 5.1(b)], [8])**.** *Let G be a finite simple graph with $J = I(G)^\vee$. Then G is bipartite if and only if $J^t = J^{(t)}$ for all $t \geq 1$.*

Proof. By Corollary 3.3, G is bipartite if and only if $J^2 = J^{(2)}$. [20, Theorem 5.1(a)] shows that the symbolic Rees vertex cover algebra is generated in degree at most two, so the result follows immediately. □

Our second result demonstrates that when G is an odd cycle, the set $\operatorname{Ass}(R/J^s)$ stabilizes when $s = 2$. We can give the irreducible decomposition of J^s explicitly, and when $s = 2$, it specializes to the decomposition given in Theorem 3.2.

Below, we will call a sequence of integers $(u_1, \ldots, u_n)$ s-**admissible** if $2 \leq u_i \leq s$ for all $1 \leq i \leq n$, and $u_1 + \cdots + u_n = 2n + \binom{n+1}{2}(s-2)$.

Theorem 5.2. *Let G be an odd cycle on the variables $x_1, \ldots, x_n$, and let $J = I(G)^\vee$. If $s \geq 2$ is an integer, then*

$$J^s = \bigcap_{x_i x_j \in E_G} \left[\bigcap_{\substack{u+v=s+1 \\ u \geq 1, v \geq 1}} (x_i^u, x_j^v) \right] \cap \bigcap_{\substack{(u_1,\ldots,u_t) \\ s\text{-admissible}}} (x_1^{u_1}, \ldots, x_n^{u_n}).$$

In particular, $\operatorname{Ass}(R/J^2) = \operatorname{Ass}(R/J^s)$ *for all* $s \geq 2$*.*

Proof. Let L be the right-hand side, and suppose $x_i x_j$ is an edge of G. First, we show that $J^s \subseteq L$. Let $M \in J^s$. Then M is an s-cover of G, that is, the product of s 1-covers (with possibly some additional variables). Note that the intersection of all the height two ideals on the right-hand side involving both x_i and x_j is $(x_i, x_j)^s$. Since M is an s-cover of G, it is divisible by some monomial $x_i^r x_j^{s-r}$, which is in $(x_i, x_j)^s$. This proves that M is in all of the height two ideals on the right-hand side.

Next, we show that M is in all the ideals of larger height in the right-hand side. Let $u_1, \ldots, u_n$ be such that $(u_1, \ldots, u_n)$ is s-admissible. The degree of M is at least $s\binom{n+1}{2}$ because it contains at least s 1-covers of G. Suppose $M \notin (x_1^{u_1}, \ldots, x_n^{u_n})$. Then $\deg M \leq (u_1 - 1) + \cdots + (u_n - 1) = n + \binom{n+1}{2}(s-2)$. Thus we have

$$n + \left(\frac{n+1}{2}\right)(s-2) \geq \deg M \geq s\left(\frac{n+1}{2}\right).$$

Hence, comparing the outside terms and rewriting n, we have

$$n\left(\frac{n+1}{2}\right)\left(\frac{2}{n+1}\right) + \left(\frac{n+1}{2}\right)(s-2) \geq s\left(\frac{n+1}{2}\right).$$

Therefore, dividing by $\frac{n+1}{2}$,

$$\frac{2n}{n+1} + s - 2 \geq s,$$

and hence $2n \geq 2(n+1)$, a contradiction. Hence $M \in (x_1^{u_1}, \ldots, x_n^{u_n})$, and thus $M \in L$.

For the other inclusion, let $M \in L$. Then M is an s-cover of G since it is in each of the $(x_i, x_j)^s$, where $x_i x_j$ is an edge of G. Assume $M \notin J^s$; then M cannot be written as the product of s 1-covers of G. Write M so that it is the product of the largest number of 1-covers possible and then 2-covers that cannot be partitioned into 1-covers. These 2-covers are just the product of all the vertices appearing in the cycle. Note that there may be at most $(s-2)$ 1-covers in the partition; if there were $s-1$, then any remaining 2-covers inside of M would force $M \in J^s$ since they would also be 1-covers.

Consider the 1-covers in our partition; say there are p of them. They must contain at least $\frac{n+1}{2}$ vertices of the cycle to be a cover. We claim there must be exactly that many. If not, any extra vertex in a 1-cover can be taken from the 1-cover and moved to a 2-cover; this extra vertex would allow the 2-cover to be split into two 1-covers of the odd cycle since we would have all of the vertices present once and one of them twice. Therefore

$$\deg M = p\left(\frac{n+1}{2}\right) + \left(\frac{s-p}{2}\right)n,$$

where the first term comes from the 1-covers, and the second comes from the 2-covers. Let v_i be the number of times that x_i appears in M. Then since M contains at least one 2-cover that cannot be split into two 1-covers, $v_i \geq 1$ for all i. In addition, $v_i \leq s-1$ because it can appear at most once in each 1-cover and appears exactly once in each 2-cover, giving an upper bound of

$$p + \frac{s-p}{2} = \frac{p}{2} + \frac{s}{2} \leq \frac{s-2}{2} + \frac{s}{2} = s-1,$$

where the inequality comes from the fact that there are at most $(s-2)$ 1-covers in M. Hence $2 \leq v_i + 1 \leq s$ for all i. Additionally, note that the degree of any of the 1-covers in M is exactly $\frac{n+1}{2}$, while the degree of a 2-cover is n. Therefore the degree of M is maximized by having as many 1-covers as possible. Thus

$$(v_1+1)+\cdots+(v_n+1) = (v_1+\cdots+v_n)+n \leq \left[(s-2)\left(\frac{n+1}{2}\right)+n\right]+n = 2n+\left(\frac{n+1}{2}\right)(s-2).$$

Moreover, if we have $(s-2)$ 1-covers and one 2-cover in M, then we have equality above, and $(v_1+1,\ldots,v_n+1)$ is s-admissible. But by construction, $M \notin (x_1^{v_1+1},\ldots,x_n^{v_n+1})$ because no $x_i^{v_i+1}$ divides M. Therefore M is not in L after all.

If instead we have fewer than $(s-2)$ 1-covers, then the inequality is strict, so $(v_1+1)+\cdots+(v_n+1)$ does not add up to the sum needed to make $(v_1+1,\ldots,v_n+1)$ s-admissible. In this case, let z be the difference

$$z = \left[2n+(s-2)\left(\frac{n+1}{2}\right)\right] - ((v_1+\cdots+v_n)+n).$$

We create new exponents v_i+1+w_i in the following way: Let $w_1 = \min\{s-(v_1+1), z\}$. The idea is to add as much to v_1+1 as possible without having the new exponent exceed s. Then let $w_2 = \min\{s-(v_2+1), z-w_1\}$; the strategy is the same, and we stop if we have used up all of the slack z. In general, we let $w_i = \min\{s-(v_i+1), z-(w_1+\cdots+w_{i-1})\}$. We claim that this procedure uses up all of the slack z. If not, then

$$z > ns - (v_1+\cdots+v_n+n) = ns + \left[z - \left(2n + \left(\frac{n+1}{2}\right)(s-2)\right)\right].$$

Hence

$$2n + \binom{n+1}{2}(s-2) > ns,$$

and therefore

$$2n + \frac{ns+s}{2} - (n+1) > ns.$$

Simplifying, we have

$$n-1 > s\left(\frac{n-1}{2}\right),$$

so $s < 2$, a contradiction.

Consequently, we have $2 \leq v_i + 1 + w_i \leq s$ for all $1 \leq i \leq n$, and the sum of the $v_i + 1 + w_i$ is $2n + \binom{n+1}{2}(s-2)$. Thus $(v_1+1+w_1, \ldots, v_n+1+w_n)$ is s-admissible, and M is not divisible by any $x_i^{v_i+1+w_i}$ because $v_i < v_i + 1 + w_i$. Hence $M \notin L$, a contradiction. Therefore $L \subseteq J^s$. $\square$

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