COHEN-MACAULAY MULTIGRADED MODULES

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Abstract. Let S be a standard $\mathbb{N}^r$-graded algebra over a local ring A, and let M be a finitely generated $\mathbb{Z}^r$-graded module over S. We characterize the Cohen-Macaulayness of M in terms of the vanishing of certain sheaf cohomology modules. As a consequence, we apply our result to study the Cohen-Macaulayness of multi-Rees modules. Our work extends previous studies on the Cohen-Macaulayness of multi-Rees algebras.

1. Introduction

The notion of Cohen-Macaulay rings and modules marks the interplay between powerful lines of research in commutative algebra, algebraic geometry, and algebraic combinatorics. It finds surprising applications in far reaching problems and topics, for instance, in duality theory, in homological theory of rings, and in the study of polytopes and simplicial complexes.

Let $(A, \mathfrak{m})$ be a local ring. Let $I \subseteq A$ be a proper ideal, and let $\mathcal{R} = A \oplus It \oplus I^2t^2 \oplus \cdots \subset A[t]$ be the Rees algebra of I. Besides encoding many algebraic properties of the ideal I as well as its powers, the Rees algebra $\mathcal{R}$ also gives an algebraic realization of the blowing up of $\operatorname{Spec} A$ at the subscheme defined by I. Thus, characterizing the Cohen-Macaulayness of $\mathcal{R}$ has always been an important problem in commutative algebra. Lipman [14] succeeded in using Sancho de Salas sequences to study the Cohen-Macaulayness of $\mathcal{R}$ via the vanishing of sheaf cohomology groups on the blowup $\operatorname{Proj} \mathcal{R}$.

In recent years, much effort has been put forward to extend our knowledge from the $\mathbb{Z}$-graded case to a more general multi-graded setting (cf. [8, 9, 10, 11, 12, 13, 16]). Lipman's method was generalized by Hyry [10] to investigate the Cohen-Macaulayness of standard $\mathbb{N}^r$-graded algebras over a local ring. More precisely, [10, Theorem 3.1] shows that if $S = \bigoplus_{\mathbf{n} \geq \mathbf{0}} S_{\mathbf{n}}$ is a standard $\mathbb{N}^r$-graded algebra over $(A, \mathfrak{m})$ such that its irrelevant ideal $S_+ = \bigoplus_{\mathbf{n} > \mathbf{0}} S_{\mathbf{n}}$ has positive height, $Z = \operatorname{Proj} S$, and $E = Z \times_A A/\mathfrak{m}$, then S is a Cohen-Macaulay ring with a negative a-invariant $\mathbf{a}(S) < \mathbf{0}$ if and only if the following conditions are satisfied:

- $\Gamma(Z, \mathcal{O}_Z(\mathbf{n})) = S_{\mathbf{n}}$ for all $\mathbf{n} \geq \mathbf{0}$,
- $H^i(Z, \mathcal{O}_Z(\mathbf{n})) = 0$ for all $i > 0$ and $\mathbf{n} \geq \mathbf{0}$,

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- $H^i_E(Z, \mathcal{O}_Z(\mathbf{n})) = 0$ for all $i < \dim Z$ and $\mathbf{n} < \mathbf{0}$.

The goal of this paper is to extend Hyry's result to study the Cohen-Macaulayness of arbitrary finitely generated $\mathbb{Z}^r$-graded modules over S. Let M be a finitely generated $\mathbb{Z}^r$-graded S-module, and let $\mathcal{M}$ be its associated coherent sheaf on Z. Our first result, Theorem 3.1, gives a characterization for the Cohen-Macaulayness of M in terms of the vanishing of sheaf cohomology groups of twisted modules $\mathcal{M}(\mathbf{n})$ on Z and with support E.

A natural generalization of Rees algebras is the notion of Rees modules, also referred to as *Rees modifications*. Although this has not yet been discussed much in the literature. The motivation of studying Rees modules originated from the finiteness of the local cohomology modules (cf. [1] and the references cited there). We apply Theorem 3.1 to study the Cohen-Macaulayness of multi-Rees modules. It is well-known (cf. [8, 9, 10]) that if $I_1, \dots, I_r \subset A$ are ideals of positive heights such that the multi-Rees algebra of $I_1, \dots, I_r$ is Cohen-Macaulay then the usual Rees algebra of the product $I_1 \cdots I_r$ is also Cohen-Macaulay. Our next result, Theorem 4.2, extends this phenomenon to multi-Rees modules.

The converse of Theorem 4.2, even in the case of multi-Rees algebras, is known to be false. It is then desirable to seek for conditions which, together with the Cohen-Macaulayness of the Rees module of $I_1 \cdots I_r$ with respect to a given A-module N, would imply that the multi-Rees module of $I_1, \dots, I_r$ with respect to N is Cohen-Macaulay. Hyry [10] solved this problem for multi-Rees algebras (i.e. when $N = A$) provided that the analytic spread of $I_1 \cdots I_r$ is small. Our last result, Theorem 4.5, shows that the general problem for multi-Rees modules, under some additional conditions, has a similar solution.

To prove Theorem 3.1, we investigate local cohomology of the Rees module of the irrelevant ideal S_+ with respect to M under various graded structures. Here is a summary of the main ideas of the proof. Let $R = \mathcal{R}_S(S_+)$ and $T = \mathcal{R}_M(S_+)$ be the Rees algebra and Rees module of S_+ with respect to M, respectively. Clearly, R is an $\mathbb{N}$-graded algebra over S and T is a finitely generated $\mathbb{Z}$-graded R-module. The ring S can also be viewed as a standard $\mathbb{N}$-graded algebra over A (by coarsening the graded structure). Let $\mathfrak{M}_R$ and $\mathfrak{M}_S$ be the maximal homogeneous ideals in R and in S, respectively. Let $Y = \operatorname{Proj} R$ and $F = Y \times_S S/\mathfrak{M}_S$. Let $\mathcal{T}$ be the associated coherent sheaf of T on Y. At the heart of our arguments is the following Sancho de Salas sequence (cf. [14, p. 150])

$$\cdots \to [H^i_{\mathfrak{M}_R}(T)]_0 \to H^i_{\mathfrak{M}_S}(M) \to H^i_F(Y, \mathcal{T}) \to [H^{i+1}_{\mathfrak{M}_R}(T)]_0 \to \cdots \tag{1.1}$$

We start by observing that R and T have a natural $\mathbb{Z}^{r+1}$-graded structure given by

$$R = \bigoplus_{\mathbf{n} \in \mathbb{N}^r, k \geq 0} R_{(\mathbf{n};k)} \text{ and } T = \bigoplus_{\mathbf{n} \in \mathbb{Z}^r, k \geq 0} T_{(\mathbf{n};k)},$$

where $R_{(\mathbf{n};k)} = S_{(n_1+k, \dots, n_r+k)} t^k$ and $T_{(\mathbf{n};k)} = M_{\mathbf{n}}[S^k_+]_{(k, \dots, k)} t^k$ for $\mathbf{n} = (n_1, \dots, n_r)$. The cohomology modules $H^i_{\mathfrak{M}_R}(T)$ then inherit this $\mathbb{Z}^{r+1}$-graded structure, and we can write

$$[H^i_{\mathfrak{M}_R}(T)]_0 = \bigoplus_{\mathbf{n}\in\mathbb{Z}^r} H^i_{\mathfrak{M}_R}(T)_{(\mathbf{n};0)}.$$

Next, we show that $H^i_{\mathfrak{M}_R}(T)_{(\mathbf{n};k)} = 0$ for $k \geq 0$ and $\mathbf{n} < \mathbf{v}(M)$. This is done in Lemma 3.2. Together with the sequence (1.1), this implies that $[H^i_{\mathfrak{M}_S}(M)]_{\mathbf{n}} = [H^i_F(Y, \mathcal{T})]_{\mathbf{n}}$ for $\mathbf{n} < \mathbf{v}(M)$. We now observe that the Cohen-Macaulayness of M is characterized by the vanishing of $H^i_{\mathfrak{M}_S}(M)$ for $i < \dim M$. Theorem 3.1 is then proved by establishing the relationship between $H^i_F(Y, \mathcal{T})$ and $H^{i-r}_E(Z, \mathcal{M})$ and the vanishing of $[H^i_{\mathfrak{M}_S}(M)]_{\mathbf{n}}$ for $\mathbf{n} \not< \mathbf{v}(M)$. These are done in Lemma 3.3.

We start our proof of Theorem 4.2 by showing that if M is the multi-Rees module of $I_1, \ldots, I_r$ with respect to an A-module N then the a-invariant of M can be calculated explicitly, namely $\mathbf{a}(M) = -\mathbf{1}$. This is done in Lemma 4.1. Observe further that $\mathbf{v}(M) = \mathbf{0} > -\mathbf{1}$ in this case, and so Theorem 3.1 can be applied. Next, we let S be the multi-Rees algebra of $I_1, \ldots, I_r$, then the Rees algebra of the product $I_1 \cdots I_r$ is a diagonal subalgebra S^Δ of S (which is $\mathbb{N}$-graded). Theorem 4.2 is now proved by noticing that there is a canonical isomorphism $f : \operatorname{Proj} S \longrightarrow \operatorname{Proj} S^\Delta$ and pushing forward through f to reduce the problem to the well known $\mathbb{Z}$-graded situation.

Our last theorem, Theorem 4.5, is proved by a straightforward generalization of Hyry's method in [10] from multi-Rees algebras to multi-Rees modules. The paper is outlined as follows. In Section 2, we collect the notation, the terminology, and the basic results that will be used throughout the paper. Section 3 is devoted to proving the main theorem, Theorem 3.1, that characterizes the Cohen-Macaulayness of a finitely generated multi-graded module using sheaf cohomology modules. Finally in Section 4, as an application, we further deduce conditions to when the Cohen-Macaulayness of multi-Rees modules of ideals $I_1, \cdots, I_r$ with respect to a module and that of its diagonal submodule become equivalent.

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2. Preliminaries

For elementary facts about schemes, graded rings, and local cohomology modules, we refer the reader to [2, 4, 5, 7].

Let $\mathbf{0} = (0, \ldots, 0)$ and $\mathbf{1} = (1, \ldots, 1)$. Let $\mathbf{e}_1, \ldots, \mathbf{e}_r$ be the standard basis vectors of $\mathbb{Z}^r$. Throughout the paper, $S = \bigoplus_{\mathbf{n}\geq\mathbf{0}} S_{\mathbf{n}}$ will denote a standard $\mathbb{N}^r$-graded algebra over a local ring $(A, \mathfrak{m})$. That is, S is generated over $S_{\mathbf{0}} = A$ by elements of $\bigoplus_{j=1}^r S_{\mathbf{e}_j}$. Define S_+ to be the *irrelevant ideal* of S which is $\bigoplus_{\mathbf{n}>\mathbf{0}} S_{\mathbf{n}}$. Let $S^\Delta = \bigoplus_{n\geq 0} S_{(n,\ldots,n)}$

denote the *diagonal subalgebra* of S. Also, $M = \bigoplus_{\mathbf{n}\in\mathbb{Z}^r} M_{\mathbf{n}}$ will denote a finitely generated $\mathbb{Z}^r$-graded S-module. Set $M^\Delta = \bigoplus_{n\in\mathbb{Z}} M_{(n,\dots,n)}$. We will call M^Δ the *diagonal submodule* of M. Clearly, M^Δ is a $\mathbb{Z}$-graded S^Δ-module.

For a vector $\mathbf{n} \in \mathbb{Z}^r$, we always use $n_1, \dots, n_r$ to represent its coordinates. For $\mathbf{n}, \mathbf{m} \in \mathbb{Z}^r$, we shall write $\mathbf{n} \geq \mathbf{m}$ if $n_j \geq m_j$ for all $j = 1, \dots, r$; similarly, we write $\mathbf{n} > \mathbf{m}$ if $n_j > m_j$ for all $j = 1, \dots, r$. We also define

$$\min\{\mathbf{n}, \mathbf{m}\} = (\min\{n_1, m_1\}, \dots, \min\{n_r, m_r\}),$$
$$\max\{\mathbf{n}, \mathbf{m}\} = (\max\{n_1, m_1\}, \dots, \{n_r, m_r\}).$$

This leads naturally to the following definition of $\mathbf{v}(M)$ which we shall often make use of throughout the paper.

Definition 2.1. Suppose M is minimally generated in degrees $\mathbf{d}_1, \dots, \mathbf{d}_u \in \mathbb{Z}^r$, then we define

$$\mathbf{v}(M) = \min\{\mathbf{d}_1, \dots, \mathbf{d}_u\}.$$

Besides the given $\mathbb{N}^r$-graded structure, S has a natural $\mathbb{N}$-graded structure defined by $S = \bigoplus_{n\in\mathbb{Z}} S_n$ where $S_n = \bigoplus_{|\mathbf{n}|=n} S_{\mathbf{n}}$ and $|\mathbf{n}|$ indicates the sum of all components in $\mathbf{n}$. Let $\mathfrak{M}_S$ be the maximal homogeneous ideal of S with respect to this grading. Observe that $\mathfrak{M}_S$ is also $\mathbb{N}^r$-homogeneous. Thus, the local cohomology modules, $H^i_{\mathfrak{M}_S}(M)$, are $\mathbb{Z}^r$-graded modules for all i. This leads to a natural multigraded analog of the usual $\mathbb{Z}$-graded a-invariant. The multigraded a-invariant is well-defined and has been studied in more detail in [6, 8, 9].

Definition 2.2. For each $j = 1, \dots, r$, let

$$a_j(M) = \max\big\{m \in \mathbb{Z} \;\big|\; [H^{\dim M}_{\mathfrak{M}_S}(M)]_{\mathbf{n}} \neq 0 \text{ for some } \mathbf{n} \in \mathbb{Z}^r \text{ with } n_j = m\big\}.$$

The *multigraded a-invariant* of M is define to be the vector

$$\mathbf{a}(M) = (a_1(M), \dots, a_r(M)) \in \mathbb{Z}^r.$$

When $r = 1$, we shall omit the vector notation and simply consider $\mathbf{a}(M)$ as an integer.

We shall now recall basic definitions of multi-Rees algebras and modules.

Definition 2.3. Let B be a Noetherian ring and let N be a B-module.

(1) Let $I \subset B$ be a proper ideal. The *Rees algebra* of I over B is defined to be the subring

$$\mathcal{R}_B(I) = B[It] = \bigoplus_{n\geq 0} I^n t^n \subset B[t].$$

The *Rees module* of I with respect to N is defined to be the $\mathcal{R}_B(I)$-module

$$\mathcal{R}_N(I) = \bigoplus_{n\geq 0} I^n t^n N.$$

(2) Let $I_1, \ldots, I_r \subset B$ be proper ideals. The *multi-Rees algebra* of $I_1, \ldots, I_r$ over B is defined to be the subring

$$\mathcal{R}_B(I_1, \ldots, I_r) = \bigoplus_{\mathbf{n} \geq \mathbf{0}} I_1^{n_1} t_1^{n_1} \ldots I_r^{n_r} t_r^{n_r} \subset B[t_1, \ldots, t_r].$$

The *multi-Rees module* of $I_1, \ldots, I_r$ with respect to N is defined to be the $R_B(I_1, \ldots, I_r)$-module

$$\mathcal{R}_N(I_1, \ldots, I_r) = \bigoplus_{\mathbf{n} \geq \mathbf{0}} I_1^{n_1} t_1^{n_1} \ldots I_r^{n_r} t_r^{n_r} N.$$

Let $Z = \operatorname{Proj} S$ with respect to the $\mathbb{N}^r$-graded structure of S. As a set, $Z = \{\mathfrak{p} \in \operatorname{Spec} S \mid \mathfrak{p}$ is $\mathbb{N}^r$-homogeneous and $S_+ \not\subset \mathfrak{p}\}$. The simplest example is when $S = A[x_{ij}]$ where $1 \leq i \leq r$ and $0 \leq j \leq N_i$ and $Z = \mathbb{P}_A^{N_1} \times \cdots \times \mathbb{P}_A^{N_r}$. Throughout the paper, let $R = \mathcal{R}_S(S_+) = S[S_+ t]$ be the Rees algebra of S_+ over S. Observe that R is a standard $\mathbb{N}$-graded algebra over $R_0 = S$ where the grading is given by the power of t appearing in each element. Let $\mathfrak{M}_R$ be the maximal homogeneous ideal of R, and let $Y = \operatorname{Proj} R$ with respect to the $\mathbb{N}$-graded structure of R. We can view Y as a vector bundle over Z by [10, Lemma 3.1] stated in the next lemma. This provides a natural projection $\pi : Y \to Z$.

Lemma 2.4. *With the above notation, we have*

$$Y = \operatorname{Proj} \operatorname{Sym}(\mathcal{O}_Z(\mathbf{e}_1) \oplus \cdots \oplus \mathcal{O}_Z(\mathbf{e}_r)).$$

One of the techniques that we employ is to view graded algebras and modules under various gradings. A simple fact we often use is that local cohomology modules behave well under a change of grading. More precisely, suppose B is a standard $\mathbb{N}^k$-graded algebra over $(A, \mathfrak{m})$ and $\mathfrak{a} \subseteq B$ is an $\mathbb{N}^k$-graded homogeneous ideal. The local cohomology functors $H_{\mathfrak{a}}^i(\bullet)$ can be defined in the category of $\mathbb{Z}^k$-graded B-modules as usual. That means if N is a finitely generated $\mathbb{Z}^k$-graded B-module, then $H_{\mathfrak{a}}^i(N)$ is also a $\mathbb{Z}^k$-graded B-module for all i. Let $\phi : \mathbb{Z}^k \to \mathbb{Z}^l$ be a group homomorphism such that $\phi(\mathbb{N}^k) \subseteq \mathbb{N}^l$, and let $B^\phi = \bigoplus_{\mathbf{m} \in \mathbb{Z}^l} \left(\bigoplus_{\phi(\mathbf{n}) = \mathbf{m}} B_{\mathbf{n}} \right)$ and $N^\phi = \bigoplus_{\mathbf{m} \in \mathbb{Z}^l} \left(\bigoplus_{\phi(\mathbf{n}) = \mathbf{m}} N_{\mathbf{n}} \right)$. Then B^ϕ is a $\mathbb{N}^l$-graded ring and N^ϕ is a $\mathbb{Z}^l$-graded B^ϕ-module. It can be seen that $\left(H_{\mathfrak{a}}^i(\bullet)\right)^\phi$ and $H_{\mathfrak{a}^\phi}^i(\bullet^\phi)$ are both δ-functors and coincide when $i = 0$. Thus, $\left(H_{\mathfrak{a}}^i(N)\right)^\phi = H_{\mathfrak{a}^\phi}^i(N^\phi)$. Hence, when a new multigraded structure is specified by a group homomorphism ϕ, we shall omit the functorial notation $\bullet^\phi$ and simply write $H_{\mathfrak{a}}^i(N)$ for $\left(H_{\mathfrak{a}}^i(N)\right)^\phi$.

A module that we consider under alternate grading is the Rees module of S_+ with respect to M, namely $T = \mathcal{R}_M(S_+)$. Clearly, T is a $\mathbb{Z}$-graded R-module. Observe that R and T both further possess a $\mathbb{Z}^{r+1}$-graded structure given by

$$R = \bigoplus_{\mathbf{n} \in \mathbb{N}^r, k \geq 0} R_{(\mathbf{n};k)} \text{ and } T = \bigoplus_{\mathbf{n} \in \mathbb{Z}^r, k \geq 0} T_{(\mathbf{n};k)},$$

where $R_{(\mathbf{n};k)} = S_{(n_1+k, \ldots, n_r+k)} t^k$ and $T_{(\mathbf{n};k)} = M_{\mathbf{n}}[S_+^k]_{(k,\ldots,k)} t^k$ for $\mathbf{n} = (n_1, \ldots, n_r)$.

The following observation shall prove useful. Let W be an arbitrary finitely generated $\mathbb{Z}^{r+1}$-graded R-module. By writing $W = \bigoplus_{k\in\mathbb{Z}} W_{\bullet;k}$ where $W_{\bullet;k} = \bigoplus_{\mathbf{n}\in\mathbb{Z}^r} W_{\mathbf{n};k}$, we can consider W as a $\mathbb{Z}$-graded R-module. Let $\widetilde{W}$ be the associated coherent sheaf of W on Y. Note the diagonal subalgebra $S^\Delta \simeq \bigoplus_{k\geq 0} R_{(\mathbf{0};k)}$, and there is a canonical isomorphism $Z = \operatorname{Proj} S \simeq \operatorname{Proj} S^\Delta$. It can be seen that $\pi_*\widetilde{W} = \bigoplus_{\mathbf{n}\in\mathbb{Z}^r} \widetilde{W_{\mathbf{n};\bullet}}$ where $W_{\mathbf{n};\bullet} = \bigoplus_{k\in\mathbb{Z}} W_{\mathbf{n};k}$ is a graded S^Δ-module. The module $\Gamma(Y,\widetilde{W}) = \bigoplus_{\mathbf{n}\in\mathbb{Z}^r} \Gamma(Z, \widetilde{W_{\mathbf{n};\bullet}})$ has a natural structure of a $\mathbb{Z}^r$-graded S-module. We, therefore, may consider $\Gamma(Y,\widetilde{\bullet})$ as a functor from the category of $\mathbb{Z}^{r+1}$-graded R-modules to the category of $\mathbb{Z}^r$-graded S-module.

Lemma 2.5. *Let W be a finitely generated $\mathbb{Z}^{r+1}$-graded R-module, $\mathcal{M}$ be the associated coherent sheaf of M on Z, $T = \mathcal{R}_M(S_+)$, and $\mathcal{T}$ be the associated coherent sheaf of T on Y. Then*

(a) *We have isomorphisms*

$$H^i(Y,\widetilde{W}) \simeq H^i(Z,\pi_*\widetilde{W}) \simeq \bigoplus_{\mathbf{n}\in\mathbb{Z}^r} H^i(Z,\widetilde{W_{\mathbf{n};\bullet}}) \text{ for all } i \geq 0.$$

In particular, for $W = T$, we get

$$H^i(Y,\mathcal{T}) \simeq \bigoplus_{\mathbf{n}\geq \mathbf{v}(M)} H^i(Z,\mathcal{M}(\mathbf{n})) \text{ for all } i \geq 0.$$

(b) *Let $E = Z \times_A A/\mathfrak{m}$. We have isomorphisms*

$$H^i_{\pi^{-1}(E)}(Y,\widetilde{W}) \simeq H^i_E(Z,\pi_*\widetilde{W}) \simeq \bigoplus_{\mathbf{n}\in\mathbb{Z}^r} H^i_E(Z,\widetilde{W_{\mathbf{n};\bullet}}) \text{ for all } i \geq 0.$$

Proof. The first statement of (a) and (b) follow from the arguments of [10, p. 322]. The second statement of (a) takes into account the canonical isomorphism $Z \simeq \operatorname{Proj} S^\Delta$ and the fact that $T_{(\mathbf{n};k)} = M_{\mathbf{n}}[S_+^k]_{(k,\dots,k)}t^k = 0$ for $\mathbf{n} \not\geq \mathbf{v}(M)$. □

3. Cohen-Macaulay multigraded modules

In this section, we prove our first main result. The theorem is stated as follows.

Theorem 3.1. *Let S be a standard $\mathbb{N}^r$-graded algebra over a local ring $(A,\mathfrak{m})$ such that S_+ has positive height, and let M be a finitely generated $\mathbb{Z}^r$-graded S-module. Let $Z = \operatorname{Proj} S$ and let $E = Z \times_A A/\mathfrak{m}$. Let $\mathcal{M}$ be the associated coherent sheaf of M on Z. Then M is a Cohen-Macaulay module with $\mathbf{a}(M) < \mathbf{v}(M)$ if and only if the following conditions are satisfied:*

(1) $\Gamma(Z,\mathcal{M}(\mathbf{n})) = M_{\mathbf{n}}$ *for all* $\mathbf{n} \geq \mathbf{v}(M)$,
(2) $H^i(Z,\mathcal{M}(\mathbf{n})) = 0$ *for all* $i > 0$ *and* $\mathbf{n} \geq \mathbf{v}(M)$,
(3) $H^i_E(Z,\mathcal{M}(\mathbf{n})) = 0$ *for all* $i < \dim M - r$ *and* $\mathbf{n} < \mathbf{v}(M)$.

To prove Theorem 3.1 we shall need some auxiliary results. As indicated in the introduction, we begin by showing that $[H^i_{\mathfrak{M}_R}(T)]_{(\mathbf{n};k)} = 0$ for $i \geq 0, k \geq 0$, and $\mathbf{n} < \mathbf{v}(M)$ in Lemma 3.2 where T denotes the multi-Rees module $\mathcal{R}_M(S_+)$. Our proof of Lemma 3.2 is based upon a simple observation that if a $\mathbb{Z}^l$-graded module P has the $\mathbb{Z}^l$-graded homogeneous decomposition being $P = \bigoplus_{m_1=t} P_{\mathbf{m}}$ for a fixed $t \in \mathbb{Z}$, then for any $\mathbf{m} \in \mathbb{Z}^l$ such that $m_1 \neq t$ we must have $P_{\mathbf{m}} = 0$.

For any $\mathbb{Z}^{r+1}$-graded R-module $N = \bigoplus_{\mathbf{n}\in\mathbb{Z}^r, k\in\mathbb{Z}} N_{(\mathbf{n};k)}$, we define the *defining region* of N to be

$$\mathcal{D}(N) = \{(\mathbf{n};k) \mid N_{(\mathbf{n};k)} \neq 0\}.$$

Lemma 3.2. *For all* $i \geq 0$, $k \geq 0$, *and* $\mathbf{n} < \mathbf{v}(M)$, *we have* $[H^i_{\mathfrak{M}_R}(T)]_{(\mathbf{n};k)} = 0$.

Proof. Let Q be the ideal $\bigoplus_{\mathbf{n}>\mathbf{0}} R_{(\mathbf{n};k)}$ of R when R is viewed as a $\mathbb{N}^{r+1}$-graded ring. Let R_+ be the irrelevant ideal of R when R is viewed as a $\mathbb{N}^r$-graded ring, i.e. $R_+ = \bigoplus_{(n_1+k,\ldots,n_r+k)>\mathbf{0}} R_{(\mathbf{n};k)}$. Then the sequences

$$0 \to QT \to T \to T/QT \to 0 \text{ and } 0 \to R_+T \to T \to T/R_+T \to 0$$

are exact. By taking the long exact sequences of cohomology, we get

$$\cdots \to H^{i-1}_{\mathfrak{M}_R}(T/QT) \to H^i_{\mathfrak{M}_R}(QT) \to H^i_{\mathfrak{M}_R}(T) \to H^i_{\mathfrak{M}_R}(T/QT) \to \cdots \tag{3.1}$$

and

$$\cdots \to H^{i-1}_{\mathfrak{M}_R}(T/R_+T) \to H^i_{\mathfrak{M}_R}(R_+T) \to H^i_{\mathfrak{M}_R}(T) \to H^i_{\mathfrak{M}_R}(T/R_+T) \to \cdots. \tag{3.2}$$

Let $\mathbf{d}_i = (d_{i,1}, \cdots, d_{i,r})$, $1 \leq i \leq u$ be the degree of a minimal generator of M as in Definition 2.1. For a region $D \subseteq \mathbb{Z}^{r+1}$, we shall denote $\bigoplus_{(\mathbf{n};k)\in D} T_{(\mathbf{n};k)}$ by T_D. Observe that the defining region of T is $\mathcal{D}(T) = \{(\mathbf{n};k) \mid k \geq 0 \text{ and } \exists j : \mathbf{n} \geq \mathbf{d}_j\}$, and the defining region of QT is $\mathcal{D}(QT) = \{(\mathbf{n};k) \mid k \geq 0 \text{ and } \exists j : \mathbf{n} > \mathbf{d}_j\}$. It is also easy to see that $[T/QT]_{(\mathbf{n};k)} \neq 0$ only if $(\mathbf{n};k) \in \mathcal{D}(T)\backslash\mathcal{D}(QT) = \{(\mathbf{n};k) \mid k \geq 0 \text{ and } \exists j,l : \mathbf{n} \geq \mathbf{d}_j, n_l = d_{j,l}\}$.

Suppose $\mathbf{v}(M) = (v_1, \ldots, v_r)$. We define $d_{\max} = \max\{d_{i,r} \mid 1 \leq i \leq u\}$ and let $[r-1] = \{1, \ldots, r-1\}$. For $1 \leq j \leq u$ and a nonempty index set $I \subset [r-1]$, let

$$A_{j,I} = \{(\mathbf{n};k) \mid k \geq 0, n_w > d_{j,w}\ \forall w \notin I, n_s = d_{j,s}\ \forall s \in I, \text{ and } n_r > d_{\max}\}.$$

For $v_r \leq t \leq d_{\max}$, let

$$B_t = \{(\mathbf{n};k) \mid k \geq 0, n_r = t\}.$$

It can be seen that the regions $\{A_{j,I}, B_t \mid 1 \leq j \leq u, \emptyset \neq I \subset [r-1], v_r \leq t \leq d_{\max}\}$ are pairwise disjoint. It also follows from the definition of $A_{j,I}$'s and B_t's that

$$\mathcal{D}(T)\backslash\mathcal{D}(QT) \subseteq \Big(\bigcup_{j=1}^{u} \bigcup_{\emptyset\neq I\subset[r-1]} A_{j,I}\Big) \bigcup \Big(\bigcup_{t=v_r}^{d_{\max}} B_t\Big).$$

Thus, we can write

$$T/QT = \Big(\bigoplus_{j=1}^{u} \bigoplus_{\emptyset \neq I \subset [r-1]} [T/QT]_{A_{j,I}}\Big) \bigoplus \Big(\bigoplus_{t=v_r}^{d_{\max}} [T/QT]_{B_t}\Big).$$

The importance of the regions $A_{j,I}$'s and B_t's lies in the fact that $[T/QT]_{A_{j,I}}$ and $[T/QT]_{B_t}$ are submodules of T/QT for all j, I, and t. Since local cohomology commutes with direct sum, this allows us to get the following decomposition of $H^i_{\mathfrak{M}_R}(T/QT)$ into a direct sum of submodules defined over the $A_{j,I}$'s and B_t's:

$$H^i_{\mathfrak{M}_R}(T/QT) = \Big(\bigoplus_{j=1}^{u} \bigoplus_{\emptyset \neq I \subset [r-1]} H^i_{\mathfrak{M}_R}([T/QT]_{A_{j,I}})\Big) \bigoplus \Big(\bigoplus_{t=v_r}^{d_{\max}} H^i_{\mathfrak{M}_R}([T/QT]_{B_t})\Big). \quad (3.3)$$

Observe that $[T/QT]_{A_{j,I}}$ is annihilated by $\bigoplus_{n_s>0 \ \forall s \in I} R_{(\mathbf{n};k)}$ in T/QT, and so $[T/QT]_{A_{j,I}}$ can be viewed as a $\mathbb{Z}^{r+1-|I|}$-graded module over $R_I = \bigoplus_{n_s=0 \ \forall s \in I} R_{(\mathbf{n};k)}$. It now follows from the definition of local cohomology that

$$H^i_{\mathfrak{M}_R}([T/QT]_{A_{j,I}}) = H^i_{\mathfrak{M}_{R_I}}([T/QT]_{A_{j,I}})$$

is a $\mathbb{Z}^{r+1-|I|}$-graded module over R_I. Moreover, $H^i_{\mathfrak{M}_{R_I}}([T/QT]_{A_{j,I}})$ has the following $\mathbb{Z}^{r+1-|I|}$-graded decomposition

$$H^i_{\mathfrak{M}_{R_I}}([T/QT]_{A_{j,I}}) = \bigoplus_{n_s=d_{j,s} \ \forall s \in I} H^i_{\mathfrak{M}_{R_I}}([T/QT]_{A_{j,I}})_{(\mathbf{n};k)}.$$

This implies that if $n_s \neq d_{j,s}$ for some $s \in I$ then the term $H^i_{\mathfrak{M}_R}([T/QT]_{A_{j,I}})_{(\mathbf{n};k)}$ is not present in the homogeneous decomposition of $H^i_{\mathfrak{M}_R}([T/QT]_{A_{j,I}})$. That is, for $\mathbf{n} \in \mathbb{Z}^r$ such that $n_s \neq d_{j,s}$ for some $s \in I$, we must have $H^i_{\mathfrak{M}_R}([T/QT]_{A_{j,I}})_{(\mathbf{n};k)} = 0$.

By a similar argument, we have $H^i_{\mathfrak{M}_R}([T/QT]_{B_t})_{(\mathbf{n};k)} = 0$ if $n_r \neq t$. Hence, it follows from (3.3) that

$$H^i_{\mathfrak{M}_R}(T/QT)_{(\mathbf{n};k)} = 0 \text{ if } n_s \neq d_{j,s} \text{ for all } 1 \leq s \leq r-1 \text{ and } n_r < v_r.$$

This, together with the definition of $\mathbf{v}(M)$, implies that

$$H^i_{\mathfrak{M}_R}(T/QT)_{(\mathbf{n};k)} = 0 \text{ for all } \mathbf{n} < \mathbf{v}(M). \quad (3.4)$$

By a similar line of arguments on the defining regions of T and R_+T, we have

$$H^i_{\mathfrak{M}_R}(T/R_+T)_{(\mathbf{n},k)} = 0 \text{ for all } k > 0. \quad (3.5)$$

Observe further that there is an obvious isomorphism $Q \to R_+(-\mathbf{1}, 1)$ which maps $R_{(\mathbf{n};k)}$ to $R_{(\mathbf{n}-\mathbf{1};k+1)}$. Hence, it follows from (3.1), (3.2), (3.4) and (3.5) that, for any $k \geq 0$ and $\mathbf{n} < \mathbf{v}(M)$,

$$H^i_{\mathfrak{M}_R}(T)_{(\mathbf{n};k)} \simeq H^i_{\mathfrak{M}_R}(QT)_{(\mathbf{n};k)} \simeq H^i_{\mathfrak{M}_R}(R_+T)_{(\mathbf{n}-\mathbf{1};k+1)} \simeq H^i_{\mathfrak{M}_R}(T)_{(\mathbf{n}-\mathbf{1};k+1)}. \quad (3.6)$$

Moreover, $H^i_{\mathfrak{M}_R}(T)_{(\mathbf{n};k)} = 0$ for $k \gg 0$. Therefore, by successively applying (3.6), we have $H^i_{\mathfrak{M}_R}(T)_{(\mathbf{n};k)} = 0$ for all $\mathbf{n} < \mathbf{v}(M)$. The lemma is proved. □

Lemma 3.2 and the sequence (1.1) imply that $[H^i_{\mathfrak{M}_S}(M)]_{\mathbf{n}} = [H^i_F(Y,\mathcal{T})]_{\mathbf{n}}$ for $\mathbf{n} < \mathbf{v}(M)$. Thus, to characterize the Cohen-Macaulayness of M, or equivalently, the vanishing of $[H^i_{\mathfrak{M}_S}(M)]_{\mathbf{n}}$ for $i < \dim M$ and $\mathbf{n} \in \mathbb{Z}^r$, we proceed by relating $H^i_F(Y,\mathcal{T})$ to $H^{i-r}_E(Z,\mathcal{M})$ and establishing the vanishing of $[H^i_{\mathfrak{M}_S}(M)]_{\mathbf{n}}$ for $\mathbf{n} \not< \mathbf{v}(M)$.

Lemma 3.3. *Let $E = Z \times_A A/\mathfrak{m}$ and $F = Y \times_S S/\mathfrak{M}_S$ where $\mathfrak{M}_S$ is the homogeneous maximal ideal of S. Then as a $\mathbb{Z}^r$-graded S-module*

$$H^i_F(Y,\mathcal{T}) = \bigoplus_{\mathbf{n}<\mathbf{v}(M)} H^{i-r}_E(Z,\mathcal{M}(\mathbf{n})) \text{ for all } i \geq 0.$$

Proof. Let $G = Y \times_S S/S_*$ where $S_* = \bigoplus_{\mathbf{n}\neq \mathbf{0}} S_{\mathbf{n}}$. As noted in the preliminaries, we consider the functor $\Gamma_F(Y,\tilde{\bullet})$ from the category of $\mathbb{Z}^{r+1}$-graded $\mathcal{R}_S(S_+)$-modules to the category of $\mathbb{Z}^r$-graded S-modules. Since $\mathfrak{M}_S = \mathfrak{m} \oplus S_*$, this functor is equal to the composition functor $\Gamma_{\pi^{-1}(E)}(Y,\mathcal{H}^0_G(\tilde{\bullet}))$. It follows that there is a spectral sequence

$$E_2^{p,q} = H^p_{\pi^{-1}(E)}(Y,\mathcal{H}^q_G(\mathcal{T})) \Rightarrow H^{p+q}_F(Y,\mathcal{T}). \tag{3.7}$$

On the other hand, by Lemma 2.5

$$H^p_{\pi^{-1}(E)}(Y,\mathcal{H}^q_G(\mathcal{T})) = H^p_E(Z,\pi_*(\mathcal{H}^q_G(\mathcal{T})))$$

as $\mathbb{Z}^r$-graded S-modules. Now, the conclusion follows from Lemma 3.4 which shows that the spectral sequence (3.7) degenerates. □

Lemma 3.4. *Let $G = Y \times_S S/S_*$ where $S_* = \bigoplus_{\mathbf{n}\neq\mathbf{0}} S_{\mathbf{n}}$. Then $\mathcal{H}^i_G(\mathcal{T}) = (H^i_{S_*R}(T))^{\sim}$. Moreover, if $Z = \operatorname{Proj} S$ and $\pi : Y \longrightarrow Z$ is the canonical projection, we have*

$$\pi_*(\mathcal{H}^i_G(\mathcal{T})) = \begin{cases} 0, & \text{if } i \neq r, \\ \bigoplus_{\mathbf{n}<\mathbf{v}(M)} \mathcal{M}(\mathbf{n}), & \text{if } i = r \end{cases}$$

as $\mathbb{Z}^r$-graded $\mathcal{O}_Z$-modules.

Proof. For any affine open set $D_+(f) \subset Y$ where $f \in R$ is a homogeneous element, we have

$$\mathcal{H}^i_G(\mathcal{T})\big|_{D_+(f)} = (H^i_{S_*R_{(f)}}(T_{(f)}))^{\sim} = ((H^i_{S_*R}(T))_{(f)})^{\sim}.$$

This proves the first claim.

To prove the second claim, we consider the shifted module $N = M(\mathbf{v}(M))$ and let $W = \mathcal{R}_N(S_+)$ be the Rees module of S_+ with respect to N. As T, W admits a $\mathbb{Z}^{r+1}$-graded structure $W = \bigoplus_{\mathbf{n}\in\mathbb{Z}^r,k\geq 0} W_{(\mathbf{n};k)}$ and there is a natural isomorphism $W \to T(\mathbf{v}(M),0)$. It then follows from Lemma 2.5 and the preceding discussion that $\pi_*(\mathcal{H}^i_G(\widetilde{W})) = \pi_*(\mathcal{H}^i_G(\mathcal{T})) \otimes \mathcal{O}_Z(\mathbf{v}(M))$.

Now cover Z with open affine sets $\{U = \operatorname{Spec} S_{(s_1\cdots s_r)} \mid s_j \in S_{\mathbf{e}_j}\ \forall j = 1,\ldots,r\}$. By construction, $\pi^{-1}(U) = \operatorname{Spec} R_{(s_1\cdots s_r t)}$. Notice that $W_k = N(S_+)^k t^k$. So $W_{(s_1\cdots s_r t)}$ contains elements in the form of $\frac{mft^k}{(s_1\cdots s_r t)^k}$ with $m \in N$ and $f \in (S_+)^k = \bigoplus_{\mathbf{n}\geq\mathbf{0}} S_{(n_1+k,\ldots,n_r+k)}$. Since $N = M(\mathbf{v}(M))$, we have $\deg(m) \geq \mathbf{0}$. Therefore, we can write $\frac{mf}{(s_1\cdots s_r)^k}$ as a sum of forms like $h(\frac{s_1}{1})^{\ell_1}\cdots(\frac{s_r}{1})^{\ell_r}$ with $h \in N_{(s_1\cdots s_r)}$ and $\ell_i \geq 0$. Set $B = N_{(s_1\cdots s_r)}$

and $t_j = s_j/1 \in W_{(s_1\cdots s_r t)}$ for $j = 1, \dots, r$. From the above observation, $W_{(s_1\cdots s_r t)} = B[t_1, \dots, t_r]$ as $S_{(s_1\cdots s_r)}$-module. Since $G \cap \pi^{-1}(U) = \mathrm{V}(t_1, \dots, t_r)$, we have

$$\mathcal{H}^i_G(\widetilde{W})\big|_{\pi^{-1}(U)} = H^i_{(t_1,\dots,t_r)}(B[t_1, \dots, t_r])^{\sim}.$$

Moreover, it follows from [5, Remarque 2.1.11 of Chapitre III] that

$$H^i_{(t_1,\dots,t_r)}(B[t_1, \dots, t_r]) = \begin{cases} 0, & \text{if } i \neq r, \\ \bigoplus_{\mathbf{n}<\mathbf{0}} Bt_1^{n_1} \cdots t_r^{n_r}, & \text{if } i = r. \end{cases}$$

Thus,

$$\pi_*(\mathcal{H}^i_G(\widetilde{W})) = \begin{cases} 0, & \text{if } i \neq r, \\ \bigoplus_{\mathbf{n}<\mathbf{0}} \widetilde{N}(\mathbf{n}), & \text{if } i = r. \end{cases}$$

The proof is completed by observing that $\widetilde{N} = \mathcal{M}(\mathbf{v}(M))$. □

We are now ready to prove Theorem 3.1.

Proof of Theorem 3.1. We first observe that the Sancho de Salas sequence (1.1) is an exact sequence of $\mathbb{Z}^r$-graded modules.

It can be seen that M is a Cohen-Macaulay module with $\mathbf{a}(M) < \mathbf{v}(M)$ if and only if the following conditions are satisfied:

(i) $H^i_{\mathfrak{M}_S}(M)_\mathbf{n} = 0$ for all $0 \le i < \dim M$ and $\mathbf{n} < \mathbf{v}(M)$,
(ii) $H^i_{\mathfrak{M}_S}(M)_\mathbf{n} = 0$ for all $i \ge 0$ and $\mathbf{n} \not< \mathbf{v}(M)$.

It follows from Lemma 3.2 that $[H^i_{\mathfrak{M}_R}(T)]_{(\mathbf{n};0)} = 0$ for all $\mathbf{n} < \mathbf{v}(M)$. By the exact sequence (1.1), this shows that $[H^i_{\mathfrak{M}_S}(M)]_\mathbf{n} = [H^i_F(Y, \mathcal{T})]_\mathbf{n}$ for all $\mathbf{n} < \mathbf{v}(M)$ and all $i \ge 0$. Using Lemma 3.3, it follows that

$$H^i_{\mathfrak{M}_S}(M)_\mathbf{n} = H^{i-r}_E(Z, \mathcal{M}(\mathbf{n})) \text{ for all } i \ge 0 \text{ and } \mathbf{n} < \mathbf{v}(M), \tag{3.8}$$

$$H^i_{\mathfrak{M}_S}(M)_\mathbf{n} = 0 \text{ for all } i \ge 0 \text{ and } \mathbf{n} \not< \mathbf{v}(M). \tag{3.9}$$

Thus, $[H^i_{\mathfrak{M}_S}(M)]_\mathbf{n} = 0$ for $i < \dim M$ and $\mathbf{n} < \mathbf{v}(M)$ if and only if (3) holds. That is, (i) is equivalent to (3).

On the other hand using Lemma 3.3 and the Sancho de Salas sequence (1.1), we have that $[H^i_{\mathfrak{M}_R}(T)]_0 \simeq \bigoplus_{\mathbf{n}\not<\mathbf{v}(M)} [H^i_{\mathfrak{M}_S}(M)]_\mathbf{n}$. By (3.9), (ii) is equivalent to the condition that $[H^i_{\mathfrak{M}_R}(T)]_0 = 0$. It then follows from the Serre-Grothendieck correspondence between local cohomology and sheaf cohomology that this is equivalent to having $\Gamma(Y, \mathcal{T}) \simeq T_0 = M$ and $H^i(Y, \mathcal{T}) = 0$ for $i > 0$. Moreover, by Lemma 2.5, $\Gamma(Y, \mathcal{T}) \simeq \bigoplus_{\mathbf{n}\ge\mathbf{v}(M)} \Gamma(Y, \mathcal{M}(\mathbf{n}))$ and $H^i(Y, \mathcal{M}) \simeq \bigoplus_{\mathbf{n}\ge\mathbf{v}(M)} H^i(Y, \mathcal{M}(\mathbf{n}))$. Thus, (ii) is equivalent to (1) and (2). □

4. Cohen-Macaulay multi-Rees modules

In this section we shall apply our main result, Theorem 3.1, to investigate the Cohen-Macaulayness of multi-Rees modules. Our work extends previous studies on the Cohen-Macaulayness of multi-Rees algebras in [10].

Throughout this section, $(A, \mathfrak{m})$ is a local ring, N is a finitely generated A-module, $I_1, \ldots, I_r \subset A$ are ideals of positive heights with respect to N, and $S = \mathcal{R}_A(I_1, \ldots, I_r)$ is the multi-Rees algebra of $I_1, \ldots, I_r$. Clearly, S is a standard $\mathbb{N}^r$-graded algebra over A. Observe further that the Rees algebra $\mathcal{R}_A(I_1 \cdots I_r)$ is the diagonal subalgebra S^Δ of S. Let $M = \mathcal{R}_N(I_1, \ldots, I_r)$ be the multi-Rees module of $I_1, \ldots, I_r$ with respect to N as defined in Section 2. Then M is a finitely generated $\mathbb{Z}^r$-graded S-module, and similarly, the Rees module $\mathcal{R}_N(I_1 \cdots I_r)$ is the diagonal submodule M^Δ of M.

Lemma 4.1. *With notation as above, we have* $\mathbf{a}(M) = -\mathbf{1}$.

Proof. We shall use induction on r. For $r = 1$, our argument is similar to that of [8, Lemma 2.1]. Observe first that when $r = 1$, $S = A[I_1 t]$ is a standard $\mathbb{N}$-graded over A and $M = \mathcal{R}_N(I_1)$ is a finitely generated $\mathbb{Z}$-graded S-module. Let S_+ be the homogeneous irrelevant ideal of S under this grading (i.e. $S_+ = (I_1 t)S$). It can be seen that $\mathfrak{M}_S = \mathfrak{m} \oplus S_+$ where $\mathfrak{M}_S$ is the maximal ideal of S. Let $\mathcal{G}_N = M/I_1 M \simeq \bigoplus_{k \geq 0} I_1^k N / I_1^{k+1} N$. We have the following exact sequences

$$0 \to S_+ M \to M \to M/S_+ M \simeq N \to 0$$

$$0 \to S_+ M(1) \to M \to \mathcal{G}_N \to 0.$$

taking the corresponding long exact sequences of local cohomology modules, for any $i \geq 0$ and $n \in \mathbb{Z}$, we have

$$\cdots \to [H^{i-1}_{\mathfrak{M}_S}(N)]_n \to [H^i_{\mathfrak{M}_S}(S_+ M)]_n \to [H^i_{\mathfrak{M}_S}(M)]_n \to [H^i_{\mathfrak{M}_S}(N)]_n \to \cdots \tag{4.1}$$

and

$$\cdots \to [H^{i-1}_{\mathfrak{M}_S}(\mathcal{G}_N)]_n \to [H^i_{\mathfrak{M}_S}(S_+ M)]_{n+1} \to [H^i_{\mathfrak{M}_S}(M)]_n \to [H^i_{\mathfrak{M}_S}(\mathcal{G}_N)]_n \to \cdots . \tag{4.2}$$

Observe that $N \simeq M/S_+ M$, as an S-module, is concentrated in degree 0. Thus, for $n \neq 0$, $[H^i_{\mathfrak{M}_S}(N)]_n$ vanish for all $i \geq 0$. The long exact sequence (4.1) implies that

$$[H^i_{\mathfrak{M}_S}(S_+ M)]_n \simeq [H^i_{\mathfrak{M}_S}(M)]_n \text{ for any } i \geq 0 \text{ and } n \neq 0. \tag{4.3}$$

By [3, Theorem 4.4.6], $\dim \mathcal{G}_N = \dim N = \dim \mathcal{R}_N(I_1) - 1 = \dim M - 1$. Thus $H^{\dim M}_{\mathfrak{M}_S}(\mathcal{G}_N) = 0$. Therefore (4.2) implies that for any $n \in \mathbb{Z}$, there is a surjection

$$[H^{\dim M}_{\mathfrak{M}_S}(S_+ M)]_n \twoheadrightarrow [H^{\dim M}_{\mathfrak{M}_S}(M)]_n. \tag{4.4}$$

It follows from (4.3) and (4.4) that for any $n \neq 0$, there is a surjection

$$[H^{\dim M}_{\mathfrak{M}_S}(M)]_n \twoheadrightarrow [H^{\dim M}_{\mathfrak{M}_S}(M)]_{n-1}. \tag{4.5}$$

By successively applying (4.5) and noting that $H^{\dim M}_{\mathfrak{M}_S}(M)_n = 0$ for $n \gg 0$, we have $[H^{\dim M}_{\mathfrak{M}_S}(M)]_n = 0$ for all $n \geq 0$. We must also have $[H^{\dim M}_{\mathfrak{M}_S}(M)]_{-1} \neq 0$; otherwise, again by successively applying (4.5) we would conclude that $H^{\dim M}_{\mathfrak{M}_S}(M) = 0$, which is impossible. Hence, $\mathbf{a}(M) = -1$. The statement holds for $r = 1$.

Assume that the statement already holds for $r - 1$ (for some $r \geq 2$). Let $A' = \mathcal{R}_A(I_r)$. Then $S = A[I_1 t_1, \ldots, I_r t_r] = A'[I_1 t_1, \ldots, I_{r-1} t_{r-1}] = \mathcal{R}_{A'}(I_1, \ldots, I_{r-1})$ can be viewed as an $\mathbb{N}^{r-1}$-graded algebra over A', where the grading is given by powers of

$t_1, \ldots, t_{r-1}$. By giving t_r degree 0, we can also view M as a $\mathbb{Z}^{r-1}$-graded S-module. Since local cohomology modules behave well under a change of grading, we have the following $\mathbb{Z}^{r-1}$-graded homogeneous decomposition of $H^i_{\mathfrak{M}_S}(M)$:

$$H^i_{\mathfrak{M}_S}(M) = \bigoplus_{\mathbf{n}'=(n_1,\ldots,n_{r-1})\in\mathbb{Z}^{r-1}} [H^i_{\mathfrak{M}_S}(M)]_{\mathbf{n}'},$$

where $[H^i_{\mathfrak{M}_S}(M)]_{\mathbf{n}'} = \bigoplus_{n\in\mathbb{Z}}[H^i_{\mathfrak{M}_S}(M)]_{(n_1,\ldots,n_{r-1},n)}$. By induction, as a $\mathbb{Z}^{r-1}$-graded module, $\mathbf{a}(M) = -\mathbf{1} \in \mathbb{Z}^{r-1}$. Thus, $a_j(M) = -1$ for all $j = 1, \ldots, r-1$. It remains to show that $a_r(M) = -1$.

Let $A'' = \mathcal{R}_A(I_1, \ldots, I_{r-1})$. Then $S = \mathcal{R}_{A''}(I_r)$ can now be viewed as an $\mathbb{N}$-graded algebra over A''. By a similar argument to last paragraph, we can view M as a $\mathbb{Z}$-graded S-module; and therefore, by induction, $a_r(M) = -1$. Hence, $\mathbf{a}(M) = (a_1(M), \ldots, a_r(M)) = -\mathbf{1} \in \mathbb{Z}^r$. □

The next theorem extends a well-known result for multi-Rees algebras to arbitrary multi-Rees modules.

Theorem 4.2. *Let $(A, \mathfrak{m})$ be a local ring and let N be a finitely generated A-module. Let $I_1, \ldots, I_r \subset A$ be ideals of positive heights with respect to N. Assume that the multi-Rees module $\mathcal{R}_N(I_1, \ldots, I_r)$ is Cohen-Macaulay. Then the Rees module $\mathcal{R}_N(I_1 \cdots I_r)$ is also Cohen-Macaulay.*

Proof. Recall that the multi-Rees algebra $S = \mathcal{R}_A(I_1, \ldots, I_r)$ is a standard $\mathbb{N}^r$-graded algebra over A, and the Rees algebra $\mathcal{R}_A(I_1 \cdots I_r)$ is the diagonal subalgebra S^Δ of S. Let $Z = \operatorname{Proj} \mathcal{R}_A(I_1, \ldots, I_r)$, and let $E = Z \times_A A/\mathfrak{m}$. As before, there is a canonical isomorphism $f : Z \xrightarrow{\simeq} \operatorname{Proj} S^\Delta = \operatorname{Proj} \mathcal{R}_A(I_1 \cdots I_r)$ given by the inclusion $S^\Delta \hookrightarrow S$.

Again, let $M = \mathcal{R}_N(I_1, \ldots, I_r)$. For simplicity, we denote $M^\Delta = \mathcal{R}_N(I_1 \cdots I_r)$ by L. By [3, Theorem 4.4.6], we have

$$\dim M = \dim N + r = \dim L + (r-1). \tag{4.6}$$

Let $\mathcal{M}$ and $\mathcal{L}$ be the associated coherent sheaves of M and L on Z and $\operatorname{Proj} S^\Delta$ respectively. It can be seen that $f_*\mathcal{M} = \mathcal{L}$ and $f^*(\mathcal{O}_{\operatorname{Proj} S^\Delta}(n)) = \mathcal{O}_Z(n, \ldots, n)$. Thus, by the projection formula we get

$$f_*\mathcal{M}(n, \ldots, n) = f_*(\mathcal{M} \otimes \mathcal{O}_Z(n, \ldots, n)) = \mathcal{L} \otimes \mathcal{O}_{\operatorname{Proj} S^\Delta}(n) = \mathcal{L}(n). \tag{4.7}$$

It follows from Lemma 4.1 that $\mathbf{a}(M) = -\mathbf{1} < \mathbf{0} = \mathbf{v}(M)$. Theorem 3.1 together with (4.6) and (4.7) imply that

(i) $\Gamma(\operatorname{Proj} S^\Delta, \mathcal{L}(n)) = M_{(n,\ldots,n)} = L_n$ for all $n \geq 0$,
(ii) $H^i(\operatorname{Proj} S^\Delta, \mathcal{L}(n)) = 0$ for all $i > 0$ and $n \geq 0$,
(iii) $H^i_E(\operatorname{Proj} S^\Delta, \mathcal{L}(n)) = 0$ for all $i < \dim L - 1$ and $n < 0$.

The Rees module $\mathcal{R}_N(I_1 \cdots I_r)$ is Cohen-Macaulay now follows from a special use of Theorem 3.1 when $r = 1$. □

The converse of Theorem 4.2 is not always true. The rest of the paper is devoted to show that when $\operatorname{Proj} S^{\Delta}$ is a Cohen-Macaulay scheme, N is free in the punctured spectrum of A, and the analytic spread of $I_1 \cdots I_r$ is small, there are conditions which, together with the Cohen-Macaulayness of the usual Rees module $\mathcal{R}_N(I_1 \cdots I_r)$, imply the Cohen-Macaulayness of the multi-Rees module $\mathcal{R}_N(I_1, \ldots, I_r)$. We recall that N is free in the punctured spectrum of A means N is a free module if localized at every prime ideal in $\operatorname{Spec} A$ with the only possible exception at the maximal ideal. This condition implies that $\mathcal{R}_N(I_1 \cdots I_r)$ is associated to a locally free sheaf on $\operatorname{Proj} S^{\Delta}$.

We shall need some preliminary results. The following lemmas are generalization of [10, Lemmas 4.4 and 4.5] from Rees algebras to Rees modules. We shall sketch the proof of Lemma 4.3 and leave that of Lemma 4.4 to the reader.

Lemma 4.3. *Assume $(A, \mathfrak{m})$ is a local ring and N is a finitely generated A-module of dimension d. Let $I \subset A$ be an ideal of positive grade with respect to N. Let $P = \mathcal{R}_A(I)$ and $L = \mathcal{R}_N(I)$. Let $\mathcal{L}$ be the associated coherent sheaf of L over $Z = \operatorname{Proj} P$. Let $E = Z \times_A A/\mathfrak{m}$, and let $\ell = \ell(I)$ be the analytic spread of I. Assume that L is Cohen-Macaulay. Then*

(a) *$H^i(Z, \mathcal{L}(\ell - 1 - i)) = 0$ for all $i > 0$;*
(b) *If Z is a Cohen-Macaulay scheme and $\mathcal{L}$ is a locally free sheaf, then*

$$H^i_E(Z, \mathcal{L}(d - \ell - i)) = 0 \text{ for all } i < d.$$

Proof. For $i > 0$, by the Serre-Grothendieck correspondence we have

$$H^i(Z, \mathcal{L}(\ell - 1 - i)) \simeq [H^{i+1}_{P_+}(L)]_{\ell-1-i}.$$

Let $\mathfrak{M}_P$ be the maximal homogeneous ideal of P, that is, $\mathfrak{M}_P = P_+ \oplus \mathfrak{m}$. By Lemma 4.1, $a(L) = -1$. This and the assumption that L is Cohen-Macaulay imply that $[H^j_{\mathfrak{M}_P}(L)]_n = 0$ for all $j \geq 0$ and $n \geq 0$. Together with [10, Proposition 3.2], this implies that $[H^j_{P_+}(L)]_n = 0$ for all $i \geq 0$ and $n \geq 0$. Hence, $H^i(Z, \mathcal{L}(\ell - 1 - i)) = 0$ for all $0 < i \leq \ell - 1$.

Observe further that by definition, the closed fiber of the canonical projection $Z \longrightarrow \operatorname{Spec} A$ has dimension $\ell - 1$. Thus, it follows (cf. [5, Corollaire 4.2.2 in Chapitre III]) that $H^i(Z, \mathcal{F}) = 0$ for every coherent sheaf $\mathcal{F}$ on Z if $i \geq \ell$. In particular, this implies that $H^i(Z, \mathcal{L}(\ell - 1 - i)) = 0$ for all $i \geq \ell$. We have proved (a).

To prove (b), we first observe that $\mathbf{v}(L) = 0$. By Theorem 3.1,

$$H^i_E(Z, \mathcal{L}(d - \ell - i)) = 0 \text{ if } i > d - \ell.$$

On the other hand, by Lipman's global-local duality theorem (cf. [15, Theorem on p. 188]), $H^i_E(Z, \mathcal{L}) \simeq \operatorname{Hom}_A(\operatorname{Ext}^{d-i}(\mathcal{L}, \omega_Z), E_A(A/\mathfrak{m}))$ where ω_Z is the dualizing sheaf on Z and $E_A(A/\mathfrak{m})$ is the injective hull of $A/\mathfrak{m}$. Since $\mathcal{L}$ is locally free, it follows from [7, Propositions III.6.3 and III.6.7] that

$$\operatorname{Ext}^{d-i}(\mathcal{L}, \omega_Z) \simeq \operatorname{Ext}^{d-i}(\mathcal{O}_Z, \mathcal{L}^{\vee} \otimes \omega_Z) \simeq H^{d-i}(Z, \mathcal{L}^{\vee} \otimes \omega_Z),$$

where $\mathcal{L}^\vee = \mathrm{Hom}_{\mathcal{O}_Z}(\mathcal{L}, \mathcal{O}_Z)$. Thus

$$H^i_E(Z, \mathcal{L}) \simeq \mathrm{Hom}_A(H^{d-i}(Z, \mathcal{L}^\vee \otimes \omega_Z), E_A(A/\mathfrak{m})).$$

This implies that if $i \leq d-\ell$ (i.e. $d-i \geq \ell$) then, since the closed fiber of the projection $Z \longrightarrow \mathrm{Spec}\, A$ has dimension $\ell - 1$, we have $H^{d-i}(Z, \mathcal{L}^\vee(-d+\ell+i) \otimes \omega_Z) = 0$. That is, if $i \leq d - \ell$ then $H^i_E(Z, \mathcal{L}(d-\ell-i)) = 0$. Hence, (b) is proved. □

Lemma 4.4. *Let $(A, \mathfrak{m})$ be a local ring, and let $I_1, \dots, I_r \subset A$ be ideals of positive grade with respect to a finitely generated A-module N. Let $S = \mathcal{R}_A(I_1, \dots, I_r)$, $Z = \mathrm{Proj}\, S$, and $M = \mathcal{R}_N(I_1, \dots, I_r)$. Let $\mathcal{M}$ be the coherent sheaf associated to M over Z. Then*

$$\Gamma(Z, \mathcal{M}(\mathbf{n} - \mathbf{m})) = \mathrm{Hom}_A(I_1^{m_1} \cdots I_r^{m_r}, \Gamma(Z, \mathcal{M}(\mathbf{n}))) \text{ for all } \mathbf{n}, \mathbf{m} \geq \mathbf{0}.$$

Moreover,

$$\Gamma(Z, \mathcal{M}(\mathbf{n} - \mathbf{m})) = \Gamma(Z, \mathcal{M}(\mathbf{n})) :_{\Gamma(Z,\mathcal{M})} (I_1^{m_1} \cdots I_r^{m_r}) \text{ for all } \mathbf{n} \geq \mathbf{m} \geq \mathbf{0}.$$

Proof. The proof goes in the same line of arguments as that of [10, Lemma 4.5]. □

The next theorem generalizes [10, Theorem 4.1] to give the converse of Theorem 4.2 in the case that $I_1 \cdots I_r$ has small analytic spread.

Theorem 4.5. *Let $(A, \mathfrak{m})$ be a local ring, and let $I_1, \dots, I_r \subset A$ be ideals of positive grades with respect to a finitely generated A-module N. Assume that $\ell = \ell(I_1 \cdots I_r) \leq 2$.*

- (a) *If $\mathcal{R}_N(I_1, \dots, I_r)$ is Cohen-Macaulay, then $\mathcal{R}_N(I_1 \cdots I_r)$ is Cohen-Macaulay and the condition $(I_{j_1} \cdots I_{j_k})N :_N I_{j_l} = (I_{j_1} \cdots I_{j_{l-1}} \cdot I_{j_{l+1}} \cdots I_{j_k})N$ holds for all $1 \leq j_1 < \cdots < j_k \leq r$ and $1 \leq l \leq k$.*
- (b) *Conversely, if $\mathcal{R}_N(I_1 \cdots I_r)$ is Cohen-Macaulay, the condition $(I_{j_1} \cdots I_{j_k})N :_N I_{j_l} = (I_{j_1} \cdots I_{j_{l-1}} \cdot I_{j_{l+1}} \cdots I_{j_k})N$ holds for all $1 \leq j_1 < \cdots < j_k \leq r$ and $1 \leq l \leq k$, and if, in addition, N is free in the punctured spectrum and $\mathrm{Proj}\, \mathcal{R}_A(I_1 \cdots I_r)$ is a Cohen-Macaulay scheme, then $\mathcal{R}_N(I_1, \dots, I_r)$ is Cohen-Macaulay.*

Proof. As before, let $S = \mathcal{R}_A(I_1, \dots, I_r)$, $M = \mathcal{R}_N(I_1, \dots, I_n)$, $Z = \mathrm{Proj}\, S$ and $E = Z \times_A A/\mathfrak{m}$. Let $\mathcal{M}$ be the associated coherent sheaf of M on Z. The first part of (a) follows from Theorem 4.2. To prove the second part of the statement, we observe that by Theorem 3.1, $\Gamma(Z, \mathcal{M}(\mathbf{n})) = M_{\mathbf{n}} = I_1^{n_1} \cdots I_k^{n_k} N$ for all $\mathbf{n} \geq \mathbf{v}(M) = \mathbf{0}$. Thus, the condition $(I_{j_1} \cdots I_{j_k})N :_N I_{j_l} = (I_{j_1} \cdots I_{j_{l-1}} \cdot I_{j_{l+1}} \cdots I_{j_k})N$ follows by substituting appropriate $\mathbf{n}$ and $\mathbf{m}$ to Lemma 4.4.

To prove (b), let $L = M^\Delta = \mathcal{R}_N(I_1 \cdots I_r)$, and notice that $S^\Delta = \mathcal{R}_A(I_1 \cdots I_r)$ and $\mathrm{Proj}\, S^\Delta$ is a Cohen-Macaulay scheme. Let $\mathcal{L}$ be the associated coherent sheaf of L on $\mathrm{Proj}\, S^\Delta$. Recall that there is a canonical isomorphism $f : Z \to \mathrm{Proj}\, S^\Delta$. Since L is Cohen-Macaulay, by Lemma 4.3(a) and (4.7), we have $H^i(Z, \mathcal{M}(\ell-1-i, \dots, \ell-1-i)) = H^i(\mathrm{Proj}\, S^\Delta, \mathcal{L}(\ell-1-i)) = 0$ for $i > 0$. In such a case, [10, Lemma 4.2.(a)] states that $H^i(Z, \mathcal{M}(m_1 - i, \dots, m_r - i)) = 0$ if $m_j \geq \ell - 1$ for all j. Moreover, for $i > 0$, since $\ell \leq 2$, we have $\ell - 1 - i \leq 0$. This implies that

$$H^i(Z, \mathcal{M}(\mathbf{n})) = 0 \text{ for any } i > 0 \text{ and } \mathbf{n} \geq \mathbf{0}. \tag{4.8}$$

Since N is free in the punctured spectrum of A, $\mathcal{L}$ is a locally free sheaf. Thus, similarly, by using Lemma 4.3(b) and [10, Lemma 4.2(b)], we get

$$H^i_E(Z, \mathcal{M}(\mathbf{n})) = 0 \text{ for any } i < \dim N \text{ and } \mathbf{n} < \mathbf{0}. \tag{4.9}$$

We shall now verify that

$$\Gamma(Z, \mathcal{M}(\mathbf{n})) = I_1^{n_1} \cdots I_r^{n_r} N \text{ for all } \mathbf{n} \geq \mathbf{0}. \tag{4.10}$$

By applying the special case of Theorem 3.1 for $r = 1$ and (4.7), we have

$$\Gamma(Z, \mathcal{M}(m, \dots, m)) = \Gamma(\operatorname{Proj} S^\Delta, \mathcal{L}(m)) = L_m = I_1^m \cdots I_r^m N \text{ for all } m \geq 0.$$

Now for any $\mathbf{n} \geq \mathbf{0}$, we can find some m such that $\mathbf{n} \leq (m, \dots, m)$. By descending induction on each coordinate and successively applying Lemma 4.4, it can be seen that (4.10) holds.

The conclusion of (b) now follows from Theorem 3.1 together with (4.8), (4.9), and (4.10). □

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