ADJOINT LINE BUNDLES AND SYZYGIES OF PROJECTIVE VARIETIES

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Abstract. Let X be a smooth projective variety and let K be the canonical divisor of X. In this paper, we study embeddings of X given by adjoint line bundles $K \otimes L$, where L is an ample invertible sheaf. When X is a regular surface, we obtain a numerical criterion for $K \otimes L$ to have property N_p. In particular, we prove Mukai's conjecture for regular anticanonical surfaces. When X is a regular variety of arbitrary dimension, under a mild condition, we give an explicit calculation for the regularity of ideal sheaves of such embeddings.

1. Introduction

A classical result of Castelnuovo showed that a curve of degree greater than $2g$ has a normal homogeneous coordinate ring (g is the genus of the curve). This result was later rediscovered and further developed by many authors including Mumford [30], St. Donat [38] and Green [21]. Making use of techniques of Koszul cohomology, Green [21, Theorem 4.a.1] proved that a curve of degree greater than $(2g + p)$ has a normal homogeneous coordinate ring, and its defining ideal is generated by quadrics and has linear syzygies up to the p-th step. This property was later formalized and called *property* N_p [22, 11].

Mukai observed that the studies on curves can be viewed as dealing with adjoint bundles of the type $K_X \otimes D$, where K_X and D are the canonical divisor and an ample line bundle. In particular, he conjectured that if X is a smooth projective surface and L is an ample line bundle on X, then for any integer $p \geq 0$, the adjoint bundle $K_X \otimes L^{\otimes(4+p)}$ has property N_p. In higher dimension, Ein and Lazarsfeld [11] extends Mukai's conjecture as follows: if X is a smooth complex projective variety of dimension n and L is an ample line bundle on X, then for any integer $p \geq 0$, the adjoint bundle $K_X \otimes L^{\otimes(n+2+p)}$ has property N_p. For abelian varieties, Lazarsfeld [28] conjectured that if X is an abelian variety and L is an ample line bundle on X, then for any integer $p \geq 0$, the adjoint bundle $K_X \otimes L^{\otimes(p+3)}$ has property N_p.

A considerable amount of efforts has since then been put forward to investigate the minimal free resolution of embeddings of projective varieties given by adjoint line bundles. Particularly, Gallego and Purnaprajna in a series of papers [14, 15, 16, 17, 18, 19] give many special situations in which Mukai's conjecture holds true; Ein and Lazarsfeld [11] prove their conjecture when the ample line bundle L is very ample; Schenck and Smith [36]

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establish Ein and Lazarsfeld's conjecture for Gorenstein projective toric varieties; Rubei [35], Pareschi [31], and Pareschi and Popa [32, 33] study syzygies of adjoint line bundles on abelian varieties in connection to Lazarsfeld's conjecture.

In this paper, we study adjoint line bundles on *regular* projective varieties. We refer the reader to Definition 2.1 for the notion of regular varieties. For surfaces, we obtain a numerical criterion for adjoint line bundles to have property N_p. As a consequence, we prove Mukai's conjecture for regular anticanonical surfaces. When the variety is of arbitrary dimension, we look at the minimal free resolution of embeddings given by adjoint line bundles in whole and study its width. Let X be a smooth regular projective variety, and let K and L be the canonical divisor and an ample line bundle on X. Let $X_{\mathcal{L}}$ be the projective embedding of X given by the adjoint line bundle $\mathcal{L} = K \otimes L$ (when $\mathcal{L}$ is every ample), and let $\mathcal{I}_{\mathcal{L}}$ be the ideal sheaf of $X_{\mathcal{L}}$. We will study property N_p of $\mathcal{L}$ when X is a surface, and investigate the regularity of $\mathcal{I}_{\mathcal{L}}$ in general.

Section 3 is devoted to studying property N_p of $\mathcal{L}$, in connection to Mukai's conjecture, when X is a surface. A result of Gallego and Purnaprajna [19, Theorem 1.3] states that if X is a rational surface, L is a globally generated ample line bundle on X, then L has property N_p provided $-K.L \geq p+3$. For adjoint line bundles, this shows that if L is an ample line bundle on a rational surface X such that $K \otimes L$ is globally generated, then $K \otimes L$ has property N_p if $-K.L \geq p+3+K^2$. We extend this result to regular surfaces, and seek for an improvement on the bound $p+3+K^2$. Let X be a regular surface, and let p_a be the arithmetic genus of X. Then, $p_a = \chi(\mathcal{O}_X) - 1 = h^2(X, \mathcal{O}_X) \geq 0$. We show in Theorem 3.2 that the bound can be taken as $p+3+K^2-2p_a$. As a consequence, we then prove Mukai's conjecture for regular anticanonical surfaces. This is done in Theorem 3.6 and Corollary 3.7.

In Section 4, we examine the regularity of $\mathcal{I}_{\mathcal{L}}$. Let $n = \dim X$, we show that when the adjoint line bundle $\mathcal{L}$ is normally generated, under a mild condition, $\operatorname{reg} \mathcal{I}_{\mathcal{L}}$ can be explicitly calculated, namely $\operatorname{reg} \mathcal{I}_{\mathcal{L}} = n+2$ if $H^0(X, K) \neq 0$ and $\operatorname{reg} \mathcal{I}_{\mathcal{L}} = n+1$ if $H^0(X, K) = 0$. This is done in Theorem 4.1. Our theorem can be interpreted as a corresponding version for adjoint line bundles of Green's Top Row theorem [21, Theorem 4.a.4]. This result also exhibits an interesting asymptotic behaviour of adjoint line bundles of the form $K \otimes L^{\otimes m}$. As Remark 4.4 indicates, when m gets large, while the beginning part of the minimal free resolution of $\mathcal{I}_{\mathcal{L}}$ becomes more linear due to Ein and Lazarsfeld's theorem, its width is always fixed; in other words, even though the beginning part of the minimal free resolution of $\mathcal{I}_{\mathcal{L}}$ becomes nicer, the level of complexity of $\mathcal{I}_{\mathcal{L}}$ remains the same.

Our approach in this paper is a step forward from our previous work [23] and Gallego and Purnaprajna's work [19]. Theorem 3.2 is proved by closely examining the Hilbert function and Betti numbers of general hyperplane sections of curves and surfaces. Theorem 3.6 is attained by extending the argument of Gallego and Purnaprajna [19] to regular anticanonical surfaces. Theorem 4.1 is established by combining vanishing theorems with the study of Koszul cohomology groups, making use of Green's work in [21] to derive the non-vanishing of certain Koszul cohomology groups.

2. Notations and terminology

Our notations and terminology mostly follow from [25, 29]. Throughout the paper, we will work over an algebraically closed field k of characteristic 0. Although most of our results hold for unmixed smooth projective schemes, we shall only discuss varieties. By a *projective variety* we shall mean a reduced irreducible smooth projective scheme over k. A *surface* is a projective variety of dimension 2. To avoid the tension between the additive language of divisors and the multiplicative formalism of line bundles, our convention is always to work multiplicatively. And so, in many occasions, we identify divisors with corresponding line bundles.

We shall define regular projective varieties. This class of varieties includes rational projective varieties, and more generally, projective varieties having arithmetically Cohen-Macaulay embeddings.

Definition 2.1. A projective variety X of dimension n is said to be *regular* if

$$H^{n-1}(X, \mathcal{O}_X) = 0.$$

By Serre's duality, a projective variety X is regular if $H^1(X, K) = 0$, where K is the canonical divisor of X.

Remark 2.2. A smooth projective variety X of dimension n is *rational* if it is birationally equivalent to $\mathbb{P}^n$, i.e. there exists a birational morphism $\pi : X \longrightarrow \mathbb{P}^n$. Since X is smooth, it follows from [7, Theorem 1.2] that

$$H^i(X, \mathcal{O}_X) \simeq H^i(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}) = 0 \ \forall \ i \geq 1.$$

Thus, a rational projective variety is regular.

Remark 2.3. A smooth projective variety X is said to have an *arithmetically Cohen-Macaulay embedding* if X has a projective embedding with Cohen-Macaulay coordinate ring, i.e. there exists a Cohen-Macaulay standard graded k-algebra A such that $X \simeq \text{Proj } A$. The class of varieties having arithmetically Cohen-Macaulay embeddings has recently attracted considerably attention (cf. [20, 6, 7, 8, 23, 24]). Since X is smooth, X is a Cohen-Macaulay scheme. Thus, it follows from [7, Lemma 1.1] that X has an arithmetically Cohen-Macaulay embedding if and only if $H^i(X, \mathcal{O}_X) = 0$ for $i = 1, \ldots, \dim X - 1$. Hence, if X has an arithmetically Cohen-Macaulay embedding then X is regular.

We now direct those who are familiar with the concepts of regularity, property N_p and Koszul cohomology to the next section. For others, we recall these notions following [30, 21, 22, 11].

Definition 2.4. Let X be a projective variety, and let $\mathcal{F}$ be a coherent sheaf on X. The *regularity* of $\mathcal{F}$, denoted by $\operatorname{reg} \mathcal{F}$, is defined to be the least integer r such that $H^i(X, \mathcal{F}(r - i)) = 0$ for all $i > 0$.

The regularity can also be interpreted in terms of minimal free resolution as follows. Let $X \hookrightarrow \mathbb{P}^n$ be an embedding of X into a projective space, and let

$$0 \longrightarrow \bigoplus_j \mathcal{O}_{\mathbb{P}^n}(-b_{sj}) \longrightarrow \ldots \longrightarrow \bigoplus_j \mathcal{O}_{\mathbb{P}^n}(-b_{0j}) \longrightarrow \mathcal{F} \longrightarrow 0$$

be the minimal free resolution of $\mathcal{F}$ as an $\mathcal{O}_{\mathbb{P}^n}$-module. Then

$$\operatorname{reg} \mathcal{F} = \max\{b_{ij} - i \mid b_{ij} \neq 0\}.$$

Definition 2.5. Let X be a smooth projective variety, and let $\mathcal{L}$ be a very ample line bundle. The sections of $\mathcal{L}$ give an embedding $X \hookrightarrow \mathbb{P}(H^0(X, \mathcal{L})^*)$. Let $R = \operatorname{Sym}^* H^0(X, \mathcal{L})$ be the coordinate ring of $\mathbb{P}(H^0(X, \mathcal{L})^*)$, and let S be the coordinate ring of X in $\mathbb{P}(H^0(X, \mathcal{L})^*)$. Let

$$0 \to F_s \to F_{s-1} \to \cdots \to F_0 \to S \to 0$$

be the minimal free resolution of S over R.

(1) $\mathcal{L}$ is said to have *property* N_0 if $\mathcal{L}$ is normally generated, i.e. $S \simeq \oplus_{i \geq 0} H^0(X, \mathcal{L}^{\otimes i})$.
(2) For an integer $p \geq 1$, $\mathcal{L}$ is said to have *property* N_p if $\mathcal{L}$ has property N_0, $F_0 = R$ and $F_i = R(-i-1)^{\alpha_i}$ for $1 \leq i \leq p$.

Roughly speaking, property N_0 says that $\mathcal{L}$ embeds X as a projectively normal variety, property N_1 requires that $\mathcal{L}$ embeds X as a projectively normal variety whose defining ideal is generated by quadrics, and more generally, for an integer $p \geq 2$, property N_p means that $\mathcal{L}$ embeds X as a projectively normal variety whose defining ideal is generated by quadrics and has linear syzygies up to the p-th step.

Definition 2.6. Let X be a projective variety. Let $\mathcal{L}$ be an ample line bundle and $\mathcal{F}$ a coherent sheaf on X. Let $W = H^0(X, \mathcal{L})$ and $S = \operatorname{Sym}^* W$. Then, S is the homogeneous coordinate ring of $\mathbb{P}(W)$. Let $B = B(\mathcal{F}, \mathcal{L}) = \oplus_{q \in \mathbb{Z}} H^0(X, \mathcal{F} \otimes \mathcal{L}^{\otimes q}) = \oplus_{q \in \mathbb{Z}} B_q$ a S-graded module. The *Koszul complex* of B is defined to be

$$\ldots \longrightarrow \bigwedge^{p+1} W \otimes B_{q-1} \xrightarrow{d_{p+1,q-1}} \bigwedge^{p} W \otimes B_q \xrightarrow{d_{p,q}} \bigwedge^{p-1} W \otimes B_{q+1} \longrightarrow \ldots$$

and the *Koszul cohomology groups* of B are defined to be

$$\mathcal{K}_{p,q}(\mathcal{F}, \mathcal{L}) = \frac{\ker\, d_{p,q}}{\operatorname{im}\, d_{p+1,q-1}}, \text{ for } p, q \in \mathbb{Z}.$$

When $\mathcal{L} = \mathcal{O}_X(D)$ is the invertible sheaf corresponding to a divisor D and $\mathcal{F} = \widetilde{M}$ is the sheaf associated to a module M, we write $\mathcal{K}_{p,q}(M, D)$ for $\mathcal{K}_{p,q}(\mathcal{F}, \mathcal{L})$. When $\mathcal{F}$ is the structure sheaf $\mathcal{O}_X$, we also write $\mathcal{K}_{p,q}(\mathcal{L})$ for $\mathcal{K}_{p,q}(\mathcal{F}, \mathcal{L})$.

3. Adjoint line bundles on surfaces

In this section, we shall derive a numerical criterion for an adjoint line bundle to have property N_p and prove Mukai's conjecture for regular anticanonical surfaces.

The next lemma is well known. We will include the proof for self-containment.

Lemma 3.1. *Let R be a standard graded k-algebra of dimension d, and let $I \subset R$ be a homogeneous ideal. Suppose $A = R/I$ is of dimension n. Then, $H^n_{\mathfrak{m}}(A)^*$ is a torsion free A-module, where $\mathfrak{m}$ is the maximal homogeneous ideal of R.*

Proof. Let $x \in A$ be a homogeneous element, which is not a zero divisor. Let δ be the degree of x. Consider the exact sequence of R-modules

$$0 \longrightarrow A(-\delta) \xrightarrow{\times x} A \longrightarrow A/xA \longrightarrow 0.$$

This sequence gives rise to the following exact sequence

$$\operatorname{Ext}^{d-n}_R(A/xA, R) \longrightarrow \operatorname{Ext}^{d-n}_R(A, R) \xrightarrow{\times x} \operatorname{Ext}^{d-n}_R(A, R)(-\delta).$$

Since $\dim A/xA = n-1$, by local duality, we have $\operatorname{Ext}^{d-n}_R(A/xA, R) = H^n_{\mathfrak{m}}(A/xA) = 0$. Thus, we have an injection $\operatorname{Ext}^{d-n}_R(A, R) \hookrightarrow \operatorname{Ext}^{d-n}_R(A, R)(-\delta)$. This implies that $\operatorname{Ext}^{d-n}_R(A, R)$ is torsion free as A-module. Moreover, by local duality again, $H^n_{\mathfrak{m}}(A)^* = \operatorname{Ext}^{d-n}_R(A, R)(-d)$. The conclusion follows. □

Recall from the definition that a smooth surface X is regular if $H^1(X, \mathcal{O}_X) = 0$. Our first main result is stated as follows. Our proof somewhat resembles that of [23, Theorem 2.5].

Theorem 3.2. *Let X be a surface with $H^1(X, \mathcal{O}_X) = 0$. Suppose K is the canonical divisor and L is an ample divisor on X such that $\mathcal{L} = K \otimes L$ is normally generated. Then, for any integer $p \geq 0$, the adjoint line bundle $\mathcal{L}$ has property N_p if $-K.L \geq p + 3 + K^2 - 2p_a$, where p_a is the arithmetic genus of X.*

Proof. Since $\mathcal{L}$ is normally generated, $\mathcal{L}$ is necessarily very ample (cf. [30]). Thus, the theorem is proved for $p = 0$. Suppose now that $p \geq 1$. Let $N = \dim_k H^0(X, \mathcal{L}) - 1$, and let $X_{\mathcal{L}}$ be the embedding of X in $\mathbb{P}^N$ given by $\mathcal{L}$. Let S, I and $\mathcal{I}$ be the coordinate ring, the defining ideal and the ideal sheaf of $X_{\mathcal{L}}$ in $\mathbb{P}^N$.

For $h > 0$, since $\mathcal{L}^{\otimes h} = K \otimes \mathcal{L}^{\otimes(h-1)} \otimes L$, by Kodaira's vanishing theorem, we have $H^i(X, \mathcal{L}^{\otimes h}) = 0$ for $i \geq 1$ and $h > 0$. This, together with Kodaira's vanishing theorem (for $h < 0$ since $\mathcal{L}$ is very ample), and the hypotheses (for $h = 0$), gives us

$$H^1(X, \mathcal{L}^{\otimes h}) = 0 \text{ for } h \in \mathbb{Z}.$$

Thus, $\mathcal{L}$ embeds X as an arithmetically Cohen-Macaulay variety. That is, S/I is a Cohen-Macaulay ring. Let

$$0 \longrightarrow F_{N-2} \longrightarrow \ldots \longrightarrow F_1 \longrightarrow I \longrightarrow 0$$

be the minimal free resolution of I, where

$$F_i = \bigoplus_{j \geq i+1} S(-j)^{\beta_{i,j}} \text{ for } i = 1, \ldots, N-2.$$

The adjoint line bundle $\mathcal{L}$ has property N_p if and only if $\beta_{i,j} = 0$ for all $1 \leq i \leq p$ and $j \geq i + 2$.

Let C be a general hyperplane section of $X_{\mathcal{L}} \subset \mathbb{P}^N$, then C is an irreducible arithmetically Cohen-Macaulay curve in $\mathbb{P}^{N-1}$ with the same minimal free resolution as that of $X_{\mathcal{L}}$. Let $T = k[x_0, \ldots, x_{N-1}]$ be the homogeneous coordinate ring of $\mathbb{P}^{N-1}$, and let A be the homogeneous coordinate ring of C in $\mathbb{P}^{N-1}$. Then, the Betti number of $X_{\mathcal{L}}$ (also of C) are given by

$$\beta_{i,j} = \mathrm{Tor}_i^T(A,k)_j, \text{ for all } i,j.$$

Let $\mathbf{H}_X$ be the Hilbert function of $X_{\mathcal{L}}$ in $\mathbb{P}^N$, i.e.

$$\mathbf{H}_X(h) = \dim_k(S/I)_h.$$

Since S/I is a Cohen-Macaulay ring, we have

$$\dim_k(S/I)_h = h^0(X_{\mathcal{L}}, \mathcal{O}_{X_{\mathcal{L}}}(h)) = h^0(X, \mathcal{L}^{\otimes h}).$$

As before, by Kodaira's vanishing theorem, $H^1(X, \mathcal{L}^{\otimes h}) = H^2(X, \mathcal{L}^{\otimes h}) = 0$ for $h > 0$. Thus, by the Riemann-Roch theorem, we get

$$\dim_k(S/I)_h = h^0(X, \mathcal{L}^{\otimes h}) = \frac{1}{2}\mathcal{L}^{\otimes h}.(\mathcal{L}^{\otimes h} - K) + 1 + p_a \ \forall\ h > 0. \tag{3.1}$$

Let d be the degree of $X_{\mathcal{L}}$ in $\mathbb{P}^N$, then by (3.1),

$$d \leq \mathcal{L}^2 = (K+L)^2 = K^2 + 2K.L + L^2. \tag{3.2}$$

Let $D = C \cap H$ be a general hyperplane section of C. Then, D is a set of d points in $\mathbb{P}^{N-2}$. Let $\mathcal{I}_C$ be the ideal sheaf of C in $\mathbb{P}^{N-1}$, and let $\mathcal{I}_D$ be the ideal sheaf of D in $H \simeq \mathbb{P}^{N-2}$. Let $\mathbf{H}_D$ be the Hilbert function of D. Since C is arithmetically Cohen-Macaulay, it follows from the exact sequence

$$0 \longrightarrow \mathcal{I}_C \longrightarrow \mathcal{I}_C(1) \longrightarrow \mathcal{I}_D(1) \longrightarrow 0$$

that

$$0 \longrightarrow H^1(\mathbb{P}^{N-2}, \mathcal{I}_D(1)) \longrightarrow H^2(\mathbb{P}^{N-1}, \mathcal{I}_C) \longrightarrow H^2(\mathbb{P}^{N-1}, \mathcal{I}_C(1)) \longrightarrow 0.$$

This implies that

$$h^2(\mathbb{P}^{N-1}, \mathcal{I}_C) - h^2(\mathbb{P}^{N-1}, \mathcal{I}_C(1)) = h^1(\mathbb{P}^{N-2}, \mathcal{I}_D(1)) = d - \mathbf{H}_D(1). \tag{3.3}$$

Claim 3.3. $h^2(\mathbb{P}^{N-1}, \mathcal{I}_C(1)) = 0$.

Proof of Claim. Suppose $h^2(\mathbb{P}^{N-1}, \mathcal{I}_C(1)) \neq 0$. By Kodaira's vanishing theorem, we have $H^1(X, \mathcal{L}) = H^2(X, \mathcal{L}) = 0$. Thus, by the Riemann-Roch theorem, we get

$$N = h^0(X, \mathcal{L}) - 1 = \frac{1}{2}\mathcal{L}.(\mathcal{L} - K) + p_a.$$

This implies that

$$2N - 3 = \mathcal{L}(\mathcal{L} - K) + 2p_a - 3 = (K+L).L + 2p_a - 3 = L^2 + K.L + 2p_a - 3.$$

Together with (3.2) we have

$$2N - 3 - d = -K.L - K^2 + 2p_a - 3 \geq p,$$

i.e. $2N-3 \geq d+p$. It now follows from [2] that

$$\mathbf{H}_D(1) \geq \min\{d, N-1\} \geq d-(N-2)+p.$$

This and (3.3) give us

$$h^2(\mathbb{P}^{N-1}, \mathcal{I}_C) - h^2(\mathbb{P}^{N-1}, \mathcal{I}_C(1)) \leq (N-2)-p < N-2.$$

Thus,

$$h^1(\mathbb{P}^{N-1}, \mathcal{O}_C) - h^2(\mathbb{P}^{N-1}, \mathcal{O}_C(1)) < N-2.$$

By Serre-Grothendieck correspondence, we get

$$\dim_k \left[H^2_{\mathfrak{m}}(A)\right]_0 - \dim_k \left[H^2_{\mathfrak{m}}(A)\right]_1 < N-2, \tag{3.4}$$

where $\mathfrak{m}$ is the maximal homogeneous ideal of A.

Let x be a non-zero linear form in A. Since A is a domain, we have the following exact sequence

$$0 \longrightarrow A(-1) \xrightarrow{\times x} A \longrightarrow A/xA \longrightarrow 0.$$

This, by taking the corresponding long exact sequence of local cohomology and since $\dim A/xA = \dim A - 1 = 1$, gives an exact sequence

$$H^2_{\mathfrak{m}}(A)(-1) \xrightarrow{\times x} H^2_{\mathfrak{m}}(A) \longrightarrow 0.$$

This and a special case of [4, Lemma 3.1] (cf. [5, Lemma 3]) imply that

$$\dim_k \left[H^2_{\mathfrak{m}}(A)\right]_1 \leq \dim_k \left[H^2_{\mathfrak{m}}(A)\right]_0 - (N-2).$$

We obtain a contradiction to (3.4). The claim is proved. □

Now, we are back to the proof of our theorem. Having $h^2(\mathbb{P}^{N-1}, \mathcal{I}_C(1)) = 0$, (3.3) gives us

$$h^2(\mathbb{P}^{N-1}, \mathcal{I}_C) = h^1(\mathbb{P}^{N-2}, \mathcal{I}_D(1)) = d - \mathbf{H}_D(1) \leq (N-2)-p.$$

Let ω_A be the canonical module of A. Then,

$$\dim_k(\omega_A)_0 = \dim_k \left[H^2_{\mathfrak{m}}(A)\right]_0 = h^1(\mathbb{P}^{N-1}, \mathcal{O}_C) = h^2(\mathbb{P}^{N-1}, \mathcal{I}_C) \leq (N-2)-p.$$

By Lemma 3.1, ω_A is a torsion free A-module. Thus, by a vanishing theorem of Eisenbud and Koh [12, Theorem 1.1] (and its proof), ω_A has initial degree at least 0 and we have $\left[\operatorname{Tor}^T_s(\omega_A, k)\right]_s = 0$ for all $s \geq (N-2)-p$. Therefore,

$$\left[\operatorname{Tor}^T_i(\omega_A, k)\right]_j = 0 \ \forall\ j < i \text{ and } \left[\operatorname{Tor}^T_s(\omega_A, k)\right]_s = 0 \ \forall\ s \geq (N-2)-p.$$

By duality, we now obtain

$$\operatorname{reg} \mathcal{I} \leq 3 \text{ and } \left[\operatorname{Tor}^T_i(A, k)\right]_{i+2} = 0 \text{ for } i \leq p.$$

Hence,

$$\beta_{i,j} = 0 \text{ for } i \leq p \text{ and } j \geq i+2.$$

This implies that $\mathcal{L}$ has property N_p. The theorem is proved. □

Corollary 3.4. *Let X be a surface with $H^1(X, \mathcal{O}_X) = 0$. Suppose K is the canonical divisor and L is an ample divisor on X such that $\mathcal{L} = K \otimes L$ is generated by global sections. Then, for any integer $p \geq 0$, the adjoint line bundle $\mathcal{L}$ has property if $-K.L \geq p+3+K^2$.*

Proof. We only need to show that $\mathcal{L}$ is normally generated; the conclusion then follows from Theorem 3.2 since $p_a \geq 0$. We need to prove that the natural maps

$$H^0(X, \mathcal{L}^{\otimes r}) \otimes H^0(X, \mathcal{L}) \longrightarrow H^0(X, \mathcal{L}^{\otimes (r+1)})$$

are surjective for all $r \geq 1$.

As before, by Kodaira's vanishing theorem and from the hypotheses, we have

$$H^1(X, \mathcal{L}^{\otimes (r-1)}) = 0 \text{ for all } r \geq 1.$$

Let $C \in |\mathcal{L}|$ be a smooth and irreducible curve. By [19, Observation 1.1], it suffices to prove that the maps

$$H^0(C, \mathcal{L}^{\otimes r} \otimes \mathcal{O}_C) \otimes H^0(C, \mathcal{L} \otimes \mathcal{O}_C) \longrightarrow H^0(C, \mathcal{L}^{\otimes (r+1)} \otimes \mathcal{O}_C)$$

are surjective for all $r \geq 1$. Since $-K.L \geq 3 + K^2$, we have $-K.\mathcal{L} \geq 3$, and so by adjunction formula, $\deg(\mathcal{L} \otimes \mathcal{O}_C) \geq 2g(C)+1$. Hence, the assertion follows from Castelnuovo's theorem. □

As a consequence of Corollary 3.4, we can now extend [19, Theorem 1.23][1] to regular surfaces. Our argument is a combination of that of [19, Theorem 1.23] and the following lemma.

Lemma 3.5. *Let X be a regular anticanonical surface. Let K and L be the canonical divisor and an ample line bundle on X. Then,*

(1) *L is generated by global sections if $-K.L \geq 2$.*
(2) *L is normally generated if $-K.L \geq 3$.*

Proof. (1) Since X is anticanonical, there exists a smooth irreducible curve $D \in |-K|$. Consider the exact sequence

$$0 \longrightarrow K \otimes L \longrightarrow L \longrightarrow L \otimes \mathcal{O}_D \longrightarrow 0. \tag{3.5}$$

Since L is ample, by Kodaira's vanishing theorem, we have $H^2(X, K \otimes L) = 0$. This, by taking the long exact sequence of cohomology from (3.5), implies that $H^2(X, L) = 0$. Moreover, since $H^1(X, \mathcal{O}_X) = 0$, by the Riemann-Roch theorem, we have $p_a = \chi(\mathcal{O}_X) - 1 = h^2(X, \mathcal{O}_X) \geq 0$. Thus, since $L^2 > 0$ and $-K.L > 0$, by the Riemann-Roch theorem, we have

$$h^0(X, L) \geq \frac{1}{2} L.(L - K) + 1 + p_a > 0.$$

That is, $H^0(X, L) \neq 0$.

Let $C \in |L|$ be a smooth irreducible curve. Consider the exact sequence

$$0 \longrightarrow \mathcal{O}_X \longrightarrow L \longrightarrow L \otimes \mathcal{O}_C \longrightarrow 0. \tag{3.6}$$

[1] Notice that the statement of [19, Theorem 1.23] missed the condition X was rational even though the proof really made use of this fact.

Since $H^1(X, \mathcal{O}_X) = 0$, by taking the long exact sequence of cohomology from (3.6), we have a surjection

$$H^0(X, L) \longrightarrow\!\!\!\!\!\rightarrow H^0(C, L \otimes \mathcal{O}_C). \tag{3.7}$$

Since $-K.L \geq 2$, by adjunction formula, $\deg(L \otimes \mathcal{O}_C) \geq 2g(C)$. It follows from [10, Proposition 7, p. 59] that $L \otimes \mathcal{O}_C$ is generated by global sections. This and (3.7) now imply that L is generated by global sections.

(2) Let $r \geq 1$ be an integer. By (1), $L^{\otimes r}$ is generated by global sections. Thus, there exists a smooth irreducible curve $C_r \in |L^{\otimes r}|$. We have an exact sequence

$$0 \longrightarrow \mathcal{O}_X \longrightarrow L^{\otimes r} \longrightarrow L^{\otimes r} \otimes \mathcal{O}_{C_r} \longrightarrow 0. \tag{3.8}$$

Taking the long exact sequence of cohomology, we get

$$0 \longrightarrow H^1(X, L^{\otimes r}) \longrightarrow H^1(C_r, L^{\otimes r} \otimes \mathcal{O}_{C_r}).$$

Since $K.L^{\otimes r} = r(K.L) < 0$, by adjunction formula, $\deg(L^{\otimes r} \otimes \mathcal{O}_{C_r}) > 2g(C_r) - 2$. Thus, $H^1(C_r, L^{\otimes r} \otimes \mathcal{O}_{C_r}) = 0$. This and (3.8) imply that $H^1(X, L^{\otimes r}) = 0$. This is true for all $r \geq 1$. Together with the hypotheses, we have

$$H^1(X, L^{\otimes (r-1)}) = 0, \ \forall\ r \geq 1.$$

Now, by (1), L is generated by global sections. We, therefore, can make use of [19, Observation 1.1] in the same line of arguments as that of Corollary 3.4, to conclude that L is normally generated. □

Theorem 3.6. *Let X be a regular anticanonical surface. Let K be the canonical divisor and let $L_1, \ldots, L_m$ be ample line bundles on X. Then, for any integer $p \geq 0$, the adjoint line bundle $K \otimes L_1 \otimes \cdots \otimes L_m$ has property N_p if $n \geq 4 + p$.*

Proof. Let $L = L_1 \otimes \cdots \otimes L_m$. Since X is anticanonical, $-K$ is in the same class with an effective divisor. Thus, $-K.L_i \geq 1$ for all $i = 1, \ldots, m$. Without loss of generality, suppose that

$$s = -K.L_1 = \min\{-K.L_i \mid i = 1, \ldots, m\}.$$

Then, $s \geq 1$ and

$$-K.L = \sum_{i=1}^{m} -K.L_i \geq ms \geq (4 + p)s.$$

It follows from Reider's theorem [34] that $K \otimes L$ is very ample. In particular, $K \otimes L$ is generated by global sections. By Corollary 3.4, it suffices to show that for any integer $p \geq 0$,

$$(4 + p)s \geq p + 3 + K^2. \tag{3.9}$$

If $-K.(K \otimes L_1) \geq 0$ then $s \geq K^2$, and so $(4 + p)s = (3 + p)s + s \geq p + 3 + K^2$, i.e. (3.9) holds. Assume that $-K.(K \otimes L_1) < 0$. By Kodaira's vanishing theorem, we have $H^1(X, K \otimes L_1) = H^2(X, K \otimes L_1) = 0$. Thus, the Riemann-Roch theorem gives us

$$h^0(X, K \otimes L_1) = \frac{1}{2}(K \otimes L_1).L_1 + 1 + p_a. \tag{3.10}$$

Suppose $(K \otimes L_1).L_1 \geq 0$. By (3.10), $h^0(X, K \otimes L_1) > 0$, i.e. $K \otimes L_1$ is in the same class with an effective divisor. Thus, $-K.(K + L_1) \geq 0$, and we are done. From now on, we shall assume that

$$-K.(K \otimes L_1) < 0 \text{ and } (K \otimes L_1).L_1 < 0. \tag{3.11}$$

If $K^2 \leq 1$, then (3.9) holds since $s \geq 1$. Suppose now that $K^2 \geq 2$. Since L_1 is ample, $L_1^2 > 0$. We will consider different cases based on the value of L_1^2.

Assume first that $L_1^2 \geq 5$. By Hodge Index inequality, we must have $-K.L_1 > 3$. By Lemma 3.5, L_1 is normally generated. This, in particular, implies that L_1 is very ample. By Reider's theorem [34], $K \otimes L_1$ is generated by global sections unless there exists a reduced irreducible curve E such that $L_1.E = 1$ and $E^2 = 0$. If $K \otimes L_1$ is generated by global sections, then $h^0(X, K \otimes L_1) > 0$, i.e. $K \otimes L_1$ is in the same class with an effective divisor, and so we have $-K.(K + L_1) \geq 0$. This is a contradiction to the assumption (3.11). If there exists a reduced and irreducible curve E such that $L_1.E = 1$ and $E^2 = 0$, then since L_1 is very ample, $E = \mathbb{P}^1$. It follows from [3, Lemma 4.1.10] that X is a $\mathbb{P}^1$-bundle over a curve of genus 0. This implies that $K^2 = 8$. We now have $(4+p)s > (4+p)3 \geq p+12 > p+3+K^2$. (3.9) holds.

Assume that $L_1^2 = 4$. By Hodge Index inequality again, we have $-K.L_1 \geq 3$. As before, by Lemma 3.5, this implies that L_1 is normally generated. Thus, L_1 embeds X as a projectively normal surface. However, the only normal smooth surfaces of degree 4 are K3 surfaces in $\mathbb{P}^3$, the Del Pezzo surface in $\mathbb{P}^4$, and the Veronese surface and the normal scroll in $\mathbb{P}^5$ (cf. [26, pp. 145, 173]). If X is embedded as a K3 surface then $K^2 = 0$, a contradiction to the assumption that $K^2 \geq 2$. If X is embedded as a Del Pezzo surface by L_1 then $K = -L_1$, and so $-K(K \otimes L_1) = 0$, a contradiction to (3.11). If X is embedded as a Veronese surface then $K^2 = 9$, and we establish (3.9) since $(4 + p)s \geq (4 + p)3 \geq p + 12 = p + 3 + K^2$. If X is embedded as a normal scroll in $\mathbb{P}^5$ then $K^2 = 8$, and again we establish (3.9).

Assume that $L_1^2 = 3$. By Hodge Index inequality, we again have $-K.L_1 \geq 3$, and so L_1 is normally generated. This implies that X is either a Del Pezzo cubic surface in $\mathbb{P}^3$ or a normal scroll in $\mathbb{P}^4$ (cf. [26, pp. 145, 173]). As before, we either arrive with a contradiction to (3.11) or establish (3.9).

Assume that $L_1^2 = 2$. By Hodge Index inequality, we have $-K.L_1 \geq 2$. If $-K.L_1 = 2$ then $(K \otimes L_1).L_1 = 0$, a contradiction to (3.11). Suppose $-K.L_1 \geq 3$. By Lemma 3.5 again, L_1 is normally generated. In particular, L_1 is very ample. This implies that $(X, L_1) = (\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1))$ (cf. [26, pp. 145, 173]). Thus, $K^2 = 8$ and we establish (3.9).

Assume finally that $L_1^2 = 1$. As before, by Kodaira's vanishing theorem and the Riemann-Roch theorem, we have $h^0(X, K \otimes L_1) = \frac{1}{2}L_1.(K \otimes L_1) + 1 + p_a$. Thus, $L_1.(K \otimes L_1)$ is an even number, i.e. $K.L_1$ is odd. By Hodge Index inequality, $-K.L_1 \geq 2$. Thus, $-K.L_1 \geq 3$. This, once again, implies that L_1 is very ample. Since $L_1^2 = 1$, we must have $(X, L_1) = (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1))$ (cf. [26, pp. 145, 173]). In this case $K^2 = 9$, and we establish (3.9).

The theorem is proved. □

Theorem 3.6 establishes Mukai's conjecture for regular anticanonical surfaces.

Corollary 3.7. *Let X be a regular anticanonical surface. Let K be the canonical divisor and let L be an ample line bundle on X. Then, for any integer $p \geq 0$, the adjoint line bundle $K \otimes L^{\otimes(4+p)}$ has property N_p.*

Proof. The assertion follows from Theorem 3.6 by taking $L_1 = \cdots = L_m = L$. □

4. Regularity of ideal sheaves of embeddings

In this section, we study width of the minimal free resolution of embeddings of a regular projective variety given by adjoint line bundles. This is done via examining the regularity of ideal sheaves of these embeddings. The reader is again referred to Definition 2.1 for the notion of regular projective varieties.

Our main result in this section is stated in the next theorem. Parts (1) and (2) follow straight forward from a standard use of Kodaira's vanishing theorem. Part (3) is what we are interested in.

Theorem 4.1. *Let X be a smooth projective variety of dimension n. Suppose K is the canonical divisor and L is an ample line bundle on X such that $\mathcal{L} = K \otimes L$ is normally generated. Let $\mathcal{I}_{\mathcal{L}}$ be the ideal sheaf of the embedding of X given by $\mathcal{L}$. Then,*

(1) $\operatorname{reg} \mathcal{I}_{\mathcal{L}} \leq n+2$. *Moreover, if $H^0(X,K) \neq 0$, then* $\operatorname{reg} \mathcal{I}_{\mathcal{L}} = n+2$.
(2) *If $H^0(X,K) = 0$, then* $\operatorname{reg} \mathcal{I}_{\mathcal{L}} \leq n+1$.
(3) *If X is regular, $H^0(X,K) = 0$ and $H^0(X, K\otimes\mathcal{L}) \neq 0$, then* $\operatorname{reg} \mathcal{I}_{\mathcal{L}} = n+1$.

Proof. (1) Since $\mathcal{L}$ is normally generated, $\mathcal{L}$ is necessarily very ample (cf. [30]). Let $N = \dim_k H^0(X,\mathcal{L}) - 1$, and let $X_{\mathcal{L}}$ be the image of X embedded in $\mathbb{P}^N$ by $\mathcal{L}$. Let S and I be the coordinate ring and the defining ideal of $X_{\mathcal{L}}$ in $\mathbb{P}^N$. We have the following exact sequence

$$0 \longrightarrow I \longrightarrow S \longrightarrow \oplus_{h\geq 0} H^0(X, \mathcal{L}^{\otimes h}) \longrightarrow \oplus_{h\geq 0} H^1(\mathbb{P}^N, \mathcal{I}_{\mathcal{L}}(h)) \longrightarrow 0.$$

Since $\mathcal{L}$ is normally generated, $\mathcal{L}$ embeds X as a projectively normal variety. Thus, we have

$$S/I \simeq \oplus_{h\geq 0} H^0(X, \mathcal{L}^{\otimes h}) \text{ and } H^1(\mathbb{P}^N, \mathcal{I}_{\mathcal{L}}(h)) = 0 \ \forall\ h \geq 0. \tag{4.1}$$

By taking the long exact sequence of sheaf cohomology associated to the exact sequence $0 \longrightarrow \mathcal{I}_{\mathcal{L}} \longrightarrow \mathbb{P}^N \longrightarrow \mathcal{O}_{X_{\mathcal{L}}} \longrightarrow 0$, we obtain

$$H^i(\mathbb{P}^N, \mathcal{I}_{\mathcal{L}}(h)) = H^{i-1}(\mathbb{P}^N, \mathcal{O}_{X_{\mathcal{L}}}(h)) \text{ for all } i \geq 2 \text{ and } h \geq -N. \tag{4.2}$$

For $h > 0$, we have $\mathcal{L}^{\otimes h} = K \otimes \mathcal{L}^{\otimes(h-1)} \otimes L$. Since L and $\mathcal{L}$ are ample, by Kodaira's vanishing theorem, we get

$$H^i(X, \mathcal{L}^{\otimes h}) = 0,\ \forall\ i \geq 1, \text{ and } h > 0. \tag{4.3}$$

Thus, $H^i(\mathbb{P}^N, \mathcal{O}_{X_{\mathcal{L}}}(n+1-i)) = H^i(X, \mathcal{L}^{\otimes(n+1-i)}) = 0$ for $1 \leq i \leq n$. Clearly, $H^i(\mathbb{P}^N, \mathcal{O}_{X_{\mathcal{L}}}(n+1-i)) = 0$ for $i > n = \dim X_{\mathcal{L}}$. Hence, (4.2) gives us

$$H^i(\mathbb{P}^N, \mathcal{I}_{\mathcal{L}}(n+2-i)) = 0 \text{ for all } 2 \leq i \leq N.$$

This and (4.1) prove that

$$\operatorname{reg} \mathcal{I}_{\mathcal{L}} \leq n+2.$$

Now, if $H^0(X,K) \neq 0$, then by Serre's duality, we have $H^n(X,\mathcal{O}_X) = H^0(X,K) \neq 0$. Thus, by (4.2),

$$H^{n+1}(\mathbb{P}^N, \mathcal{I}_{\mathcal{L}}) = H^n(\mathbb{P}^N, \mathcal{O}_{X_{\mathcal{L}}}) = H^n(X,\mathcal{O}_X) \neq 0.$$

This implies that $\operatorname{reg} \mathcal{I}_{\mathcal{L}} \geq n+2$. Hence, $\operatorname{reg} \mathcal{I}_{\mathcal{L}} = n+2$.

(2) By Serre's duality, we have $H^n(\mathbb{P}^N, \mathcal{O}_{X_{\mathcal{L}}}) = H^n(X,\mathcal{O}_X) = H^0(X,K) = 0$. This and (4.3) imply that $H^i(\mathbb{P}^N, \mathcal{O}_{X_{\mathcal{L}}}(n-i)) = H^i(X, \mathcal{L}^{\otimes(n-i)}) = 0$ for all $1 \leq i \leq n$. It's again clear that $H^i(\mathbb{P}^N, \mathcal{O}_{X_{\mathcal{L}}}(n-i)) = 0$ for $i > n = \dim X_{\mathcal{L}}$. It now follows from (4.2) that

$$H^i(\mathbb{P}^N, \mathcal{I}_{\mathcal{L}}(n+1-i)) = 0 \text{ for all } 2 \leq i \leq N. \tag{4.4}$$

The conclusion follows from (4.1) and (4.4).

(3) Suppose that X is regular, $H^0(X,K) = 0$ and $H^0(X, K\otimes\mathcal{L}) \neq 0$. By (2), $\operatorname{reg} \mathcal{I}_{\mathcal{L}} \leq n+1$. We need to show that

$$\operatorname{reg} \mathcal{I}_{\mathcal{L}} \geq n+1. \tag{4.5}$$

By (4.1), $S/I \simeq \oplus_{h\geq 0} H^0(X, \mathcal{L}^{\otimes h})$. Thus, by Green's Syzygy theorem [21, 1.b.4], S/I admits a minimal free resolution

$$0 \longrightarrow F_s \longrightarrow \ldots \longrightarrow F_1 \longrightarrow F_0 = S \longrightarrow S/I \longrightarrow 0,$$

for some $s \geq N-n$ (the codimension of $X_{\mathcal{L}}$ in $\mathbb{P}^N$) and

$$F_i = \bigoplus_{q\geq 1} \mathcal{K}_{i,q}(\mathcal{L}) \otimes S(-i-q), \text{ for } i = 1, \ldots, s,$$

where $\mathcal{K}_{i,q}(\mathcal{L})$ is the Koszul cohomology group associated to $\mathcal{L}$ as in Definition 2.6. It suffices to show that there exists an integer $p \geq 0$ such that

$$\mathcal{K}_{p,n}(\mathcal{L}) \neq 0.$$

As in (4.3), we have $H^i(X, \mathcal{L}^{\otimes h}) = 0$ for $i \geq 1$ and $h > 0$. This, together with the hypothesis that X is regular, gives us

$$H^i(X, \mathcal{L}^{\otimes(n-i)}) = H^i(X, \mathcal{L}^{\otimes(n-i-1)}) = 0 \text{ for } i = 1, \ldots, n-1.$$

Therefore, Green's Duality theorem [21, 2.c.6] applies for $p \geq 0$ and $q = n$, and gives

$$\mathcal{K}_{N-n,n}(\mathcal{L})^* \simeq \mathcal{K}_{0,1}(K,\mathcal{L}).$$

We shall prove that $\mathcal{K}_{0,1}(K,\mathcal{L}) \neq 0$ under the given hypotheses. Indeed, let $W = H^0(X,\mathcal{L})$. The Koszul complex of $\oplus_{h\geq 0} H^0(X, K\otimes\mathcal{L}^{\otimes h})$ at degree $(0,1)$ is

$$\bigwedge W \otimes H^0(X,K) \xrightarrow{d_{1,0}} W \otimes H^0(X, K\otimes\mathcal{L}) \xrightarrow{d_{0,1}} 0.$$

Hence, since $H^0(X,K) = 0$ and $H^0(X, K\otimes\mathcal{L}) \neq 0$,

$$\mathcal{K}_{0,1}(K,\mathcal{L}) = W \otimes H^0(X, K\otimes\mathcal{L}) \neq 0.$$

The theorem is proved. □

Theorem 4.1, combined with Fujita's freeness and very ampleness conjectures (cf. [13, 27, 37]), gives rise to several interesting corollaries. It is well known that Fujita's freeness and very ampleness conjectures hold for ample line bundles which are generated by global sections. For completeness, we shall still include the proof in the next proposition.

Proposition 4.2. *Let X be a smooth projective variety of dimension n with canonical divisor K. Let L be an ample line bundle which is generated by global sections. Then, $K \otimes L^{\otimes m}$ is generated by global sections for $m \geq n+1$, and is very ample for $m \geq n+2$.*

Proof. Let $\mathcal{L}_m = K \otimes L^{\otimes m}$ for $m \geq 0$. For $h \geq n+1$, we have $\mathcal{L}_h \otimes L^{\otimes(-i)} = K \otimes L^{\otimes(h-i)}$. It follows from Kodaira's vanishing theorem that

$$H^i(X, \mathcal{L}_h \otimes L^{\otimes(-i)}) = 0$$

for $i = 1, \ldots, n$. Thus, $\mathcal{L}_h$ is 0-regular in the sense of Castelnuovo-Mumford with respect to L. Since L is generated by global sections, this implies (cf. [29, Theorem 1.8.5 and Example 1.8.22]) that $\mathcal{L}_h$ is generated by global sections and $\mathcal{L}_h \otimes L$ is very ample. In other words, $\mathcal{L}_m$ is generated by global sections for $m \geq n+1$, and is very ample for $m \geq n+2$. □

Corollary 4.3. *Let X be a smooth regular projective variety of dimension n. Let K be the canonical divisor and let L be a very ample line bundle on X. Let $\mathcal{L}_m = K \otimes L^{\otimes m}$, and let $\mathcal{I}_m$ be the ideal sheaf of the embedding of X given by $\mathcal{L}_m$ for $m \geq n+2$. The following statements hold.*

(1) *If $H^0(X, K) \neq 0$, then $\operatorname{reg} \mathcal{I}_m = n+2$ for $m \geq n+2$.*
(2) *If $H^0(X, K) = 0$, then $\operatorname{reg} \mathcal{I}_m = n+1$ for all $m \geq 2n+2$.*

Proof. By [11, Theorem 1], $\mathcal{L}_m$ is normally generated for $m \geq n+2$. The first conclusion follows from Theorem 4.1. Observe further that by Proposition 4.2 for $m \geq 2n+2$, $\mathcal{L}_{n+1}$ and $\mathcal{L}_{m-n-1}$ are generated by global sections, and so $K \otimes \mathcal{L}_m = \mathcal{L}_{n+1} \otimes \mathcal{L}_{m-n-1}$ is generated by global sections. Thus, $H^0(K \otimes \mathcal{L}_m) \neq 0$ for $m \geq 2n+2$. The second assertion is also a consequence of Theorem 4.1. □

Remark 4.4. It is worth noting that having the same hypotheses as in Corollary 4.3, by [11, Theorem 1] we know that for $m \geq 2n+2$, $\mathcal{L}_m$ has property N_{m-n-2}. This and Corollary 4.3 say that as m gets large, while the beginning part of the minimal free resolution of $\mathcal{I}_m$ becomes more linear, the width of the resolution is always fixed, either $n+2$ or $n+1$.

Corollary 4.5. *Let X be a smooth rational projective variety of dimension n, and let K be the canonical divisor of X. Suppose L is a very ample line bundle on X. Let $\mathcal{L}_m = K \otimes L^{\otimes m}$, and let $\mathcal{I}_m$ be the ideal sheaf of the embedding of X given by $\mathcal{L}_m$ for $m \geq n+2$. Then,*

$$\operatorname{reg} \mathcal{I}_m = n+1 \text{ for } m \geq 2n+2.$$

Proof. Since X is rational, X is obtained from $\mathbb{P}^n$ by blowing up along an ideal sheaf. Let $\pi : X \longrightarrow \mathbb{P}^n$ be the blowing up along an ideal sheaf $\mathcal{I}$ associated to an ideal $I \subset k[x_0, \ldots, x_n]$.

Let E_0 and E be the pull-back of a general hyperplane in $\mathbb{P}^n$ and the exceptional divisor of π. Let r be the height of I. We have

$$K = -(n+1)E_0 + (r-1)E.$$

Thus, $H^0(X, K) = 0$. Moreover, it follows from Remark 2.2 that X is regular. Hence, the conclusion follows from Corollary 4.3. □

Remark 4.6. For surfaces, making use of Reider's theorem [34] and results in a series of papers of Gallego and Purnaprajna [14, 15, 16, 17, 18, 19], we can relax the condition that L is very ample in Corollary 4.3. Yet, the bounds obtained would be slightly weaker since Mukai's conjecture for projective normality (property N_0) is not completely known to hold.

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