PROJECTIVE EMBEDDINGS OF PROJECTIVE SCHEMES BLOWN UP AT SUBSCHEMES

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Abstract. Suppose X is a nonsingular projective scheme, Z a nonsingular closed subscheme of X. Let $\tilde{X}$ be the blowup of X centered at Z, E_0 the pull-back of a general hyperplane in X, and E the exceptional divisor. In this paper, we study projective embeddings of $\tilde{X}$ given by divisors $\mathcal{D}_{e,t} = tE_0 - eE$. When X satisfies a necessary condition, we give explicit values of d and δ such that for all $e > 0$ and $t > ed + \delta$, $\mathcal{D}_{e,t}$ embeds $\tilde{X}$ as a projectively normal and arithmetically Cohen-Macaulay scheme. We also give a uniform bound for the regularities of the ideal sheaves of these embeddings, and study their asymptotic behaviour as t gets large compared to e. When X is a surface and Z is a 0-dimensional subscheme, we further show that these embeddings possess property N_p for all $t \gg e > 0$.

Dedicated to the sixtieth birthday of Prof. A.V. Geramita

0. Introduction

Let R be a finitely generated standard graded k-algebra of dimension $(n+1)$, and $X = \text{Proj } R$ a nonsingular projective scheme ($\dim X = n$). Suppose Z is a nonsingular closed subscheme of X, and $I \subseteq R$ its defining ideal (I saturated). Let $\pi : \tilde{X} \to X$ be the blowup of X centered at Z. For positive integers $e, t \in \mathbb{N}$, consider the divisor $\mathcal{D}_{e,t} = tE_0 - eE$ on $\tilde{X}$, where E_0 is the pull-back to $\tilde{X}$ of a general hyperplane in X and E is the exceptional divisor of the blowing up. $\mathcal{D}_{e,t}$ corresponds to the linear system of hypersurfaces of degree t containing Z with multiplicity at least e. For $t \gg e > 0$, it is known (cf. [3, Lemma 1.1]) that $\mathcal{D}_{e,t}$ is very ample on $\tilde{X}$. This gives a projective embedding $\tilde{X} \hookrightarrow \mathbb{P}^{N_{e,t}}$. Let $\tilde{X}_{e,t}$ be the image of $\tilde{X}$ in this embedding (i.e. $\tilde{X} \xrightarrow{\sim} \tilde{X}_{e,t} \subseteq \mathbb{P}^{N_{e,t}}$). In the last fifteen years, there have been many studies on these projective embeddings of $\tilde{X}$ in various situations depending upon the scheme X, its subscheme Z, and the values of e and t. For instance, [3, 8, 9, 10, 11, 12, 15, 16, 18, 19]. This line of works also has a close relation with the studies of diagonal subalgebras of a bi-graded algebra, such as in [2, 24, 26, 27]. In this paper, we push the study on projective embeddings of blowup schemes a step forward. We

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will investigate the arithmetically Cohen-Macaulayness, the regularity and syzygies of $\tilde{X}_{e,t}$. We further examine how these properties behave asymptotically (i.e. when $t \gg e > 0$).

It was shown by [3] (see also [26]) that, when R is Cohen-Macaulay, under certain conditions, there exists an f such that for all $e > 0$ and $t > fe$, the embedding $\tilde{X}_{e,t}$ of $\tilde{X}$ is arithmetically Cohen-Macaulay, i.e. it has a Cohen-Macaulay coordinate ring. However, no explicit bound for f could be found unless one is in special situations, for example if it is known that the Rees algebra $\mathcal{R}(I)$ of I is Cohen-Macaulay (see [26]), or when char $k = 0$, $X = \mathbb{P}^n$, Z is a scheme of fat points in X and $e = 1$ (see [10, Theorem 2.4]). In the first section of this paper, we address this problem again, and generalize the method of [10] to give an explicit bound for f for any X and Z, when the characteristic of k is 0. More precisely, we give a constant $\delta = \delta(I) > 0$ such that for all $e > 0$ and $t > ed(I) + \delta$ (where $d(I)$ is the maximum degree of a minimal system of homogeneous generators of I), the projective embedding $\tilde{X}_{e,t}$ of $\tilde{X}$ is arithmetically Cohen-Macaulay (Theorem 1.4). An explicit bound for f then can be taken to be $d(I) + \delta(I)$. We also replace the condition on the Cohen-Macaulayness of R (which has been the case in most of previous works) by a slightly weaker condition that $H^i(X, \mathcal{O}_X) = 0$ for $i = 1, \ldots, n-1$. This turns out to be a necessary condition for the existence of any arithmetically Cohen-Macaulay $\tilde{X}_{e,t}$ (Remark 1.5). Our result is stated as follows.

Theorem 0.1 (Theorem 1.4)**.** *Suppose $H^i(X, \mathcal{O}_X) = 0$ for $i = 1, \ldots, n-1$. Then, for $t > ed(I) + \delta$, the projective embedding $\tilde{X}_{e,t}$ of $\tilde{X}$ is arithmetically Cohen-Macaulay.*

In the second section of this paper, we study the regularity of the ideal sheaf and syzygies of $\tilde{X}_{e,t}$. Let $\mathcal{J}_{e,t}$ denote the ideal sheaf of $\tilde{X}_{e,t}$. We prove that, under a mild condition, the regularity of $\mathcal{J}_{e,t}$ is always bounded above by the dimension of R, when $t > ed(I) + \delta$ (Theorem 2.2). This gives a uniform bound on the shifts of the minimal free resolution of $\tilde{X}_{e,t}$. A very interesting asymptotic behaviour is that, with the additional condition $H^i(X, \mathcal{O}_X) = 0$ for $i = 1, \ldots, n-1$, the regularity of $\mathcal{J}_{e,t}$ stabilizes as t gets large compared to e. We also prove this fact in Theorem 2.2.

Theorem 0.2 (Theorem 2.2)**.** *Suppose $H^0(\tilde{X}, w_{\tilde{X}}) = 0$, where $w_{\tilde{X}}$ is the dualizing sheaf on $\tilde{X}$. Then,*

$$\text{reg } \mathcal{J}_{e,t} \leq n+1, \ \forall \ t > ed(I) + \delta.$$

If, in addition, $H^i(X, \mathcal{O}_X) = 0$ for $i = 1, \ldots, n-1$, then for all $t \gg e$,

$$\text{reg } \mathcal{J}_{e,t} = n+1.$$

We further restrict our attention to the case when X is a surface and Z is a 0-dimensional subscheme of X, and show that for any $p \in \mathbb{N}$, the embedding $\tilde{X}_{e,t}$ possesses property N_p for all $t \gg e > 0$ (Theorem 2.5). This is a property, introduced by Green and Lazarsfeld ([13, 14]), that encodes a lot of information about the scheme.

Results in this paper exhibit and support the philosophy that when embedding a variety into large projective spaces, it seems to have nicer algebraic properties, and its algebraic invariants seem to stabilize as the dimension of the ambient space gets large. This philosophy is the guideline of many recent studies since the impact of works of Mumford [21], its amplification by St. Donat [25], and the strong theorems and conjectures of Green [13] and Green and Lazarsfeld [14].

Throughout this paper, our base field k will be algebraically closed and of characteristic 0.

1. Cohen-Macaulayness

Let us first briefly recall some notations and terminology. Unexplained terminology follow from that of [6] and [17]. Suppose $Y = \text{Proj } T \subseteq \mathbb{P}^l$ is a projective scheme. Y is said to be *arithmetically Cohen-Macaulay* if T is a Cohen-Macaulay ring. Suppose $\mathcal{J}$ is the ideal sheaf of Y in $\mathbb{P}^l$. The *regularity* of $\mathcal{J}$, denoted by reg $\mathcal{J}$, is defined to be the smallest integer r' such that $H^i(\mathbb{P}^l, \mathcal{J}(r'-i)) = 0$ for all $i > 0$. The regularity can also be interpreted in terms of the minimal free resolution as follows. Let

$$0 \to \bigoplus_j \mathcal{O}_{\mathbb{P}^l}(-b_{sj}) \to \ldots \to \bigoplus_j \mathcal{O}_{\mathbb{P}^l}(-b_{1j}) \to \bigoplus_j \mathcal{O}_{\mathbb{P}^l}(-b_{0j}) \to \mathcal{J} \to 0$$

be the minimal free resolution of $\mathcal{J}$, then reg $\mathcal{J} = \max\{b_{ij} - i \mid b_{ij} \neq 0\}$.

Back to our setup, let R be a finitely generated standard graded k-algebra of dimension (n+1), $X = \text{Proj } R$ a nonsingular projective scheme. Let Z be a nonsingular closed subscheme of X defined by a homogeneous ideal $I \subseteq R$ of positive height (I saturated). Let $\mathcal{I}$ be the associated sheaf of I on X, and let $\pi : \tilde{X} = \text{Proj } (\oplus_{n \geq 0} \mathcal{I}^n) \to X$ be the blowup of X centered at Z. Let E be the exceptional divisor of this blowing up, and E_0 the pull-back to $\tilde{X}$ of a general hyperplane in X. For each pair of positive integers e and t, consider the divisor $\mathcal{D}_{e,t} = tE_0 - eE$ on $\tilde{X}$. Suppose I is generated in degrees at most $d(I)$. By [3, Lemma 1.1], for each $e > 0$ and $t \geq ed(I) + 1$, the divisor $\mathcal{D}_{e,t}$ is very ample on $\tilde{X}$, so from now on, we only consider values of $e > 0$ and t such that $t \geq ed(I) + 1$. For such values of e and t, let $\tilde{X} \xrightarrow{\sim} \tilde{X}_{e,t} \subseteq \mathbb{P}^{N_{e,t}}$ be the projective embedding of $\tilde{X}$ given by $\mathcal{D}_{e,t}$, where $N_{e,t} = \dim_k H^0(\tilde{X}, \mathcal{D}_{e,t}) - 1$. We also denote by r the height of I.

Now, suppose R is generated as a k-algebra by $x_1, \ldots, x_g \in R_1$, and I is generated by $f_1, \ldots, f_m \in R$, where the degree of f_j is d_j for all $j = 1, \ldots, m$. Let $\mathcal{R}(I) = \oplus_{n \geq 0} I^n t^n$ be the Rees algebra of I. Let $S = k[X_1, \ldots, X_g, Y_1, \ldots, Y_m]$ be a polynomial ring, then S has a natural bi-gradation (induced by $\deg X_i = (1, 0),\ \forall i = 1, \ldots, g$, and $\deg Y_j = (d_j, 1),\ \forall j = 1, \ldots, m$). Consider the k-algebra homomorphism

$$S \to \mathcal{R}(I),$$

given by sending X_i to x_i for all $i = 1, \ldots, g$, and Y_j to $f_j t$ for all $j = 1, \ldots, m$. It is clear that this homomorphism is surjective and gives $\mathcal{R}(I)$ the structure of a bi-graded module over S. Suppose

$$0 \to \ldots \to \oplus_j S(-a_{ij}, -b_{ij}) \to \ldots \to \oplus_j S(-a_{0j}, -b_{0j}) \to \mathcal{R}(I) \to 0$$

is the bi-graded minimal free resolution of $\mathcal{R}(I)$ over S. This minimal free resolution exists and is finite due to Hilbert's syzygy theorem. Let

$$c_i = \max_j \{a_{ij} - b_{ij} d(I)\}.$$

Clearly, $\max_i\{c_i - i\}$ is an invariant of I. We denote this invariant by $c(I)$. Define

$$\delta = \delta(I) = \max\{c(I), 1\}.$$

A straight forward adaptation of [4, Theorem 2.4 and Proposition 4.1] gives us the following result.

Lemma 1.1. *With $\delta = \delta(I)$ defined as above, for any natural number l, we have*

$$\text{reg } \mathcal{I}^l \leq \text{reg } I^l \leq l d(I) + \delta.$$

The following proposition is an extension of [10, Proposition 2.2].

Proposition 1.2. *For $t > ed(I) + \delta$, the projective embedding $\tilde{X}_{e,t}$ of $\tilde{X}$ is projectively normal.*

Proof. Suppose $t > ed(I) + \delta$. By [3, Lemma 1.1], $\mathcal{D}_{e,t}$ is very ample on $\tilde{X}$. Let $S_{e,t} = \text{Sym}^*(H^0(\tilde{X}, \mathcal{D}_{e,t}))$ be the coordinate ring of $\mathbb{P}^N = \mathbb{P}^{N_{e,t}}$ into which $\tilde{X}$ is embedded by $\mathcal{D}_{e,t}$. Let $I_{e,t} \subseteq S_{e,t}$ be the defining ideal of $\tilde{X}_{e,t}$, and $\mathcal{J}_{e,t}$ its associated ideal sheaf in $\mathbb{P}^N$. From now on, when working with $\tilde{X}_{e,t}$, we shall always use these notations of $S_{e,t}, I_{e,t}, \mathcal{J}_{e,t}$ and $N = N_{e,t}$. One has the following exact sequence

$$0 \to I_{e,t} \to S_{e,t} \to \oplus_{h \geq 0} H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}(h\mathcal{D}_{e,t})) \to \oplus_{h \geq 0} H^1(\mathbb{P}^N, \mathcal{J}_{e,t}(h)) \to 0.$$

To prove $\tilde{X}_{e,t}$ is projectively normal, it is enough to show that for each $h \geq 0$, there is an isomorphism of k-vector spaces

$$\left(\frac{S_{e,t}}{I_{e,t}}\right)_h \simeq H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}(h\mathcal{D}_{e,t})),$$

where $\left(\frac{S_{e,t}}{I_{e,t}}\right)_h$ is the k-vector space of monomials of degree h in $\frac{S_{e,t}}{I_{e,t}}$. This is clear for $h = 0$. Suppose $h \geq 1$. It can be observed that a hyperplane in $\mathbb{P}^N$ when restricted to $\tilde{X}_{e,t}$ pulls back to a hypersurface of degree t in X containing Z with multiplicity at least e. Thus,

$$\left(\frac{S_{e,t}}{I_{e,t}}\right)_h \simeq [(I_t^e)^h]_{ht} = [(I^e)^h]_{ht} = (I^{he})_{ht}, \tag{1.1}$$

where $[(I_t^e)^h]_{ht}$ is the k-vector space of monomials of degree ht in the h-th power of the ideal generated by I_t^e.

Let $\mathcal{M} = \mathcal{I}.\mathcal{O}_{\tilde{X}} = \mathcal{O}_{\tilde{X}}(-E)$, then

$$\mathcal{O}_{\tilde{X}}(h\mathcal{D}_{e,t}) = \mathcal{M}^{he} \otimes \mathcal{O}_{\tilde{X}}(htE_0) = \mathcal{M}^{he} \otimes \pi^*\mathcal{O}_X(ht).$$

By [20, Proposition 10.2], we have $\pi_*\mathcal{M}^{he} = \mathcal{I}^{he}$. Hence, by the projection formula, we get

$$\pi_*\mathcal{O}_{\tilde{X}}(h\mathcal{D}_{e,t}) = \pi_*(\mathcal{M}^{he}) \otimes \mathcal{O}_X(ht) = \mathcal{I}^{he}(ht).$$

Thus,

$$H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}(h\mathcal{D}_{e,t})) = H^0(X, \pi_*\mathcal{O}_{\tilde{X}}(h\mathcal{D}_{e,t})) = H^0(X, \mathcal{I}^{he}(ht)) = \dim_k I_{ht}^{(he)}, \tag{1.2}$$

where $I^{(he)}$ is the saturation ideal of I^{he}.

It follows from Lemma 1.1 that

$$\text{reg } I^l \leq ld(I) + \delta \text{ for any } l \geq 1.$$

Thus, for $t > ed(I) + \delta$, we have

$$ht > hed(I) + h\delta \geq hed(I) + \delta \geq \text{reg } I^{he} \geq \text{sat } (I^{he}),$$

where sat (I^{he}) is the saturation degree of I^{he} (i.e. the least degree starting from which I^{he} and its saturation ideal $I^{(he)}$ agree). Therefore, for $t > ed(I) + \delta$, we have

$$\dim_k [I^{he}]_{ht} = \dim_k I_{ht}^{(he)}.$$

This, together with (1.1) and (1.2), proves the proposition. □

The following lemma plays an important role in the study of arithmetically Cohen-Macaulayness of $\tilde{X}_{e,t}$.

Lemma 1.3. *For $e > 0, t \geq ed(I) + 1$, we have*

(1) $R^j\pi_*\mathcal{O}_{\tilde{X}}(h\mathcal{D}_{e,t}) = 0$ *for all* $j > 0$ *and* $h \ge 0$.
(2) $H^i(\tilde{X}, \mathcal{O}_{\tilde{X}}(h\mathcal{D}_{e,t})) = H^i(X, \mathcal{I}^{he}(ht))$ *for all* $i \ge 0$ *and* $h \ge 0$.

Here, $\mathcal{O}_{\tilde{X}}$ *is the structure sheaf of* $\tilde{X}$.

Proof. Again, we let $\mathcal{M} = \mathcal{I}\mathcal{O}_{\tilde{X}} = \mathcal{O}_{\tilde{X}}(-E)$.

(1) For each $l \ge 0$, we have

$$\mathcal{O}_{\tilde{X}}(h\mathcal{D}_{e,t}) \otimes \pi^*\mathcal{O}_X(l) = \mathcal{O}_{\tilde{X}}(h(tE_0 - eE)) \otimes \pi^*\mathcal{O}_X(l) = \mathcal{M}^{he} \otimes \pi^*\mathcal{O}_X(ht + l).$$

Now we will use the argument of [3, Proposition 1.5] to get

$$H^i(\tilde{X}, \mathcal{O}_{\tilde{X}}(h\mathcal{D}_{e,t}) \otimes \pi^*\mathcal{O}_X(l)) = 0, \forall i > 0, l \gg 0.$$

Indeed, let $w_{\tilde{X}}$ and w_X be the dualizing sheaves on $\tilde{X}$ and X, respectively. Since the height of I is r, $w_{\tilde{X}} = \pi^*w_X \otimes \mathcal{M}^{1-r}$. Thus,

$$\mathcal{M}^{he} \otimes \pi^*\mathcal{O}_X(ht + l) = \mathcal{M}^{he+r-1} \otimes \pi^*\mathcal{O}_X(ht + l) \otimes (\pi^*w_X)^{-1} \otimes w_{\tilde{X}}.$$

Observe that for any $c, d \in \mathbb{N}$, $\mathcal{M}^d \otimes \pi^*\mathcal{O}_X(c) = I_c^d.\mathcal{O}_{\tilde{X}}$. It is known that $w_X^{-1}(g)$ is very ample on X for some $g > 0$. Thus, by [3, Lemma 1.1], for $l > (he + r - 1)d(I) + g - ht$ (i.e. $ht + l > (he + r - 1)d(I) + g$), the divisor $\mathcal{M}^{he+r-1} \otimes \pi^*\mathcal{O}_X(ht + l) \otimes (\pi^*w_X)^{-1}$ is very ample on $\tilde{X}$. By Kodaira's vanishing theorem, one has for $l > (he + r - 1)d(I) + g - ht$ and $i > 0$,

$$H^i(\tilde{X}, \mathcal{O}_{\tilde{X}}(h\mathcal{D}_{e,t}) \otimes \pi^*\mathcal{O}_X(l)) = 0. \tag{1.3}$$

Next, for each $i > 0$ and $h \ge 0$, since $\mathcal{O}_X(1)$ is very ample on X, by Serre's theorem, there exists an integer $l_{i,h}$ such that for all $l \ge l_{i,h}, p > 0$ and $q \le i$,

$$H^p(X, R^q\pi_*\mathcal{O}_{\tilde{X}}(h\mathcal{D}_{e,t}) \otimes \mathcal{O}_X(l)) = 0.$$

By the projection formula, we then have, for all $l \ge l_{i,h}, p > 0$ and $q \le i$,

$$H^p(X, R^q\pi_*(\mathcal{O}_{\tilde{X}}(h\mathcal{D}_{e,t}) \otimes \pi^*\mathcal{O}_X(l))) = 0. \tag{1.4}$$

Consider the Leray spectral sequence

$$E_2^{p,q} = H^p(X, R^q\pi_*(\mathcal{O}_{\tilde{X}}(h\mathcal{D}_{e,t}) \otimes \pi^*\mathcal{O}_X(l))) \Rightarrow H^{p+q}(\tilde{X}, \mathcal{O}_{\tilde{X}}(h\mathcal{D}_{e,t}) \otimes \pi^*\mathcal{O}_X(l)).$$

By (1.4), this Leray spectral sequence (when used to compute $H^i(\tilde{X}, \mathcal{O}_{\tilde{X}}(h\mathcal{D}_{e,t}) \otimes \pi^*\mathcal{O}_X(l))$) concentrates on its vertical boundary. This and (1.3) imply that, for all $l > \max\{l_{i,h}, (he + r - 1)d(I) + g\}$ and all $i > 0$,

$$\Gamma(X, R^i\pi_*(\mathcal{O}_{\tilde{X}}(h\mathcal{D}_{e,t}) \otimes \pi^*\mathcal{O}_X(l))) = H^i(\tilde{X}, \mathcal{O}_{\tilde{X}}(h\mathcal{D}_{e,t}) \otimes \pi^*\mathcal{O}_X(l)) = 0. \tag{1.5}$$

Since $\mathcal{O}_X(1)$ is very ample on X, we also know that for $l \gg 0$,

$$R^i\pi_*(\mathcal{O}_{\tilde{X}}(h\mathcal{D}_{e,t}) \otimes \pi^*\mathcal{O}_X(l)) = R^i\pi_*\mathcal{O}_{\tilde{X}}(h\mathcal{D}_{e,t}) \otimes \mathcal{O}_X(l)$$

is generated by global sections. Thus, (1.5) implies $R^i\pi_*(\mathcal{O}_{\tilde{X}}(h\mathcal{D}_{e,t})) = 0,\ \forall i > 0$. The result is proved.

(2) It follows from the Leray spectral sequence

$$E_2^{p,q} = H^p(X, R^q\pi_*\mathcal{O}_{\tilde{X}}(h\mathcal{D}_{e,t})) \Rightarrow H^{p+q}(\tilde{X}, \mathcal{O}_{\tilde{X}}(h\mathcal{D}_{e,t})),$$

and the result in part (1) that

$$H^i(X, \pi_*\mathcal{O}_{\tilde{X}}(h\mathcal{D}_{e,t})) = H^i(\tilde{X}, \mathcal{O}_{\tilde{X}}(h\mathcal{D}_{e,t})),\ \forall i \geq 0 \text{ and } h \geq 0.$$

We also have, by the projection formula and [20, Proposition 10.2],

$$\pi_*\mathcal{O}_{\tilde{X}}(h\mathcal{D}_{e,t}) = \pi_*(\mathcal{M}^{he} \otimes \pi^*\mathcal{O}_X(ht)) = \pi_*\mathcal{M}^{he} \otimes \mathcal{O}_X(ht) = \mathcal{I}^{he}(ht).$$

Thus, the result follows, and the lemma is proved. □

The main result of this section is stated as follows.

Theorem 1.4. *Suppose $H^i(X, \mathcal{O}_X) = 0$ for $i = 1, \ldots, n-1$. Then, for $t > ed(I) + \delta$, the projective embedding $\tilde{X}_{e,t}$ of $\tilde{X}$ is arithmetically Cohen-Macaulay.*

Proof. Having Lemma 1.3, the proof now follows in the same line as that of [10, Theorem 2.4] with a slight modification.

Suppose $t > ed(I) + \delta$. As in Proposition 1.2, we let $S_{e,t} = \operatorname{Sym}^* H^0(\tilde{X}, \mathcal{D}_{e,t})$ be the coordinate ring of $\mathbb{P}^N = \mathbb{P}^{N_{e,t}}$, let $I_{e,t} \subseteq S_{e,t}$ be the defining ideal of $\tilde{X}_{e,t}$, and $\mathcal{J}_{e,t}$ its ideal sheaf in $\mathbb{P}^N$. Since $\dim \tilde{X}_{e,t} = \dim X = n$, to prove $\tilde{X}_{e,t}$ is arithmetically Cohen-Macaulay, it is enough to show that $H^1(\mathbb{P}^N, \mathcal{J}_{e,t}(h)) = 0$ for all $h \in \mathbb{Z}$ and $H^i(\mathbb{P}^N, \mathcal{O}_{\tilde{X}_{e,t}}(h)) = 0$ for all $i = 1, \ldots, n-1$ and for all $h \in \mathbb{Z}$.

By Proposition 1.2, we know $H^1(\mathbb{P}^N, \mathcal{J}_{e,t}(h)) = 0$ for all $h \geq 0$. For $h < 0$, it is clear that $\mathcal{O}_{\tilde{X}_{e,t}}(-h)$ is a very ample invertible sheaf on $\tilde{X}_{e,t}$. Therefore, by Kodaira's vanishing theorem, we get $H^0(\mathbb{P}^N, \mathcal{O}_{\tilde{X}_{e,t}}(h)) = 0$ for all $h < 0$. This implies that $H^1(\mathbb{P}^N, \mathcal{J}_{e,t}(h)) = 0$ for all $h < 0$. Thus,

$$H^1(\mathbb{P}^N, \mathcal{J}_{e,t}(h)) = 0, \text{ for all } h \in \mathbb{Z}.$$

Let us now consider $H^i(\mathbb{P}^N, \mathcal{O}_{\tilde{X}_{e,t}}(h))$ $(i = 1, \ldots, n-1)$ for $h \geq 0$. By Lemma 1.3, we have, for all $i = 1, \ldots, n-1$,

$$H^i(\mathbb{P}^N, \mathcal{O}_{\tilde{X}_{e,t}}(h)) = H^i(\tilde{X}, \mathcal{O}_{\tilde{X}}(h\mathcal{D}_{e,t})) = H^i(X, \mathcal{I}^{he}(ht)).$$

Moreover, it follows from Lemma 1.1 that, for $h > 0$,

$$\text{reg } \mathcal{I}^{he} \leq hed(I) + \delta \leq hed(I) + h\delta < ht.$$

Thus, $H^i(X, \mathcal{I}^{he}(ht)) = 0$ for all $i = 1, \ldots, n-1$, and $h > 0$. Together with the hypothesis, we have

$$H^i(\mathbb{P}^N, \mathcal{O}_{\tilde{X}_{e,t}}(h)) = 0, \text{ for } i = 1, \ldots, n-1, \text{ and } h \geq 0.$$

For $h < 0$, we can again use the Kodaira's vanishing theorem to obtain

$$H^i(\mathbb{P}^N, \mathcal{O}_{\tilde{X}_{e,t}}(h)) = 0, \text{ for } i = 1, \ldots, n-1.$$

This is due to the fact that $\mathcal{O}_{\tilde{X}_{e,t}}(-h)$ is a very ample invertible sheaf on $\tilde{X}_{e,t}$ for all $h < 0$. The theorem is proved. □

Corollary 1.4.1. *Suppose R is Cohen-Macaulay. Suppose I is generated by s elements and the Rees algebra $\mathcal{R}(I)$ of I is Cohen-Macaulay. Then, for all $e > 0$ and $t > ed(I) + (s-1)(d-1)$, the projective embedding $\tilde{X}_{e,t}$ of $\tilde{X}$ is arithmetically Cohen-Macaulay.*

Proof. It follows from [26, Proposition 4.1] that, in the minimal free resolution of the Rees algebra $\mathcal{R}(I)$ of I, the bi-graded Betti numbers satisfy

$$a_{ij} \leq sd - (s-1) + i.$$

Thus, $\delta(I) \leq (s-1)[d(I) - 1]$. The result now follows from Theorem 1.4. □

Remark 1.5. It follows from [5, Theorem 1.3] that the condition $H^i(X, \mathcal{O}_X) = 0$ for all $i = 1, \ldots, n-1$ is necessary. This is because an arithmetically Cohen-Macaulay $\tilde{X}_{e,t}$ would yield an *arithmetic Macaulayfication* of X (see [5] for the definition and further results on arithmetic Macaulayfication of projective schemes).

Remark 1.6. We should point out that with exactly the same proofs, all our results in this section are true even if Z is only a locally complete intersection subscheme of X.

2. Regularity and N_p property

In this section, we study how the regularity and the syzygies of $\tilde{X}_{e,t}$ behave asymptotically. We use the same notations and terminology as in the previous section.

Let us also recall the notions of Koszul complex and Koszul cohomology which were studied in [13].

Let Y be a projective scheme. Let $\mathcal{L}$ be a very ample line bundle and $\mathcal{F}$ a coherent sheaf on Y. Let $W = H^0(Y, \mathcal{L})$ and $S = \text{Sym}^* W$. Then, S is the homogeneous coordinate ring of $\mathbb{P}(W)$, the projective space into which $\mathcal{L}$ embeds Y. Let $B = B(\mathcal{F}, \mathcal{L}) = \oplus_{q\in\mathbb{Z}} H^0(Y, \mathcal{F} \otimes q\mathcal{L}) = \oplus_{q\in\mathbb{Z}} B_q$ a S-graded module.

Definition 2.1. The *Koszul complex* of B is defined to be

$$\ldots \to \wedge^{p+1} W \otimes B_{q-1} \overset{d_{p+1,q-1}}{\to} \wedge^p W \otimes B_q \overset{d_{p,q}}{\to} \wedge^{p-1} W \otimes B_{q+1} \to \ldots$$

and the *Koszul cohomology groups* of B are defined to be

$$\mathcal{K}_{p,q}(\mathcal{F}, \mathcal{L}) = \frac{\ker\, d_{p,q}}{\mathrm{im}\, d_{p+1,q-1}}, \text{ for } p, q \in \mathbb{Z}.$$

When $\mathcal{L} = \mathcal{L}(D)$ is the invertible sheaf corresponding to a divisor D and $\mathcal{F} = \tilde{M}$ is the sheaf associated to a module M, we write $\mathcal{K}_{p,q}(M, D)$ for $\mathcal{K}_{p,q}(\mathcal{F}, \mathcal{L})$. When $\mathcal{F}$ is the structure sheaf $\mathcal{O}_Y$, we also write $\mathcal{K}_{p,q}(\mathcal{L})$ for $\mathcal{K}_{p,q}(\mathcal{F}, \mathcal{L})$.

Recall further that $\mathcal{J}_{e,t}$ denotes the ideal sheaf of $\tilde{X}_{e,t}$. The following theorem gives an upper bound for the regularity of $\mathcal{J}_{e,t}$, and shows that this regularity stabilizes for $t \gg e$.

Theorem 2.2. *Suppose $H^0(\tilde{X}, w_{\tilde{X}}) = 0$, where $w_{\tilde{X}}$ is the dualizing sheaf on $\tilde{X}$. Then,*

$$\mathrm{reg}\ \mathcal{J}_{e,t} \leq n+1,\ \forall\ t > ed(I) + \delta.$$

If, in addition, $H^i(X, \mathcal{O}_X) = 0$ for $i = 1, \ldots, n-1$, then for all $t \gg e$,

$$\mathrm{reg}\ \mathcal{J}_{e,t} = n+1.$$

Proof. Let $S_{e,t}$ and $I_{e,t}$ be as before (see Proposition 1.2). It follows from Proposition 1.2 that for $t > ed(I) + \delta$,

$$\frac{S_{e,t}}{I_{e,t}} \simeq \oplus_{h\geq 0} H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}(h\mathcal{D}_{e,t})).$$

Thus, by Green's syzygy theorem ([13, 1.b.4]), the minimal free resolution of $S_{e,t}/I_{e,t}$ is given by

$$0 \to F_s \to \ldots \to F_1 \to F_0 = S_{e,t} \to S_{e,t}/I_{e,t} \to 0,$$

for some $s \geq N - n$ and

$$F_i = \bigoplus_{q\geq 1} \mathcal{K}_{i,q}(\mathcal{D}_{e,t}) \otimes S(-i-q),\ \text{for } i = 1, \ldots, s.$$

We first prove reg $\mathcal{J}_{e,t} \leq n+1$ for all $t > ed(I) + \delta$. This is equivalent to showing that $\mathcal{K}_{i,q}(\mathcal{D}_{e,t}) = 0$ for all $1 \leq i \leq s$ and $q \geq n+1$, when $t > ed(I) + \delta$. Consider $1 \leq i \leq s$ and $q \geq n+1$. Let $K_{\tilde{X}}$ be the canonical divisor of $\tilde{X}$, then $w_{\tilde{X}}$ is the sheaf associated to $K_{\tilde{X}}$ on $\tilde{X}$. It follows from the proof of Theorem 1.4 that

$$H^i(\tilde{X}, \mathcal{O}_{\tilde{X}}(h\mathcal{D}_{e,t})) = H^i(X, \mathcal{I}^{he}(ht)) = 0,$$

for all $i = 1, \ldots, n-1$, and $h > 0$. Thus, Green's Duality theorem ([13, 2.c.6]) gives us

$$\mathcal{K}_{i,q}(\mathcal{D}_{e,t})^* \simeq \mathcal{K}_{N-n-i,n+1-q}(K_{\tilde{X}}, \mathcal{D}_{e,t}),\ \forall\ q \geq n+1.$$

This implies that, if $i > N-n$, then $\mathcal{K}_{i,q}(\mathcal{D}_{e,t}) = 0$. Suppose $i \leq N-n$. By Green's Vanishing theorem ([13, 3.a.1]), it suffices now to show that

$$h^0(\tilde{X}, w_{\tilde{X}} \otimes \mathcal{O}_{\tilde{X}}((n+1-q)\mathcal{D}_{e,t})) \quad \leq \quad N-n-i,\ \forall\ t > ed(I)+\delta. \tag{2.1}$$

This is indeed true. If $q = n+1$, this follows from the hypothesis that $H^0(\tilde{X}, w_{\tilde{X}}) = 0$. Otherwise, if $q > n+1$, then by Serre's duality one has $H^0(\tilde{X}, w_{\tilde{X}} \otimes \mathcal{O}_{\tilde{X}}((n+1-q)\mathcal{D}_{e,t})) = H^n(\tilde{X}, \mathcal{O}_{\tilde{X}}((q-n-1)\mathcal{D}_{e,t}))$. Thus, by Lemma 1.3 and [20, Proposition 10.2], we have

$$H^0(\tilde{X}, w_{\tilde{X}} \otimes \mathcal{O}_{\tilde{X}}((n+1-q)\mathcal{D}_{e,t})) = H^n(X, \mathcal{I}^{e(q-n-1)}(t(q-n-1))).$$

It follows from Lemma 1.1 that for $t > ed(I)+\delta$,

$$\operatorname{reg} \mathcal{I}^{e(q-n-1)} < t(q-n-1),$$

whence

$$H^n(X, \mathcal{I}^{e(q-n-1)}(t(q-n-1))) = 0.$$

Therefore,

$$H^0(\tilde{X}, w_{\tilde{X}} \otimes \mathcal{O}_{\tilde{X}}((n+1-q)\mathcal{D}_{e,t})) = 0,\ \forall\ t > ed(I)+\delta,$$

and (2.1) is proved.

Suppose, the additional condition $H^i(X, \mathcal{O}_X) = 0$ for $i = 1, \ldots, n-1$ is satisfied. This, together with the proof of Theorem 1.4, shows that

$$H^i(\tilde{X}, \mathcal{O}_{\tilde{X}}(h\mathcal{D}_{e,t})) = H^i(X, \mathcal{I}^{he}(ht)) = 0,$$

for all $i = 1, \ldots, n-1$, and $h \geq 0$. Thus, Green's Duality theorem ([13, 2.c.6]) holds for $\mathcal{K}_{N-n,n}(\mathcal{D}_{e,t})$, and we have

$$\mathcal{K}_{N-n,n}(\mathcal{D}_{e,t})^* \quad \simeq \quad \mathcal{K}_{0,1}(K_{\tilde{X}}, \mathcal{D}_{e,t}). \tag{2.2}$$

Let $W = H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}(\mathcal{D}_{e,t}))$, the Koszul complex of $\oplus_{h \geq 0} H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}(h\mathcal{D}_{e,t}))$ at degree $(0,1)$ is

$$\wedge W \otimes H^0(\tilde{X}, w_{\tilde{X}}) \stackrel{d_{1,0}}{\to} W \otimes H^0(w_{\tilde{X}} \otimes \mathcal{O}_{\tilde{X}}(\mathcal{D}_{e,t})) \stackrel{d_{0,1}}{\to} 0.$$

By the hypothesis, since $H^0(\tilde{X}, w_{\tilde{X}}) = 0$, we have

$$\mathcal{K}_{0,1}(K_{\tilde{X}}, \mathcal{D}_{e,t}) \quad = \quad W \otimes H^0(w_{\tilde{X}} \otimes \mathcal{O}_{\tilde{X}}(\mathcal{D}_{e,t})). \tag{2.3}$$

Recall that r is the height of I, w_X is the dualizing sheaf on X, and $\mathcal{M} = \mathcal{I}\mathcal{O}_{\tilde{X}} = \mathcal{O}_{\tilde{X}}(-E)$, we have

$$H^0(\tilde{X}, w_{\tilde{X}} \otimes \mathcal{O}_{\tilde{X}}(\mathcal{D}_{e,t})) = H^0(X, \pi_*(w_{\tilde{X}} \otimes \mathcal{O}_{\tilde{X}}(h\mathcal{D}_{e,t}))) = H^0(\tilde{X}, \pi_*(\mathcal{M}^{e+1-r} \otimes \pi^* w_X(t))).$$

By the projection formula and by [20, Proposition 10.2], we now obtain

$$H^0(\tilde{X}, w_{\tilde{X}} \otimes \mathcal{O}_{\tilde{X}}(\mathcal{D}_{e,t})) = H^0(X, \mathcal{I}^{e+1-r} \otimes w_X \otimes \mathcal{O}_X(t)),$$

where it is understood that $\mathcal{I}^{e+1-r} = \mathcal{O}_X$ if $e \leq r-1$.

It can be seen that $\mathcal{O}_X(1)$ is an ample invertible sheaf on X, so for $t \gg e$, $\mathcal{I}^{e+1-r} \otimes w_X \otimes \mathcal{O}_X(t)$ is generated by its global sections. Therefore, since $\mathcal{I}^{e+1-r} \otimes w_X \otimes \mathcal{O}_X(t)$ cannot be the zero sheaf on X, we must have $H^0(X, \mathcal{I}^{e+1-r} \otimes w_X \otimes \mathcal{O}_X(t)) \neq 0$ for all $t \gg e$. It now follows from (2.2) and (2.3) that, for all $t \gg e$,

$$\mathcal{K}_{0,1}(K_{\tilde{X}}, \mathcal{D}_{e,t}) \neq 0, \text{ i.e. } \mathcal{K}_{N-n,n}(\mathcal{D}_{e,t}) \neq 0.$$

This implies that reg $\mathcal{J}_{e,t} \geq n+1$. The result now follows. The theorem is proved. □

Remark 2.3. One can observe that even without the condition $H^0(\tilde{X}, w_{\tilde{X}}) = 0$, the proof of Theorem 2.2 still shows reg $\mathcal{J}_{e,t} \leq n+1$ for all $t \gg e > 0$. This is because for $t \gg e$, $N = N_{e,t} \gg 0$.

Before moving on to study the syzygies of $\tilde{X}_{e,t}$, let us recall the notion of *property* N_p (from [14]).

Definition 2.4. Let Y be a projective variety and let $\mathcal{L}$ be a very ample line bundle on Y defining an embedding $\varphi_{\mathcal{L}} : Y \hookrightarrow \mathbb{P} = \mathbb{P}(H^0(Y, \mathcal{L})^*)$.

(1) The line bundle $\mathcal{L}$, or the embedding $\varphi_{\mathcal{L}}(Y)$ of Y, is said to have *property* N_0 if $\varphi_{\mathcal{L}}(Y)$ is projectively normal, i.e. $\mathcal{L}$ is normally generated.
(2) Let $S = \mathrm{Sym}^* H^0(Y, \mathcal{L})$, the homogeneous coordinate ring of the projective space $\mathbb{P}$. Suppose A is the homogeneous coordinate ring of $\varphi_{\mathcal{L}}(Y)$ in $\mathbb{P}$, and

$$0 \to F_n \to F_{n-1} \to \ldots \to F_0 \to A \to 0$$

is a minimal free resolution of A. The line bundle $\mathcal{L}$, or the embedding $\varphi_{\mathcal{L}}(Y)$ of Y, is said to have *property* N_p (for $p \in \mathbb{N}$) if and only if it has property N_0, $F_0 = S$ and $F_i = S(-i-1)^{\alpha_i}$ with $\alpha_i \in \mathbb{N}$ for all $1 \leq i \leq p$.

In what comes next, we restrict our attention to the case when X is a surface, i.e. $\dim R = 3$, and Z is a nonsingular 0-dimensional subscheme of X. We will show that for any $p \in \mathbb{N}$, the projective embedding $\tilde{X}_{e,t}$ of $\tilde{X}$ possesses property N_p for all $t \gg e > 0$. We apply the method which was demonstrated in [23].

Theorem 2.5. *Suppose X is a nonsingular surface such that $H^1(X, \mathcal{O}_X) = 0$, and Z is a nonsingular 0-dimensional subscheme of X. Let $\tilde{X}$ be the blowup of X centered at Z. Then, for any $p \in \mathbb{N}$, the projective embedding $\tilde{X}_{e,t}$ of $\tilde{X}$ possesses property N_p for all $t \gg e > 0$.*

Proof. For $p = 0$, the theorem is proved in Proposition 1.2. Suppose now that $p > 0$. Let $S = S_{e,t}, I_{e,t}$ and $\mathcal{J}_{e,t}$ be as before. It follows from Theorem 2.2 and Remark 2.3 that for all $t \gg e$,

$$\text{reg } \mathcal{J}_{e,t} \leq 3.$$

It also follows from Theorem 1.4 that for all $t \gg e$, $\tilde{X}_{e,t}$ is arithmetically Cohen-Macaulay. Thus, for all $t \gg e$, the defining ideal $I_{e,t}$ of $\tilde{X}_{e,t}$ has the following minimal free resolution

$$\begin{array}{ccccccccccc} & & S(-N)^{\beta_{N-2,N}} & & & & S(-4)^{\beta_{2,4}} & & S(-3)^{\beta_{1,3}} & & \\ & & \bigoplus & & & & \bigoplus & & \bigoplus & & \\ 0 & \rightarrow & S(-(N-1))^{\beta_{N-2,N-1}} & \rightarrow & \ldots & \rightarrow & S(-3)^{\beta_{2,3}} & \rightarrow & S(-2)^{\beta_{1,2}} & \rightarrow & I_{e,t} \rightarrow 0, \end{array}$$

where $N = N_{e,t} = \dim_k H^0(\tilde{X}, \mathcal{D}_{e,t}) - 1$. To prove $\tilde{X}_{e,t}$ possesses property N_p, it is enough to show that $\beta_{i,i+2} = 0$ for all $1 \leq i \leq p$.

Let C be a general hyperplane section of $\tilde{X}_{e,t}$, then C is an arithmetically Cohen-Macaulay curve in $\mathbb{P}^{N-1}$ with the same minimal free resolution as that of $\tilde{X}_{e,t}$. Let A be the homogeneous coordinate ring of C in $\mathbb{P}^{N-1}$. Then, the Betti numbers of C (also of $\tilde{X}_{e,t}$) are given by

$$\beta_{i,j} = \text{Tor}_i^T(A, k)_j \ \forall i, j,$$

where $T = k[x_0, \ldots, x_{N-1}]$ is the coordinate ring of $\mathbb{P}^{N-1}$.

Now, let $i(E) = E.E$ be the self-intersection number of E. Let $A_{e,t} = S_{e,t}/I_{e,t}$ be the homogeneous coordinate ring of $\tilde{X}_{e,t}$, and $\mathbb{H}_{\tilde{X}_{e,t}}$ the Hilbert function of $\tilde{X}_{e,t}$, i.e.

$$\mathbb{H}_{\tilde{X}_{e,t}}(\lambda) = \dim_k (A_{e,t})_\lambda.$$

For $t \gg e$, since $\tilde{X}_{e,t}$ is arithmetically Cohen-Macaulay, we have

$$\dim_k (A_{e,t})_\lambda = \dim_k H^0(\tilde{X}_{e,t}, \mathcal{O}_{\tilde{X}_{e,t}}(\lambda)) = \dim_k H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}(\lambda \mathcal{D}_{e,t})).$$

Since $\tilde{X}_{e,t}$ is arithmetically Cohen-Macaulay, we also have $H^1(\tilde{X}, \mathcal{O}_{\tilde{X}}(\lambda \mathcal{D}_{e,t})) = 0$ for all $\lambda \in \mathbb{Z}$. Thus, by the Riemann-Roch theorem, we have

$$\dim_k (A_{e,t})_\lambda = h^0(\tilde{X}, \lambda \mathcal{D}_{e,t}) \leq \frac{1}{2}[(\lambda \mathcal{D}_{e,t})^2 - \lambda \mathcal{D}_{e,t}.K_{\tilde{X}}] + 1 + p_{\tilde{X}},$$

where $K_{\tilde{X}}$ and $p_{\tilde{X}}$ are the canonical divisor and the arithmetic genus of X. Let $\delta_{e,t}$ be the degree of $\tilde{X}_{e,t}$, then we now have

$$\delta_{e,t} = \deg \tilde{X}_{e,t} \leq (\mathcal{D}_{e,t})^2 = (tE_0 - eE)^2 = t^2 + e^2 i(E).$$

Let $D = C \cap H$ be a general hyperplane section of C, then D is a set of $\delta_{e,t}$ points in $\mathbb{P}^{N-2}$. Let $\mathcal{J}_C$ be the ideal sheaf of C in $\mathbb{P}^{N-1}$, $\mathcal{J}_D$ the ideal sheaf of D in $H \simeq \mathbb{P}^{N-2}$, and $\mathbb{H}_D$ the

Hilbert function of D. Since C is arithmetically Cohen-Macaulay, it follows from the exact sequence

$$0 \to \mathcal{J}_C \to \mathcal{J}_C(1) \to \mathcal{J}_D(1) \to 0$$

that

$$0 \to H^1(\mathbb{P}^{N-2}, \mathcal{J}_D(1)) \to H^2(\mathbb{P}^{N-1}, \mathcal{J}_C) \to H^2(\mathbb{P}^{N-1}, \mathcal{J}_C(1)) \to 0.$$

This implies

$$h^2(\mathbb{P}^{N-1}, \mathcal{J}_C) - h^2(\mathbb{P}^{N-1}, \mathcal{J}_C(1)) \quad = \quad h^1(\mathbb{P}^{N-2}, \mathcal{J}_D(1)) = \delta_{e,t} - \mathbb{H}_D(1). \qquad (2.4)$$

Let Z_e be the subscheme of X defined by I^e. Then, Z_e is also a 0-dimensional subscheme of X. We, therefore, know that the Hilbert function of Z_e eventually stabilizes at deg Z_e. Since R is a standard graded k-algebra (i.e. R is generated as a k-algebra by R_1) and $\dim R = 3$, we have $\dim_k R_t \geq \binom{t+2}{2}$. Thus, for $t \gg e$,

$$N = \dim_k R_t - \deg Z_e \geq \binom{t+2}{2} - \deg Z_e.$$

Clearly, for $t \gg e$ (more precisely, when $t \geq \dfrac{e^2 i(E) + 2\deg Z_e + 1 + p}{3}$),

$$2N - 3 \geq t^2 + e^2 i(E) + p \geq \delta_{e,t} + p.$$

By [1], we now have, for $t \gg e$,

$$\mathbb{H}_D(1) \quad \geq \quad \min\{\delta_{e,t}, N-1\} \geq \delta_{e,t} - (N-2) + p. \qquad (2.5)$$

(2.4) then implies

$$h^2(\mathbb{P}^{N-1}, \mathcal{J}_C) - h^2(\mathbb{P}^{N-1}, \mathcal{J}_C(1)) \leq (N-2) - p < N-2.$$

By [22, Corollary 3.3], this only happens if $h^2(\mathbb{P}^{N-1}, \mathcal{J}_C(1)) = 0$. Thus, for $t \gg e$, (2.4) and (2.5) give

$$h^2(\mathbb{P}^{N-1}, \mathcal{J}_C) = h^1(\mathbb{P}^{N-2}, \mathcal{J}_D(1)) = \delta_{e,t} - \mathbb{H}_D(1) \leq (N-2) - p.$$

Now, let w_A be the canonical module of A. Then,

$$\dim_k(w_A)_0 = \dim_k[H^2_{\mathfrak{m}}(A)]_0 = h^1(\mathbb{P}^{N-1}, \mathcal{O}_C) = h^2(\mathbb{P}^{N-1}, \mathcal{J}_C) \leq (N-2) - p.$$

Furthermore, by [23, Lemma 2.4], w_A is torsion free as an A-module. Thus, the vanishing theorem of [7, Theorem 1.1] gives us, for $t \gg e$,

$$[\mathrm{Tor}^T_s(w_A, k)]_s = 0, \ \forall \ s \geq (N-2) - p.$$

By duality, we now obtain, for $t \gg e$,

$$[\mathrm{Tor}^T_i(A, k)]_{i+2} = 0, \ \forall \ i \leq p,$$

i.e. for $t \gg e$,

$$\beta_{i,i+2} = 0 \ \forall \ i \leq p.$$

This implies that $\tilde{X}_{e,t}$ possesses property N_p. The theorem is proved. □

Remark: It would be interesting to have an explicit bound $d_{e,p}$ such that for any $p \in \mathbb{N}$ and $e > 0$, the projective embedding $\tilde{X}_{e,t}$ of $\tilde{X}$ has property N_p for all $t \geq d_{e,p}$. It follows from Theorem 1.4 and the proof of Theorem 2.5 that a bound could be given as

$$d_{e,p} = \max\Big\{e[d(I) + \delta(I)] + 1, \frac{e^2 i(E) + 2\deg Z_e + 1 + p}{3}\Big\},$$

i.e. for any $p \in \mathbb{N}$ and $e > 0$, $\tilde{X}_{e,t}$ has property N_p for all

$$t \geq \max\Big\{e[d(I) + \delta(I)] + 1, \frac{e^2 i(E) + 2\deg Z_e + 1 + p}{3}\Big\}.$$

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