THE DEPTH OF THE ASSOCIATED GRADED RING OF IDEALS WITH ANY REDUCTION NUMBER

IAN ABERBACH, LAURA GHEZZI AND HUY TÀI HÀ

Abstract. Let R be a local Cohen-Macaulay ring, let I be an R-ideal, and let $\mathcal{G}$ be the associated graded ring of I. We give an estimate for the depth of $\mathcal{G}$ when $\mathcal{G}$ is not necessarily Cohen-Macaulay. We assume that I is either equimultiple, or has analytic deviation one, but we do not have any restriction on the reduction number. We also give a general estimate for the depth of $\mathcal{G}$ involving the first $\mathfrak{r}+\ell$ powers of I, where $\mathfrak{r}$ denotes the Castelnuovo regularity of $\mathcal{G}$ and ℓ denotes the analytic spread of I.

Key words. depth, associated graded ring, Rees algebra, reduction number, Castelnuovo regularity.

0. Introduction

Let R be a Noetherian local ring with infinite residue field k, and let I be an R-ideal. The *Rees algebra* $\mathcal{R} = R[It] \cong \oplus_{i\geq 0} I^i$ and the *associated graded ring* $\mathcal{G} = gr_I(R) = \mathcal{R} \otimes_R R/I \cong \oplus_{i \geq 0} I^i/I^{i+1}$ are two graded algebras that reflect various algebraic and geometric properties of the ideal I. For example, $\mathrm{Proj}(\mathcal{R})$ is the blow-up of $\mathrm{Spec}(R)$ along $V(I)$ and $\mathrm{Proj}(\mathcal{G})$ corresponds to the exceptional fiber of the blow-up. Many authors have extensively studied the Cohen-Macaulay property of $\mathcal{R}$ and $\mathcal{G}$. The most general results have been obtained by Johnson and Ulrich [6, 3.1] and by Goto, Nakamura and Nishida [4, 1.1]. The goal of this paper is to estimate the depth of $\mathcal{G}$ and $\mathcal{R}$ when these rings are not necessarily Cohen-Macaulay. We can focus on the study of depth $\mathcal{G}$, since if $\mathcal{G}$ is not Cohen-Macaulay, we have that depth $\mathcal{R}$ = depth $\mathcal{G}+1$ [5, 3.10]. In order to state and motivate our results, we first need to recall some definitions and background.

A very useful tool in the study of blow-up rings is the notion of reduction of an ideal, with the reduction number measuring how closely the two ideals are related. This approach is due to Northcott and Rees [7]. An ideal $J \subseteq I$ is called a *reduction* of I if the morphism $R[Jt] \hookrightarrow R[It]$ is finite, or equivalently if $I^{r+1} = JI^r$ for some $r \geq 0$. The least such r is denoted by $r_J(I)$. A reduction is *minimal* if it is minimal with respect to inclusion, and the

2000 *Mathematics Subject Classification.* 13A30, 13C15, 14J26.

reduction number $r(I)$ is defined as $\min\{r_J(I) \mid J$ a minimal reduction of $I\}$. One of the big advantages of reductions is that they contain a lot of information about the ideal I, but often require fewer generators. More precisely, every minimal reduction of I is generated by ℓ elements, where $\ell = \ell(I)$ is the analytic spread of I; i.e., the Krull dimension of the ring $\mathcal{R} \otimes_R k \cong \mathcal{G} \otimes_R k$. The analytic spread is at least the height g of I, and at most the dimension of R. The difference $\ell - g$ is the *analytic deviation* of I. Ideals for which the analytic deviation is zero are said to be *equimultiple*. For further details see [11].

Cortadellas and Zarzuela came up with formulas for depth $\mathcal{G}$ in [1], in the special cases of ideals with analytic deviation at most one and reduction number at most two. Ghezzi in [2] found a general estimate of depth $\mathcal{G}$ involving the depth of the powers of the ideal I up to the reduction number (see [2, 2.1] for the precise statement). This theorem recovers the formulas of [1] and generalizes the results of [6] and [4]. However, in the set-up of [2] (as well as in [6], [4], and [1]), the reduction number is at most the "expected" one. Namely, the assumptions of [2, 2.1] imply that $r(I) \leq \ell - g + 1$. The main goal of this paper is to find an estimate of depth $\mathcal{G}$ without any restriction on $r(I)$. In Section 1 we treat the cases in which the ideal is either equimultiple, or has analytic deviation one. We make an assumption on depth $_{\mathcal{G}_+}\mathcal{G}$, where $\mathcal{G}_+$ denotes the ideal of $\mathcal{G}$ generated by homogeneous elements of positive degree. We are now ready to state our main results.

Theorem 1.1. *Let R be a local Cohen-Macaulay ring with infinite residue field, and let I be an equimultiple ideal with height g and reduction number r. Let $t = \min\{\text{depth } R/I^j - r + j | 1 \leq j \leq r\}$.*

(1) *If* depth $_{\mathcal{G}_+}\mathcal{G} = g$, *then* depth $\mathcal{G} \geq g + \max\{0, t\}$.

(2) *If* depth $_{\mathcal{G}_+}\mathcal{G} = g - 1$, *then* depth $\mathcal{G} \geq g + \max\{-1, t\}$.

In particular, if the reduction number is small, we have a formula for depth $\mathcal{G}$.

Corollary 1.3. *Let R be a local Cohen-Macaulay ring with infinite residue field, and let I be an equimultiple ideal with height g and reduction number two. Assume that* depth $R/I^2 <$ depth R/I. *If either* depth $_{\mathcal{G}_+}\mathcal{G} = g$ *or* depth $_{\mathcal{G}_+}\mathcal{G} = g-1$, *then* depth $\mathcal{G} = g +$ depth R/I^2.

Theorem 1.5. *Let R be a local Cohen-Macaulay ring with infinite residue field, and let I be an analytic deviation one ideal with height g and reduction number r. Assume that I is generically a complete intersection, and that* depth $_{\mathcal{G}_+}\mathcal{G} = g$. *Let $t = \min\{\text{depth } R/I^j - r + j | 1 \leq j \leq r\}$. Then,* depth $\mathcal{G} \geq g + 1 + \max\{-1, t\}$.

The key fact in the proofs of Theorem 1.1 and of Theorem 1.5 is that we can reduce to the case where depth $_{\mathcal{G}_+}\mathcal{G} = 0$ and so the reduction of the ideal is principal. The case of a reduction generated by two elements is more complicated (see Proposition 1.6 for a special case).

In Section 2 we give a lower bound for depth $\mathcal{G}$ in terms of the depth of the first $\mathfrak{r} + \ell$ powers of the ideal I. Here $\mathfrak{r}$ denotes the Castelnuovo-Mumford regularity of the associated graded ring of I. In general it is known that $\mathfrak{r} \geq r(I)$, but our results of Section 2 are valid for ideals with any reduction number (not just ideals with the expected reduction number). We first recall the definition and some notation that we will use throughout Section 2.

Let $S = \oplus_{n\geq 0} S_n$ be a finitely generated standard graded ring over a Noetherian ring S_0. For any graded S-module $M = \oplus_{n\geq 0} M_n$, we define

$$a(M) := \begin{cases} \max\{n | M_n \neq 0\} & \text{if } M \neq 0, \\ -\infty & \text{if } M = 0. \end{cases}$$

Let $S_+ = \oplus_{n>0} S_n$ be the ideal generated by the homogeneous elements of positive degree of S. For $i \geq 0$, set

$$a_i(S_+, S) := a(H^i_{S_+}(S)),$$

where $H^i_{S_+}(.)$ denotes the ith local cohomology functor with respect to the ideal S_+. The *Castelnuovo-Mumford regularity* of S is defined as the number

$$\text{reg } S := \max\{a_i(S_+, S) + i | i \geq 0\}.$$

This is an important invariant of the graded ring S (see for instance [9] and the literature cited there).

The main result of Section 2 can be stated as follows.

Theorem 2.4. *Let R be a local Cohen-Macaulay ring with infinite residue field, and let I be an R-ideal with analytic spread ℓ. Let $\mathcal{G}$ be the associated graded ring of I, and $\mathfrak{r} = \text{reg } \mathcal{G}$. Then,* depth $\mathcal{G} \geq \min(\{\text{depth } R/I^j | 1 \leq j \leq \mathfrak{r}+1\} \cup \{\text{depth } R/I^j + j - \mathfrak{r} | 2+\mathfrak{r} \leq j \leq \ell+\mathfrak{r}\})$.

The proof of Theorem 2.4 uses the techniques of [2]. The result is inspired by work of Trung [9], that shows that we can find a minimal reduction of I with "good intersection properties" (see Lemma 2.1).

The authors would like to thank the referee for a careful reading of the paper.

1. Main Results

The following theorem gives a lower bound of depth $\mathcal{G}$ for equimultiple ideals with any reduction number.

Theorem 1.1. *Let R be a local Cohen-Macaulay ring of dimension d with infinite residue field, and let I be an equimultiple ideal with height g and reduction number r. Let* $t = \min\{\text{depth } R/I^j - r + j | 1 \leq j \leq r\}$.

(1) *If* $\text{depth}_{\mathcal{G}_+} \mathcal{G} = g$*, then* $\text{depth } \mathcal{G} \geq g + \max\{0, t\}$.
(2) *If* $\text{depth}_{\mathcal{G}_+} \mathcal{G} = g - 1$*, then* $\text{depth } \mathcal{G} \geq g + \max\{-1, t\}$.

Proof. We prove the results by induction on $\text{depth}_{\mathcal{G}_+} \mathcal{G}$.

(1) Suppose that $\text{depth}_{\mathcal{G}_+} \mathcal{G} = 0$. Then $I^{r+1} = 0$. Hence, $\mathcal{G} = R/I \oplus I/I^2 \oplus \ldots \oplus I^{r-1}/I^r \oplus I^r$, and $\text{depth } \mathcal{G} = \min\{\text{depth } R/I, \text{depth } I/I^2, \ldots, \text{depth } I^{r-1}/I^r, \text{depth } I^r\}$. Since $\text{depth } I^i/I^{i+1} \geq \min\{\text{depth } R/I^{i+1}, \text{depth } R/I^i + 1\}$ for each $0 \leq i \leq r-1$, and $\text{depth } I^r \geq \text{depth } R/I^r$, we have that $\text{depth } \mathcal{G} \geq \min\{\text{depth } R/I^j | 1 \leq j \leq r\} \geq t$.

Now assume that $\text{depth}_{\mathcal{G}_+} \mathcal{G} > 0$. Let $x \in I$ be an element such that $\overline{x} \in I/I^2$ is regular on $\mathcal{G}$. By [10, 2.7] x is regular on R and $I^j \cap (x) = xI^{j-1}$ for every $j \geq 1$. Let $\overline{R} = R/(x)$, $\overline{I} = I/(x)$ and $\overline{\mathcal{G}} = \mathcal{G}/(\overline{x}) = \mathcal{G}_{\overline{R}}(\overline{I})$. By the induction hypothesis, we have that $\text{depth } \overline{\mathcal{G}} \geq g - 1 + \min\{\text{depth } \overline{R}/\overline{I}^j - r + j | 1 \leq j \leq r\}$, and so $\text{depth } \mathcal{G} \geq g + \min\{\text{depth } \overline{R}/\overline{I}^j - r + j | 1 \leq j \leq r\}$. For $2 \leq j \leq r$, consider the exact sequence

$$0 \to R/xI^{j-1} \to R/I^j \oplus R/(x) \to \overline{R}/\overline{I}^j \to 0.$$

We have that $\text{depth } \overline{R}/\overline{I}^j \geq \min\{\text{depth } R/I^j, \text{depth } R/xI^{j-1} - 1\}$, since $\text{ht } I > 0$ and $\text{depth } R/(x) = d - 1$. From the exact sequence

$$0 \to xR/xI^{j-1} \to R/xI^{j-1} \to R/(x) \to 0$$

and the isomorphism $xR/xI^{j-1} \simeq R/I^{j-1}$ it follows that $\text{depth } R/xI^{j-1} \geq \text{depth } R/I^{j-1}$. We conclude that $\text{depth } \overline{R}/\overline{I}^j \geq \min\{\text{depth } R/I^j, \text{depth } R/I^{j-1} - 1\}$ for $1 \leq j \leq r$. Hence we have that $\text{depth } \mathcal{G} \geq g + t$.

(2) Suppose that $\text{depth}_{\mathcal{G}_+} \mathcal{G} = 0$. Let $J = (a)$ be a minimal reduction of I with $r_J(I) = r$. For every $j \geq r + 1$ we have an exact sequence

$$0 \to R/I^{j-1} \to R/I^j \to R/(a) \to 0.$$

Using induction on j we see that depth $R/I^j \geq$ depth R/I^r for every $j \geq r$. Hence, depth $\mathcal{G} \geq \inf\{\text{depth } R/I^j | j \geq 1\} = \min\{\text{depth } R/I^j | 1 \leq j \leq r\} \geq t$. We may assume that $t \geq 0$. If $t > 0$, let $x_1, \ldots, x_t \in R$ be a regular sequence on R and on R/I^j for all $j = 1, \ldots, r$. Write $\overline{R} = R/(x_1, \ldots, x_t)$, $\overline{I} = I\overline{R}$. Since depth $\overline{R}/\overline{I}^j =$ depth $R/I^j - t$ for all $j = 1, \ldots, r$, we have that $\min\{\text{depth } \overline{R}/\overline{I}^j - r + j | 1 \leq j \leq r\} = 0$. Hence we can reduce the problem to the case where $t = 0$. Now, choose $x \in R$ such that x is regular on R and on R/I^j for all $j = 1, \ldots, r-1$. Let x^* and a^* be the initial forms of x and a in $\mathcal{G}$ (x^* has degree 0 and a^* has degree 1). We claim that $x^* + a^*$ is regular on $\mathcal{G}$, which proves the assertion. If not, there exists $v^* = v_0^* + \ldots + v_n^* \in \mathcal{G}$, with $v_i^* \in I^i/I^{i+1}$ and at least one $v_i^* \neq 0$, such that $(x^* + a^*)v^* = 0$ in $\mathcal{G}$. Suppose that $n \geq r - 1$. Then, $a^* v_n^* = 0$ implies that $av_n \in I^{n+2} = aI^{n+1}$, and so $v_n \in I^{n+1}$ since a is regular on R. Hence $v_n^* = 0$. Let $v_k^* \neq 0$ be the lowest degree term of v^*. Then $x^* v_k^* = 0$ implies that $xv_k \in I^{k+1}$. Since $k + 1 \leq r - 1$, x is regular on R/I^{k+1}, and so $v_k \in I^{k+1}$, i.e., $v_k^* = 0$, a contradiction. This finishes the proof of the case depth $_{\mathcal{G}_+}\mathcal{G} = 0$.

If depth $_{\mathcal{G}_+}\mathcal{G} > 0$, we can follow the same induction step as that of part (1) to prove the theorem. $\square$

The following remark gives an upper bound for depth $\mathcal{G}$ in a general context.

Remark 1.2. [2, 2.11] *Let R be a Noetherian local ring, let I be an R-ideal with analytic spread ℓ. Then* depth $\mathcal{G} \leq \inf\{\text{depth } R/I^j \mid j \geq 1\} + \ell$.

The next corollary is a special case of Theorem 1.1, for reduction number two. Combining Theorem 1.1 with Remark 1.2, we have a formula for depth $\mathcal{G}$.

Corollary 1.3. *Let R be a local Cohen-Macaulay ring with infinite residue field, and let I be an equimultiple ideal with height g and reduction number two. Assume that* depth $R/I^2 <$ depth R/I. *If either* depth $_{\mathcal{G}_+}\mathcal{G} = g$ *or* depth $_{\mathcal{G}_+}\mathcal{G} = g - 1$, *then* depth $\mathcal{G} = g +$ depth R/I^2.

In the next example we compute the depth of the associated graded ring.

Example 1.4. Let $R = k[[X, Y, T_1, \ldots, T_n]]/(X^3Y) = k[[x, y, t_1, \ldots, t_n]]$, where k is a field and $n \geq 2$. R is Cohen-Macaulay and $\dim R = n + 1$. Let $I = (xy, t_1, \ldots, t_{n-1})$, and let $J = (t_1, \ldots, t_{n-1})$. I is equimultiple with ht $I = n - 1$ and reduction number 2. J is a minimal reduction of I. We have that depth $_{\mathcal{G}_+}\mathcal{G} = n-1$ by [10, 2.7], since $I^2 \cap J = JI$. Furthermore, depth $R/I = 2$ and depth $R/I^2 = 1$. Corollary 1.3 implies that depth $\mathcal{G} = n$.

In the next theorem we treat the case of analytic deviation one ideals with any reduction number. We obtain a lower bound for depth $\mathcal{G}$ similar to that of Theorem 1.1.

Theorem 1.5. *Let R be a local Cohen-Macaulay ring of dimension d with infinite residue field, and let I be an analytic deviation one ideal with height g and reduction number r. Assume that $r(I_\wp) < r$ for every prime $\wp$ containing I with* ht $\wp = g$*, and that* depth $_{\mathcal{G}_+}\mathcal{G} = g$*. Let $t = \min\{\text{depth } R/I^j - r + j | 1 \leq j \leq r\}$. Then,*

$$\text{depth } \mathcal{G} \geq g + 1 + \max\{-1, t\}.$$

Proof. We prove the result by induction on depth $_{\mathcal{G}_+}\mathcal{G}$. Suppose that depth $_{\mathcal{G}_+}\mathcal{G} = 0$, and let $J = (a)$ be a minimal reduction of I with $r_J(I) = r$. We have that $(0 : a) \cap I^r = 0$. If R/I^r is not Cohen-Macaulay, by [1, 3.4] we have that depth $R/I^j =$ depth R/I^r for every $j \geq r$. Hence depth $\mathcal{G} \geq \min\{\text{depth } R/I^j | 1 \leq j \leq r\} \geq t$.

If R/I^r is Cohen-Macaulay, by [1, 3.4] we have that depth $R/I^{r+1} =$ depth $R/I^r - 1 = d-1$, and that depth $R/I^j =$ depth R/I^{r+1} for every $j \geq r+1$. Hence depth $\mathcal{G} \geq \min(\{\text{depth } R/I^j | 1 \leq j \leq r-1\} \cup \{d-1\}) \geq \min\{t, d-1\}$. If $r \geq 2$, we have that $\min\{t, d-1\} = t$, and so depth $\mathcal{G} \geq t$. If $r \leq 1$, by [12, 3.1] we still have that depth $\mathcal{G} \geq t$. Now we proceed as in the proof of part (2) of Theorem 1.1 to obtain that depth $\mathcal{G} \geq 1 + t$. This finishes the proof of the case depth $_{\mathcal{G}_+}\mathcal{G} = 0$.

Suppose now that depth $_{\mathcal{G}_+}\mathcal{G} > 0$. Let $\wp$ be a prime containing I with ht $\wp = g$. Since $\mathcal{G}_\wp$ is Cohen-Macaulay, we have that $r(I_\wp)$ is independent of the minimal reduction of $I_\wp$ [8, 1.2 (i)]. Choose $x \in I$ such that x is part of a minimal reduction of I, $\overline{x} \in I/I^2$ is regular on $\mathcal{G}$, and for every prime $\wp$ of height g containing I, $x_\wp$ is part of a minimal generating set of a minimal reduction of $I_\wp$. By [10, 2.7] x is regular on R and $I^j \cap (x) = xI^{j-1}$ for every $j \geq 1$. Let $\overline{R} = R/(x)$, $\overline{I} = I/(x)$ and $\overline{\mathcal{G}} = \mathcal{G}/(\overline{x}) = \mathcal{G}_{\overline{R}}(\overline{I})$. Then $\overline{I}$ is an analytic deviation one ideal with height $g-1$. Let $\mathfrak{q} \subset \overline{R}$ be a prime ideal of height $g-1$ containing $\overline{I}$. Then $\mathfrak{q} = \wp/(x)$, where $\wp$ is a prime of R of height g containing I. Let K be a minimal reduction of $I_\wp$, such that $x_\wp$ is part of a minimal generating set of K. Then $K/(x)_\wp$ is a minimal reduction of $\overline{I}_\mathfrak{q}$, and so $r(\overline{I}_\mathfrak{q}) \leq r(I_\wp) < r$. In other words, $\overline{I}$ generically has reduction number less than r. The conclusion now follows by induction, as in the proof of Theorem 1.1. $\square$

We remark again that the key fact in the proofs of Theorem 1.1 and of Theorem 1.5 is that we can reduce to the case where the reduction of the ideal is principal. Even when the reduction is generated by a regular sequence of two elements the situation is much more complicated. The next proposition treats a special case.

Proposition 1.6. *Let R be a local Cohen-Macaulay ring with infinite residue field, and let I be an equimultiple ideal of height two and reduction number r. Let $J = (a_1, a_2)$ be a minimal reduction of I such that $I^r : a_1 = I^r : a_2$. Then,* depth $R/I^j \geq \min\{\text{depth } R/I^r - 1, \text{depth } R/I^{r+1}\}$ *for every $j \geq r+1$, and*

$$\text{depth } \mathcal{G} \geq \min(\{\text{depth } R/I^j | 1 \leq j \leq r-1\} \cup \{\text{depth } R/I^r - 1, \text{depth } R/I^{r+1}\})$$

Proof. For $j \geq r+1$ we have that $I^{j-1} = J^{j-r-1}I^r$, and so we have the following exact sequences

$$0 \to \frac{R}{a_1 J^{j-r-1} I^r \cap a_2^{j-r} I^r} \quad \to \quad \frac{R}{a_1 I^{j-1}} \oplus \frac{R}{a_2^{j-r} I^r} \to R/I^j \to 0. \tag{1.1}$$

Since $I^r : a_1 = I^r : a_2$, we have that $a_1 J^{j-r-1} I^r \cap a_2^{j-r} I^r = a_1 a_2^{j-r}(I^r : J)$. The sequence (1.1) for $j = r+1$ implies that

$$\text{depth } \frac{R}{I^r : J} \geq \min\{\text{depth } R/I^r, \text{depth } R/I^{r+1} + 1\}.$$

Now we prove by induction on j that depth $R/I^j \geq \min\{\text{depth } R/I^r - 1, \text{depth } R/I^{r+1}\}$ for all $j \geq r+1$. The claim is clear for $j = r+1$. Suppose that $j \geq r+2$. From the sequence (1.1), we have that depth $R/I^j \geq \min\{\text{depth } R/I^r - 1, \text{depth } R/I^{r+1}, \text{depth } R/I^{j-1}\} \geq \min\{\text{depth } R/I^r - 1, \text{depth } R/I^{r+1}\}$. The first assertion is proved. The second assertion follows from the first one. $\square$

Remark 1.7. Let R be a local Cohen-Macaulay ring with infinite residue field, and let I be an analytic deviation one ideal of height one and reduction number r. Let $J = (a_1, a_2)$ be a minimal reduction of I such that $(a_1 : a_2^n) \cap I^r \subseteq (a_1)$ for every $n \geq 1$, and $I^r : a_1 = I^r : a_2$. Then, by the proof of Proposition 1.6, we have that depth $R/I^j \geq \min\{\text{depth } R/I^r - 1, \text{depth } R/I^{r+1}\}$ for every $j \geq r+1$, and depth $\mathcal{G} \geq \min(\{\text{depth } R/I^j | 1 \leq j \leq r-1\} \cup \{\text{depth } R/I^r - 1, \text{depth } R/I^{r+1}\})$.

2. Depth of $\mathcal{G}$ and its Castelnuovo-Mumford regularity

Let R be a local Cohen-Macaulay ring, let I be an R-ideal with analytic spread ℓ, let $\mathcal{G}$ be the associated graded ring of I, and let $\mathfrak{r} = \operatorname{reg} \mathcal{G}$. The purpose of this section is to give a lower bound for depth $\mathcal{G}$ involving the depth of the first $\mathfrak{r}+\ell$ powers of I. We first prove some technical results (Lemmas 2.1, 2.2, and 2.3), that will play a crucial role for the proof of Theorem 2.4.

Lemma 2.1. *Let R be a local ring with infinite residue field, let I be an ideal of R, $\mathcal{G} = gr_I(R)$ and $\mathfrak{r} = \operatorname{reg} \mathcal{G}$. For $a \in I$ let a^* denote the image of a in $[\mathcal{G}]_1$. Let J be a reduction of I. Then there exists a minimal basis $a_1, \ldots, a_s$ of J satisfying the following conditions:*

$$[(a_1, \ldots, a_i) : a_{i+1}] \cap I^j \quad = \quad (a_1, \ldots, a_i) I^{j-1}, \ \forall\ 0 \leq i \leq s-1, \ j \geq \mathfrak{r}+1. \tag{2.1}$$

$$[(a_1^*, \ldots, a_i^*) : a_{i+1}^*]_j \quad = \quad (a_1^*, \ldots, a_i^*)_j, \ \forall\ 0 \leq i \leq s-1, \ j \geq \mathfrak{r}+1. \tag{2.2}$$

Proof. By [9, 1.1] there exists a minimal basis $a_1, \ldots, a_s$ of J such that $[(a_1, \ldots, a_i) : a_{i+1}] \cap I^{\mathfrak{r}+1} = (a_1, \ldots, a_i) I^{\mathfrak{r}}$, whenever $0 \leq i \leq s-1$. Thus, [9, 4.7] implies (2.1). By [9, 4.8], $a_1^*, \ldots, a_s^*$ is a filter-regular sequence of $\mathcal{G}$. We have that $[(a_1^*, \ldots, a_i^*) : a_{i+1}^*]_j = (a_1^*, \ldots, a_i^*)_j$ whenever $0 \leq i \leq s-1$ and $j \geq a(a_1^*, \ldots, a_s^*) + 1$, where $a(a_1^*, \ldots, a_s^*)$ is defined to be $\max\{a[(a_1^*, \ldots, a_i^*) : a_{i+1}^*/(a_1^*, \ldots, a_i^*)] \mid i = 0, \ldots, s-1\}$. By [9, 2.4] we have that $\mathfrak{r} \geq a(a_1^*, \ldots, a_s^*)$. This implies (2.2). $\square$

In particular, (2.1) implies that $[0 : a_1] \cap I^j = 0\ \forall\ j \geq \mathfrak{r}+1$. We remark that since k is infinite, we can choose the basis $a_1, \ldots, a_s$ such that each a_i satisfies $[0 : a_i] \cap I^j = 0\ \forall\ j \geq \mathfrak{r}+1$.

Lemma 2.2. (see [2, 2.3]) *Let R be a local Cohen-Macaulay ring of dimension d with infinite residue field, and let I be an R-ideal. Let J be a reduction of I with basis $a_1, \ldots, a_s$ satisfying* (2.1)*. Write $\mathfrak{a}_i = (a_1, \ldots, a_i)$ for all $i = 0, \ldots, s$. Then,* depth $R/\mathfrak{a}_i I^j \geq \min(\{d - i\} \cup \{\text{depth } R/I^{j-n} - n \mid 0 \leq n \leq i-1\})$*, for $0 \leq i \leq s$ and $j \geq \mathfrak{r}+i$.*

Proof. We use induction on i. For $i = 0$ the result is trivial. Assume that $0 \leq i \leq s-1$. We need to show that the inequality holds for $i+1$. Let $j \geq \mathfrak{r}+i+1$. By (2.1), $\mathfrak{a}_i I^j \cap a_{i+1} I^j = a_{i+1}[(\mathfrak{a}_i I^j : a_{i+1}) \cap I^j] \subseteq a_{i+1}[(\mathfrak{a}_i : a_{i+1}) \cap I^j] = a_{i+1}\mathfrak{a}_i I^{j-1} \subseteq \mathfrak{a}_i I^j \cap a_{i+1} I^j$. Hence we obtain an exact sequence

$$0 \to a_{i+1}\mathfrak{a}_i I^{j-1} \to \mathfrak{a}_i I^j \oplus a_{i+1} I^j \to \mathfrak{a}_{i+1} I^j \to 0. \tag{2.3}$$

On the other hand, by (2.1) for $i = 0$, $[0 : a_{i+1}] \cap \mathfrak{a}_i I^{j-1} \subseteq [0 : a_{i+1}] \cap I^j = 0$, and therefore $a_{i+1}\mathfrak{a}_i I^{j-1} \cong \mathfrak{a}_i I^{j-1}$, $a_{i+1} I^j \cong I^j$. The conclusion follows from (2.3) and the induction hypothesis. □

Lemma 2.3. [2, 2.5] *Let R be a local Cohen-Macaulay ring with infinite residue field, and let I be an R-ideal. Let J be a reduction of I with basis $a_1, \dots, a_s$ satisfying* (2.2)*. Then,* $\operatorname{depth} [\mathcal{G}/(a_1^*, \dots, a_i^*)]_j \geq \min(\{\operatorname{depth} R/I^n + n - j - 1 | j - i + 1 \leq n \leq j + 1\} \cup \{\operatorname{depth} R/I^{j-i} - i + 1\})$*, whenever $0 \leq i \leq s$ and $j \geq \mathfrak{r} + i + 1$.*

Proof. We show by induction on i that $\operatorname{depth} [\mathcal{G}/(a_1^*, \dots, a_i^*)]_j \geq \min\{\operatorname{depth}\ I^n/I^{n+1} + n - j \mid j - i \leq n \leq j\}$, whenever $0 \leq i \leq s$ and $j \geq \mathfrak{r} + i + 1$. The assertion being trivial for $i = 0$, we may assume that $0 \leq i \leq s - 1$. We need to show that the inequality holds for $i + 1$. By (2.2) we have an exact sequence

$$0 \to [\mathcal{G}/(a_1^*, \dots, a_i^*)]_j \to [\mathcal{G}/(a_1^*, \dots, a_i^*)]_{j+1} \to [\mathcal{G}/(a_1^*, \dots, a_{i+1}^*)]_{j+1} \to 0$$

whenever $0 \leq i \leq s - 1$ and $j \geq \mathfrak{r} + 1$. The conclusion follows from the exact sequence and the induction hypothesis. □

The goal of this section is to prove the following theorem.

Theorem 2.4. *Let R be a local Cohen-Macaulay ring of dimension d with infinite residue field, let I be an R-ideal, and let J be a reduction of I generated by s elements. Let $\mathcal{G}$ be the associated graded ring of I, and $\mathfrak{r} = \operatorname{reg} \mathcal{G}$. Then,*

$$\operatorname{depth} \mathcal{G} \geq \min(\{\operatorname{depth} R/I^j | 1 \leq j \leq \mathfrak{r} + 1\} \cup \{\operatorname{depth} R/I^j + j - \mathfrak{r} | 2 + \mathfrak{r} \leq j \leq s + \mathfrak{r}\}).$$

To prove Theorem 2.4, we will apply the methods of [2]. We first need some preliminary notation and lemmas.

Let J be a reduction of I with basis $a_1, \dots, a_s$ satisfying the conclusions of Lemma 2.1. If $s > 0$, for $0 \leq i \leq s$ consider the graded $\mathcal{G}$-modules:

$$M_{(i)} = [\mathcal{G}/(a_1^*, \dots, a_i^*)]_{\geq \mathfrak{r}+i+1} = \frac{\mathcal{G}_+^{\mathfrak{r}+i+1}}{(a_1^*, \dots, a_i^*)\mathcal{G}_+^{\mathfrak{r}+i}}$$

$$N_{(i)} = \frac{\mathcal{G}_+^{\mathfrak{r}+i}}{a_i^*\mathcal{G}_+^{\mathfrak{r}+i} + (a_1^*, \ldots, a_{i-1}^*)\mathcal{G}_+^{\mathfrak{r}+i-1}}.$$

Then, $[N_{(i)}]_{\geq \mathfrak{r}+i+1} = M_{(i)}$ and $[N_{(i)}]_{\mathfrak{r}+i} = [\mathcal{G}/(a_1^*, \ldots, a_{i-1}^*)]_{\mathfrak{r}+i}$. Hence, for $0 \leq i \leq s$ we have the exact sequences

$$0 \to M_{(i)} \to N_{(i)} \to [\mathcal{G}/(a_1^*, \ldots, a_{i-1}^*)]_{\mathfrak{r}+i} \to 0. \tag{2.4}$$

Furthermore, if $0 \leq i \leq s-1$, then $N_{(i+1)} = M_{(i)}/a_{i+1}^* M_{(i)}$ and by (2.2) we have that $0 :_{M_{(i)}} (a_{i+1}^*) = 0$. Thus, in the range $0 \leq i \leq s-1$ we have exact sequences

$$0 \to M_{(i)}(-1) \to M_{(i)} \to N_{(i+1)} \to 0. \tag{2.5}$$

Notice that $M_{(s)} = 0$, since $I^{\mathfrak{r}+s+1} = JI^{\mathfrak{r}+s}$.

Let $\lambda = \min(\{\text{depth } R/I^j | 1 \leq j \leq \mathfrak{r}+1\} \cup \{\text{depth } R/I^j + j - \mathfrak{r} | 2 + \mathfrak{r} \leq j \leq s + \mathfrak{r}\})$. Recall that our goal is to show that depth $\mathcal{G} \geq \lambda$. The next lemma gives an estimate of depth $M_{(i)}$. In particular we show that depth $M_{(i)} \geq \lambda - i - 1$.

Lemma 2.5. *In addition to the assumptions of Theorem 2.4, assume that $s > 0$. Let $M_{(i)}$ be defined as above. Then,*

$$\text{depth } M_{(i)} \geq \min(\{d-i, \text{depth } R/I^{\mathfrak{r}+1} - i + 1\} \cup \{\text{depth } R/I^j + j - \mathfrak{r} - i - 1 | 2 + \mathfrak{r} \leq j \leq s + \mathfrak{r}\}).$$

Proof. We use decreasing induction on i. For $i = s$, the assertion is true since $M_{(s)} = 0$. Suppose that $0 \leq i \leq s-1$. Consider the exact sequence (2.4)

$$0 \to M_{(i+1)} \to N_{(i+1)} \to [\mathcal{G}/(a_1^*, \ldots, a_i^*)]_{\mathfrak{r}+i+1} \to 0.$$

It follows from Lemma 2.3 and the induction hypothesis that depth $N_{(i+1)} \geq \min(\{d - i - 1, \text{depth } R/I^{\mathfrak{r}+1} - i\} \cup \{\text{depth } R/I^j + j - \mathfrak{r} - i - 2 | 2 + \mathfrak{r} \leq j \leq s + \mathfrak{r}\} \cup \{\text{depth } R/I^j + j - \mathfrak{r} - i - 2 | 2 + \mathfrak{r} \leq j \leq \mathfrak{r} + i + 2\})$. If $i \leq s-2$, then depth $N_{(i+1)} \geq \min(\{d - i - 1, \text{depth } R/I^{\mathfrak{r}+1} - i\} \cup \{\text{depth } R/I^j + j - \mathfrak{r} - i - 2 | 2 + \mathfrak{r} \leq j \leq s + \mathfrak{r}\})$. If $i = s-1$, then by Lemma 2.2 we have that depth $R/I^{\mathfrak{r}+s+1} = \text{depth } R/JI^{\mathfrak{r}+s} \geq \min(\{d - s\} \cup \{\text{depth } R/I^{\mathfrak{r}+s-n} - n | 0 \leq n \leq s-1\})$. It follows that depth $N_{(s)} \geq \min(\{d - s, \text{depth } R/I^{\mathfrak{r}+1} - s + 1\} \cup \{\text{depth } R/I^j + j - \mathfrak{r} - s - 1 | 2 + \mathfrak{r} \leq j \leq s + \mathfrak{r}\})$. In any case depth $N_{(i+1)} \geq \min(\{d - i - 1, \text{depth } R/I^{\mathfrak{r}+1} - i\} \cup \{\text{depth } R/I^j + j - \mathfrak{r} - i - 2 | 2 + \mathfrak{r} \leq j \leq s + \mathfrak{r}\})$, and since depth $M_{(i)} = \text{depth } N_{(i+1)} + 1$, the conclusion follows. □

Let S be a homogeneous Noetherian ring with S_0 local and homogeneous maximal ideal $\mathfrak{M}$, let $H^\bullet(-)$ denote local cohomology with support in $\mathfrak{M}$. For a graded S-module N and an integer j, let

$$a_j(N) = \max\{n \mid [H^j(N)]_n \neq 0\}.$$

The following lemma is well known, but we recall it for convenience.

Lemma 2.6. [2, 2.6] *Let $0 \to A \to B \to C \to 0$ be an exact sequence of graded S-modules, let n and j be integers.*

(a) *If $a_j(A) \leq n$ and $a_j(C) \leq n$, then $a_j(B) \leq n$.*
(b) (i) *If $H^j(A) = 0$, then $a_j(C) \geq a_j(B)$.*
 (ii) *If $H^j(B) = 0$, then $a_{j+1}(A) \geq a_j(C)$.*
 (iii) *If $H^j(C) = 0$, then $a_{j+1}(B) \geq a_{j+1}(A)$.*

Lemma 2.7. *In addition to the assumptions of Theorem 2.4, assume that $s > 0$. Let $M_{(i)}$ and λ be defined as above. Then,*

(1) $a_j(M_{(i)}) \leq \mathfrak{r} + i$ *for any integer j and $0 \leq i \leq s$.*
(2) depth $M_{(i)} \geq \lambda - i - 1$ *and if* depth $M_{(i)} = \lambda - i - 1$ *then* $a_{\lambda-i-1}(M_{(i)}) = \mathfrak{r} + i$.

Proof. **(1)** We prove the claim by decreasing induction on i. For $i = s$ the assertion is trivial, since $M_{(s)} = 0$. Suppose that $0 \leq i \leq s-1$ and that $a_j(M_{(i+1)}) \leq \mathfrak{r} + i + 1$ for any integer j. Consider the exact sequence (2.4)

$$0 \to M_{(i+1)} \to N_{(i+1)} \to [\mathcal{G}/(a_1^*, \ldots, a_i^*)]_{\mathfrak{r}+i+1} \to 0.$$

By [3, 2.2], for any integer j, $H^j([\mathcal{G}/(a_1^*, \ldots, a_i^*)]_{\mathfrak{r}+i+1})$ is concentrated in degree $\mathfrak{r} + i + 1$. Hence, Lemma 2.6 (a) implies that $a_j(N_{(i+1)}) \leq \mathfrak{r} + i + 1$ for any j. Applying the local cohomology functor to the sequence (2.5)

$$0 \to M_{(i)}(-1) \to M_{(i)} \to N_{(i+1)} \to 0$$

it follows that for any j, $[H^j(M_{(i)})]_n = 0$ whenever $n > \mathfrak{r} + i$. Hence $a_j(M_{(i)}) \leq \mathfrak{r} + i$ and the proof of (a) is completed.

(2) It follows from Lemma 2.5 that depth $M_{(i)} \geq \lambda - i - 1$. To prove the last assertion, we again use decreasing induction on i. For $i = s$, the assertion is vacuous. Suppose that $0 \leq i \leq s-1$, and that depth $M_{(i)} = \lambda - i - 1$. It follows from (2.5) that depth $N_{(i+1)} = \lambda - i - 2$, and so $H^{\lambda-i-2}(N_{(i+1)}) \neq 0$. Applying the local cohomology functor to (2.4) we obtain the exact sequence

$$\cdots \to H^{\lambda-i-2}(M_{(i+1)}) \to H^{\lambda-i-2}(N_{(i+1)}) \to H^{\lambda-i-2}\big([\mathcal{G}/(a_1^*,\ldots,a_i^*)]_{\mathfrak{r}+i+1}\big) \to \ldots.$$

If depth $M_{(i+1)} > \lambda - i - 2$, then $H^{\lambda-i-2}(N_{(i+1)}) \hookrightarrow H^{\lambda-i-2}([\mathcal{G}/(a_1^*,\ldots,a_i^*)]_{\mathfrak{r}+i+1})$, and so $a_{\lambda-i-2}(N_{(i+1)}) = \mathfrak{r}+i+1$. If depth $M_{(i+1)} = \lambda - i - 2$, then by induction hypothesis we have that $a_{\lambda-i-2}(M_{(i+1)}) = \mathfrak{r}+i+1$. Again, we consider the exact sequence (2.4)

$$0 \to M_{(i+1)} \to N_{(i+1)} \to [\mathcal{G}/(a_1^*,\ldots,a_i^*)]_{\mathfrak{r}+i+1} \to 0.$$

It follows from Lemma 2.3 (and Lemma 2.2 when $i = s-1$) that depth $[\mathcal{G}/(a_1^*,\ldots,a_i^*)]_{\mathfrak{r}+i+1} \geq \lambda-i-2$. Thus, applying Lemma 2.6 (b) (iii) with $j = \lambda-i-3$ to the exact sequence above we get that $a_{\lambda-i-2}(N_{(i+1)}) \geq \mathfrak{r}+i+1$. In any case, we have that $a_{\lambda-i-2}(N_{(i+1)}) \geq \mathfrak{r}+i+1$. Since depth $M_{(i)} = \lambda-i-1$, applying Lemma 2.6 (b) (ii) to the sequence (2.5) we conclude that $a_{\lambda-i-1}(M_{(i)}) \geq \mathfrak{r}+i$. □

We are now ready to prove Theorem 2.4.

Proof of Theorem 2.4. We need to show that depth $\mathcal{G} \geq \lambda$. Let J be a reduction of I with basis $a_1,\ldots,a_s$ satisfying the conclusions of Lemma 2.1. If $s = 0$, then $\mathcal{G} = R/I \oplus I/I^2 \oplus \ldots \oplus I^{r-1}/I^r \oplus I^r$, where $r = r_J(I)$ is the reduction number of I with respect to J. Hence depth $\mathcal{G} = \min\{\text{depth } R/I^j \mid 1 \leq j \leq r\}$. The result follows from the fact that $\mathfrak{r} \geq r$.

Suppose now that $s > 0$. From the definition of $M_{(0)}$, we have the exact sequence

$$0 \to M_{(0)} \to \mathcal{G} \to \oplus_{n=0}^{\mathfrak{r}} I^n/I^{n+1} \to 0. \tag{2.6}$$

Let $C = \oplus_{n=0}^{\mathfrak{r}} I^n/I^{n+1}$. Since depth $C \geq \lambda$, and depth $M_{(0)} \geq \lambda - 1$ by Lemma 2.7, it follows that depth $\mathcal{G} \geq \lambda-1$. Applying local cohomology to (2.6) we see that $H^{\lambda-1}(M_{(0)}) \cong H^{\lambda-1}(\mathcal{G})$. Furthermore, by Lemma 2.7 and by Lemma 2.6 (a), we have that $a_j(\mathcal{G}) \leq \mathfrak{r}$ for any integer j. If depth $\mathcal{G} = \lambda - 1$, then depth $M_{(0)} = \lambda - 1$, and so $a_{\lambda-1}(M_{(0)}) = \mathfrak{r}$ by Lemma 2.7. On the other hand, since $\lambda - 1 < d$, by [2, 2.9] we have that $a_{\lambda-1}(\mathcal{G}) < \mathfrak{r}$, a contradiction. Hence, depth $\mathcal{G} \geq \lambda$. □

References

[1] T. Cortadellas and S. Zarzuela, Burch's inequality and the depth of the blowup rings of an ideal, J. Pure Appl. Algebra, **157**, Nos. 2-3 (2001), 183-204.
[2] L. Ghezzi, On the depth of the associated graded ring of an ideal, J. Algebra **248** (2002), 688-707.

[3] S. Goto and S. Huckaba, On graded rings associated to analytic deviation one ideals, Amer. J. Math. **116** (1994), 905-919.

[4] S. Goto, Y. Nakamura and K. Nishida, Cohen-Macaulay graded rings associated to ideals, Amer. J. Math. **118** (1996), 1197-1213.

[5] S. Huckaba and T. Marley, Depth formulas for certain graded rings associated to an ideal, Nagoya Math. J. **133** (1994), 57-69.

[6] M. Johnson and B. Ulrich, Artin-Nagata properties and Cohen-Macaulay associated graded rings, Compositio Math. **103** (1996), 7-29.

[7] D.G. Northcott and D. Rees, Reductions of ideals in local rings, Math. Proc. Camb. Phil. Soc. **50** (1954), 145-158.

[8] N.V. Trung, Reduction exponent and degree bound for the defining equations of graded rings, Proc. Amer. Math. Soc. **101**, No. 2 (1987), 229-236.

[9] N.V. Trung, The Castelnuovo regularity of the Rees algebra and the associated graded ring, Trans. Amer. Math. Soc. **350**, No. 7 (1988), 2813-2832.

[10] P. Valabrega and G. Valla, Form Rings and regular sequences, Nagoya Math. J. **72** (1978), 93-101.

[11] W.V. Vasconcelos, Arithmetic of blow up algebras, London Mathematical Society lecture Notes Series **195**, Cambridge University Press, 1994.

[12] S. Zarzuela, On the depth of blow up algebras of ideals with analytic deviation one, Proc. Amer. Math. Soc. **123**, No. 2 (1995), 3639-3647.

Department of Mathematics, University of Missouri, Columbia MO 65211.

E-mail address: aberbach@math.missouri.edu, ghezzi@math.missouri.edu, tai@math.missouri.edu