

Resolutions of square-free monomial ideals via facet ideals: a survey

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ABSTRACT. We survey some recent results on the minimal graded free resolution of a square-free monomial ideal. The theme uniting these results is the point-of-view that the generators of a monomial ideal correspond to the maximal faces (the facets) of a simplicial complex Δ . This correspondence gives us a new method, distinct from the Stanley-Reisner correspondence, to associate to a square-free monomial ideal a simplicial complex. In this context, the monomial ideal is called the facet ideal of Δ . Of particular interest is the case that all the facets have dimension one. Here, the simplicial complex is a simple graph G , and the facet ideal is usually called the edge ideal of G . Many people have been interested in understanding how the combinatorial data or structure of Δ appears in or affects the minimal graded free resolution of the associated facet ideal. In the first part of this paper, we describe the current state-of-the-art with respect to this program by collecting together many of the relevant results. We sketch the main details of many of the proofs and provide pointers to the relevant literature for the remainder. In the second part we introduce some open questions which will hopefully inspire future research on this topic.

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2000 *Mathematics Subject Classification.* 13D02, 13D40, 05C90, 05E99.

Key words and phrases. simple graphs, simplicial complexes, monomial ideals, edge ideals, facet ideals, resolutions, Betti numbers.

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1. Introduction

Let I be a *monomial ideal* in a polynomial ring $R = k[x_1, \dots, x_n]$. Associated to I is a *minimal graded free resolution* of the form

$$0 \rightarrow \bigoplus_j R(-j)^{\beta_{l,j}(I)} \rightarrow \bigoplus_j R(-j)^{\beta_{l-1,j}(I)} \rightarrow \dots \rightarrow \bigoplus_j R(-j)^{\beta_{0,j}(I)} \rightarrow I \rightarrow 0$$

where $l \leq n$ and $R(-j)$ is the free R -module obtained by shifting the degrees of R by j . The number $\beta_{i,j}(I)$, the ij th *graded Betti number* of I , equals the number of minimal generators of degree j in the i th syzygy module of I .

A classical topic in commutative algebra is to understand how the graded Betti numbers in the minimal free resolution of a monomial ideal depend upon the generators of the ideal. This problem continues to inspire current research; we refer the reader to Miller and Sturmfels' book [23] and Villarreal's book [33] for background, descriptions of various approaches to the problem, and many relevant references to ongoing work.

This paper surveys a new perspective on the problem of understanding the resolution of a monomial ideal that has appeared since [23]. The new point-of-view relates the graded Betti numbers of monomial ideals to combinatorial objects described by the generators of the monomial ideal. More precisely, we know from work of Fröberg [12] that the study of graded Betti numbers of monomial ideals can be reduced to understanding the case that I is generated by *square-free monomials*. We then have a bijection between the sets

$$\{\text{simplicial complexes } \Delta \text{ on } n \text{ vertices}\} \leftrightarrow \left\{ \begin{array}{l} \text{square-free monomial} \\ \text{ideals } I \subseteq R = k[x_1, \dots, x_n] \end{array} \right\}$$

given by

$$\Delta \xrightarrow{1-1} \mathcal{I}(\Delta) = \left\langle \prod_{x \in F} x \mid F \text{ is a } \textit{facet} \text{ (a maximal face) of } \Delta \right\rangle.$$

The ideal $\mathcal{I}(\Delta)$ is called the *facet ideal* of Δ . We then wish to describe the minimal graded free resolution of $\mathcal{I}(\Delta)$ in terms of the combinatorial data of Δ . The novelty of this approach is to view the generators of a square-free monomial ideal as the maximal faces of the simplicial complex. This contrasts with the usual Stanley-Reisner correspondence which associates to a simplicial complex Δ the square-free monomial ideal I_Δ generated by the minimal *non-faces* of Δ .

When all the facets of Δ have dimension 1, Δ can be viewed as a simple graph G (a graph with no loops or multiple edges) on n vertices. We shall usually write $\mathcal{I}(G)$ in this case for $\mathcal{I}(\Delta)$, and we call $\mathcal{I}(G)$ the *edge ideal* of G . Observe that $\mathcal{I}(G)$ is generated by square-free quadratic monomials and is the first non-trivial case of a square-free monomial ideal. Historically, edge ideals were introduced by Villarreal in [32] before facet ideals. Facet ideals, which can be seen as a generalization of edge ideals, were introduced later by Faridi in [5]. To learn more about the properties of edge ideals, one should see [10, 15, 16, 17, 25, 27, 29, 28, 31, 33]; further properties of facet ideals can be found in the sequels [6, 7, 8] to Faridi's paper cited above.

Eliahou and Villarreal [4] provided one of the first examples showing that the numbers $\beta_{i,j}(\mathcal{I}(\Delta))$ could be described in terms of the combinatorial data of Δ . Specifically, it was shown that the number $\beta_{1,3}(\mathcal{I}(G))$ could be computed from the degrees of the vertices of G and the number of triangles of G . Although Fröberg's paper [11] predates the notion of an edge ideal, Fröberg demonstrated that a facet ideal point-of-view could provide information about the resolution of a monomial ideal; using the language of edge ideals, the ideal $\mathcal{I}(G)$ has a linear resolution if and only if G^c , the complement of G , is a *chordal* graph. Since 2003, there has been a flurry of results on the resolutions of facet (and edge) ideals. We mention in particular Eisenbud, Green, Hulek, Popescu [2], Jacques [20], Jacques and Katzman [21], Katzman [22], Zheng [34], the second author and Roth [26], and the two authors [14].

The goal of this survey paper is two-fold. Our first goal is to provide a summary of the state-of-the-art on the resolutions of facet and edge ideals. This is a relatively new area, and we hope to gather together most of the relevant results currently available in the literature. We will not concentrate on proving these results, but rather, we try to develop a unified and systematic perspective to these studies. From time to time, we shall give the reader a flavor of the proofs by sketching out important ideas.

As we see it, there are two main themes in the literature. The first theme is to study the structure of the resolution of edge and facet ideals. For example, Katzman [22] showed that the minimal free resolution of edge ideals of certain graphs depends upon the characteristic of k ; Eisenbud, et. al. [2] characterized property $N_{2,p}$ of edge ideals via the cycle structure of complement graphs; and Zheng [34] calculated the regularity of the edge ideal of a forest from the number of disconnected edges in the graph. The second theme is to give explicit computations for various graded Betti numbers. For instance, Jacques [20] provided formulas for the graded Betti numbers of edge ideals of special classes graphs including cycles and complete graphs; Roth and the second author [26] calculated the graded Betti numbers in the linear strand of edge ideals of graphs having no minimal cycle of length 4; Zheng [34] computed the graded Betti numbers in the linear strand of facet ideals of simplicial forests; and the two authors [14] derived recursive-like formulas for graded Betti numbers of the edge ideal of a graph in terms of its subgraphs.

To present the known results, we have divided the results into two main categories based upon the techniques used in their proof. In Section 3, we describe what results have been obtained using the theory of reduced simplicial homology.

In particular, we describe results that one can obtain using Hochster's formula and its variant, the formula of Eagon and Reiner. In Section 4 we describe the results on resolutions of facet ideals which rely on the notion of a splittable ideal (first introduced by Eliahou and Kervaire [3]).

The second goal of this paper is based upon the belief that a good survey should also inspire future research on a topic. To this end, we provide in Section 5 a collection of open questions that we would like answered. The questions have been grouped into four broad categories. The first category is related to building a dictionary between the graded Betti numbers $\beta_{i,j}(\mathcal{I}(\Delta))$ and the combinatorial data of Δ . The second category discusses questions on how the characteristic of the ground field affects the numbers $\beta_{i,j}(\mathcal{I}(\Delta))$. The third group of questions deal with other homological invariants such as regularity and projective dimension. The final category is a series of questions based upon the authors' paper [14] and the notion of splittable ideals.

It is our hope that the reader will be convinced that there is an advantage of viewing square-free monomial ideals as the facet ideal of a simplicial complex. However, we do wish to point out some natural limitations of this approach. G. Reisner [24] showed that the square-free monomial ideal

$$\begin{aligned} \mathcal{I}(\Delta) = & (x_1x_2x_3, x_1x_2x_4, x_1x_3x_5, x_1x_4x_6, x_1x_5x_6, x_2x_3x_6, \\ & x_2x_4x_5, x_2x_5x_6, x_3x_4x_5, x_3x_4x_6) \end{aligned}$$

in $R = k[x_1, \dots, x_6]$ is an ideal whose resolution depends upon $\text{char}(k)$. Specifically, $\mathcal{I}(\Delta)$ has a linear resolution except in the case that $\text{char}(k) = 2$. This example illustrates that knowing the combinatorial object Δ will not always be enough to understand the minimal graded free resolution (see also [22] for an example involving an edge ideal). So although we cannot expect a theory that relates the data of every simplicial complex Δ to the resolution of $\mathcal{I}(\Delta)$, we do not feel that this diminishes the importance of this new perspective. As shown in this paper, the combinatorial data is enough to describe the entire (or part of the) resolution in many interesting cases, and we are of the belief that many more interesting cases remain to be discovered using this approach.

2. Preliminaries

To make this paper as self-contained as possible, we have included the relevant definitions about graphs, simplicial complexes, and minimal resolutions.

2.1. Graphs. Let G denote a graph with vertex set V_G and edge set E_G . We shall say that G is *simple* if G has no loops or multiple edges. A simple graph need not be connected. We shall let $\#\text{comp}(G)$ denote the number of connected components of G . The *degree* of a vertex $x \in V_G$, denoted by $\deg x$, is the number of edges incident to x .

If $S \subseteq V_G$, then the *induced subgraph* of G on the vertex set S , denoted by G_S , is the subgraph of G whose vertex set is S and whose edge set consists of edges of G connecting two vertices in S . The *complement* of a graph G , denoted by G^c , is the graph whose vertex set is the same as G , but whose edge set is defined by the rule: $\{x_i, x_j\} \in E_{G^c}$ if and only if $\{x_i, x_j\} \notin E_G$.

We write xy as a shorthand for the edge $\{x, y\}$. A *cycle* of length q in a graph G is a sequences of edges $\{e_1 = x_1x_2, e_2 = x_2x_3, \dots, e_{q-1} = x_{q-1}x_q, e_q = x_qx_1\}$ in G (where $x_i \neq x_j$ for $i \neq j$). We use $(x_1x_2 \dots x_qx_1)$ to denote a cycle of length q with vertices x_1, \dots, x_q . We sometimes also use C_q to refer to a cycle of length q . We say that a cycle $C = (x_1x_2 \dots x_qx_1)$ of G has a *chord* if there exists some $j \not\equiv i + 1 \pmod{q}$ such that x_ix_j is an edge of G . We call a cycle C in G a *minimal cycle* if C has length at least 4 and contains no chord.

A *forest* is any graph with no cycles; a *tree* is a connected forest. The *wheel* W_n is the graph obtained by adding a vertex z to C_n and then adjoining an edge between z and every vertex in V_{C_n} . Note that W_n has $n + 1$ vertices. The *complete graph* on n vertices, denoted \mathcal{K}_n , is the graph with the property that for all $x_i, x_j \in V_{\mathcal{K}_n}$ with $i \neq j$, the edge $x_ix_j \in E_{\mathcal{K}_n}$. The *complete bipartite graph*, denoted $\mathcal{K}_{n,m}$, is the graph with vertex set $V_G = \{x_1, \dots, x_n, y_1, \dots, y_m\}$ and edge set $E_G = \{x_ix_j \mid 1 \leq i \leq n, 1 \leq j \leq m\}$. We shall write $c_n(G), w_n(G), k_n(G)$, and $k_{n,m}(G)$ for the number of induced subgraphs of G isomorphic to C_n, W_n, \mathcal{K}_n , and $\mathcal{K}_{n,m}$, respectively.

2.2. Simplicial complexes. A *simplicial complex* Δ on a vertex set V_Δ is a collection of subsets of V_Δ such that for all $x \in V_\Delta$, $\{x\} \in \Delta$, and for each $F \in \Delta$, if $G \subseteq F$, then $G \in \Delta$. Note that $\emptyset \in \Delta$, except in the case that $\Delta = \{\}$ is the void complex (see [23, Definition 1.4]).

An element of a simplicial complex Δ is called a *face* of Δ . The *dimension* of a face F of Δ , denoted $\dim F$, is defined to be $|F| - 1$, where $|F|$ denotes the number of vertices in F . The *dimension* of Δ , denoted by $\dim \Delta$, is defined to be the maximal dimension of a face in Δ . The maximal faces of Δ under inclusion are called *facets* of Δ . If all facets of Δ have the same dimension d , then Δ is said to be *pure d -dimensional*.

We usually denote the simplicial complex Δ with facets F_1, \dots, F_q by

$$\Delta = \langle F_1, \dots, F_q \rangle;$$

here, the set $\mathcal{F}(\Delta) = \{F_1, \dots, F_q\}$ is often referred to as the *facet set* of Δ . There is a one-to-one correspondence between simplicial complexes and their facet sets. If we realize a simplicial complex Δ by its facet set $\mathcal{F}(\Delta)$, then a graph G can be thought of as a simplicial complex with the facet set being its edge set. Suppose F is a facet of Δ , say $F = F_q$, then we denote by $\Delta \setminus F$ the simplicial complex obtained by removing F from $\mathcal{F}(\Delta)$, i.e., $\Delta \setminus F = \langle F_1, \dots, F_{q-1} \rangle$. Throughout the paper, by a *subcomplex* of a simplicial complex Δ , we shall mean a simplicial complex whose facet set is a subset of the facet set of Δ . If Δ' is a subcomplex of Δ , then we denote by $\Delta \setminus \Delta'$ the simplicial complex obtained from Δ by removing from its facet set all facets of Δ' .

We say that two facets F and G of Δ are *connected* if there exists a *chain* of facets of Δ , $F = F_0, F_1, \dots, F_m = G$, such that $F_i \cap F_{i+1} \neq \emptyset$ for any $i = 0, \dots, m - 1$. The simplicial complex Δ is said to be *connected* if any two of its facets are connected.

A facet F of a simplicial complex Δ is a *leaf* of Δ if either F is the only facet of Δ , or there exists a facet G in Δ , $G \neq F$, such that $F \cap G' \subseteq F \cap G$ for every facet $G' \in \Delta$, $G' \neq F$. It follows from [5, Remark 2.3] that if F is a leaf of Δ , then F must contain a vertex that does not belong to any other facet of the complex

(the converse is not true though). A simplicial complex Δ is called a *tree* if Δ is connected and every nonempty connected subcomplex of Δ (including Δ itself) has a leaf. We call Δ a *forest* if every connected component of Δ is a tree.

If Δ is a simplicial complex over a vertex set $V_\Delta = \{x_1, \dots, x_n\}$, then we can associate to Δ two ideals in the polynomial ring $R = k[x_1, \dots, x_n]$, the facet ideal and the Stanley-Reisner ideal (by abuse of notation, we shall use x_i s to denote both the vertices of Δ and the variables in the polynomial ring). For a face F of Δ , we write x^F to denote the monomial $\prod_{x \in F} x$ in R . The *facet ideal* of Δ is defined to be

$$\mathcal{I}(\Delta) = \langle x^F \mid F \in \mathcal{F}(\Delta) \rangle \subseteq R,$$

and the *Stanley-Reisner ideal* of Δ is defined to be

$$I_\Delta = \langle x^F \mid F \subseteq V_\Delta, F \notin \Delta \rangle \subseteq R.$$

Finally, we can associate to any graph a simplicial complex. If G is any graph, then the *clique complex* of G is the simplicial complex $\Delta(G)$ where $F = \{x_{i_1}, \dots, x_{i_j}\} \in \Delta(G)$ if and only if G_F is a complete graph. Note that if G is a simple graph with edge ideal $\mathcal{I}(G)$, then the ideal $\mathcal{I}(G)$ is generated by square-free monomials. So $\mathcal{I}(G)$ is also the Stanley-Reisner ideal of a simplicial complex Δ , that is, $\mathcal{I}(G) = I_\Delta$. Specifically, $\mathcal{I}(G)$ is the Stanley-Reisner ideal associated to the clique complex $\Delta = \Delta(G^c)$ constructed from the complement of G .

2.3. Minimal resolutions. Let $R = k[x_1, \dots, x_n]$ and let M be a graded R -module. Associated to M is a *minimal graded free resolution* of the form

$$0 \rightarrow \bigoplus_j R(-j)^{\beta_{l,j}(M)} \rightarrow \bigoplus_j R(-j)^{\beta_{l-1,j}(M)} \rightarrow \dots \rightarrow \bigoplus_j R(-j)^{\beta_{0,j}(M)} \rightarrow M \rightarrow 0$$

where $l \leq n$ and $R(-j)$ is the free R -module obtained by shifting the degrees of R by j . The number $\beta_{i,j}(M)$, the ij th *graded Betti number* of M , equals the number of minimal generators of degree j in the i th syzygy module of M .

If $\dots \rightarrow \mathcal{F}_1 \rightarrow \mathcal{F}_0 \rightarrow I \rightarrow 0$ is a minimal graded free resolution of I , then $\dots \rightarrow \mathcal{F}_1 \rightarrow \mathcal{F}_0 \rightarrow R \rightarrow R/I \rightarrow 0$ is a minimal graded free resolution of R/I . From this observation it follows that $\beta_{i,j}(R/I) = \beta_{i-1,j}(I)$ for all $i, j \geq 0$. Here, we will adopt the convention that $\beta_{-1,0}(I) = 1$ and $\beta_{-1,j}(I) = 0$ for any $j > 0$.

The *regularity* of I , denoted $\text{reg}(I)$, is defined to be

$$\text{reg}(I) := \max\{j - i \mid \beta_{i,j}(I) \neq 0\}.$$

The *projective dimension* of I , denoted $\text{pd}(I)$, is defined to be

$$\text{pd}(I) := \max\{i \mid \beta_{i,j}(I) \neq 0\}.$$

These invariants measure the “size” of the minimal graded free resolution.

If d is the smallest degree of a generator of an ideal I , then the Betti numbers $\beta_{i,i+d}(I)$ form the so-called *linear strand* of I (see [2, 18]). An ideal I generated by elements all of degree d is said to have a *linear resolution* if $\beta_{i,j}(I) = 0$ for all $j \neq i + d$, that is, the only nonzero graded Betti numbers are those in the linear strand.

EXAMPLE 2.3.1. (The numbers $\beta_{0,j}(\mathcal{I}(\Delta))$) The number $\beta_{0,j}(\mathcal{I}(\Delta))$ is simply the number of generators of $\mathcal{I}(\Delta)$ of degree j . From our construction of $\mathcal{I}(\Delta)$ it follows that $\beta_{0,j}(\mathcal{I}(\Delta)) =$ number of facets of Δ of dimension $j - 1$. When $\Delta = G$

is a simple graph, then $\beta_{0,j}(\mathcal{I}(G))$ equals $|E_G|$, the number of edges if $j = 2$, and equals 0 if $j \neq 2$. Since the relation between the 0th graded Betti numbers and the combinatorics of Δ is well understood, we can restrict to studying the numbers $\beta_{i,j}(\mathcal{I}(\Delta))$ with $i \geq 1$.

EXAMPLE 2.3.2. (Disconnected simplicial complexes) Suppose that Δ is a simplicial complex that is the disjoint union of two components, i.e. $\Delta = \Delta_1 \cup \Delta_2$ with $\Delta_1 \cap \Delta_2 = \emptyset$. If $V_{\Delta_1} = \{x_1, \dots, x_n\}$ and $V_{\Delta_2} = \{y_1, \dots, y_m\}$, then $\mathcal{I}(\Delta) = \mathcal{I}(\Delta_1) + \mathcal{I}(\Delta_2)$ and

$$\frac{k[x_1, \dots, x_n, y_1, \dots, y_m]}{\mathcal{I}(\Delta)} \cong \frac{k[x_1, \dots, x_n]}{\mathcal{I}(\Delta_1)} \otimes_k \frac{k[y_1, \dots, y_m]}{\mathcal{I}(\Delta_2)}.$$

It then follows that the graded Betti numbers of $\mathcal{I}(\Delta)$ can be calculated by finding the resolutions of $\mathcal{I}(\Delta_1)$ and $\mathcal{I}(\Delta_2)$ and then tensoring the two resolutions together (see [21, Lemma 2.1] for details). The upshot is that to study the numbers $\beta_{i,j}(\mathcal{I}(\Delta))$ one can make the harmless assumption that Δ is connected.

EXAMPLE 2.3.3. (Resolutions of $\mathcal{I}(\mathcal{K}_{1,d})$) Suppose $G = \mathcal{K}_{1,d}$ is the complete bipartite graph of size $1, d$. If x denotes the vertex of degree d , and x_i one of the vertices of degree 1 for $i = 1, \dots, d$, then $\mathcal{I}(G) = (xx_1, \dots, xx_d) \subseteq R = k[x, x_1, \dots, x_d]$. Since $\mathcal{I}(G) = x(x_1, \dots, x_d)$ we have $\beta_{i,j}(\mathcal{I}(G)) = \beta_{i,j-1}((x_1, \dots, x_d))$. But because x_1, \dots, x_d is a regular sequence on R , the numbers $\beta_{i,j-1}((x_1, \dots, x_d))$ are a consequence of the Koszul resolution; specifically

$$\beta_{i,j}(\mathcal{I}(G)) = \begin{cases} \binom{d}{i+1} & \text{if } j = i + 2 \\ 0 & \text{otherwise.} \end{cases}$$

3. Results via reduced simplicial homology

Given a square-free monomial ideal I , one can associate to I two simplicial complexes. The first is the simplicial complex Δ with $I = \mathcal{I}(\Delta)$; the second is the simplicial complex Δ' with $I = I_{\Delta'}$ via the standard Stanley-Reisner correspondence. The starting point for most of the known results in the literature about the graded Betti numbers of a facet ideal $\mathcal{I}(\Delta)$ is to first describe the associated simplicial complex Δ' with the property that $\mathcal{I}(\Delta) = I_{\Delta'}$. Then one appeals to such results as the formulas of Hochster and Eagon-Reiner to describe the graded Betti numbers of $I_{\Delta'}$ in terms of the reduced simplicial homology of Δ' . Finally, one translates these results back in terms of the simplicial complex Δ . This approach has proven extremely useful in describing the graded Betti numbers of edge ideals of a simple graphs G due, in part, to the fact that we know that $\Delta' = \Delta(G^c)$, the clique complex of the corresponding complement graph. In this section, we shall describe what formulas and results about facet ideals have been derived via this approach.

3.1. Hochster's formula and Eagon-Reiner's formula. Hochster's and Eagon-Reiner's formulas describe the graded Betti numbers of a square-free monomial ideal I in terms of the simplicial complex Δ where $I = \mathcal{I}(\Delta)$. We recall these methods below.

Let Δ be a simplicial complex over a vertex set $V_\Delta = \{x_1, \dots, x_n\}$. Then

$$I_\Delta = \langle x^F \mid F \subseteq V_\Delta, F \notin \Delta \rangle \subseteq R = k[x_1, \dots, x_n]$$

is the Stanley-Reisner ideal of Δ . We can view R as an \mathbb{N}^n -graded k -algebra and I_Δ as a \mathbb{Z}^n -graded R -module. For a monomial m of $R = k[x_1, \dots, x_n]$ having multidegree $\alpha \in \mathbb{N}^n$, we define

$$\mathrm{Tor}_i^R(I_\Delta, k)_m := \mathrm{Tor}_i^R(I_\Delta, k)_\alpha.$$

Also, for $W \subseteq V_\Delta$ we write $\Delta_W = \{F \in \Delta \mid F \subseteq W\}$ for the restriction of Δ to W . It follows that Δ_W is a simplicial complex on W . If m is a square-free monomial ideal, and W denotes the set of variables that divide m , then we will write $|m|$ for Δ_W . Hochster [19] provided the following formula to calculate $\beta_{i,j}(I_\Delta)$.

THEOREM 3.1.1 (Hochster's Formula). *Let Δ be a simplicial complex on the vertex set $V_\Delta = \{x_1, \dots, x_n\}$ and let m be a monomial of R . If m is square-free, then*

$$\dim_k \mathrm{Tor}_i^R(I_\Delta, k)_m = \dim_k \tilde{H}_{\deg(m)-i-2}(|m|, k)$$

where $\tilde{H}_j(|m|, k)$ denotes the j th reduced homology of $|m|$. If m is not square-free, then $\mathrm{Tor}_i^R(I_\Delta, k)_m$ vanishes. In particular

$$\beta_{i,j}(I_\Delta) = \sum_{\deg m = j, m \text{ is square-free}} \dim_k \tilde{H}_{j-i-2}(|m|, k) \text{ for all } i, j \geq 0.$$

Eagon and Reiner [1] introduced a variant of Hochster's formula that uses Alexander duality. The *Alexander dual* of a simplicial complex Δ is given by

$$\Delta^\vee := \{\{x_1, \dots, x_n\} \setminus F \mid F \notin \Delta\}.$$

Moreover, if $F \in \Delta$, then the *link* of F in Δ , denoted by $\mathrm{Link}_\Delta F$, is the simplicial complex defined by

$$\mathrm{Link}_\Delta F := \{G \in \Delta \mid G \cup F \in \Delta \text{ and } G \cap F = \emptyset\}.$$

THEOREM 3.1.2 (Eagon-Reiner's Formula). *Let Δ be a simplicial complex. Then*

$$\beta_{i,j}(I_\Delta) = \sum_{F \in \Delta^\vee, |F|=n-j} \dim_k \tilde{H}_{i-1}(\mathrm{Link}_{\Delta^\vee} F, k) \text{ for all } i, j \geq 0.$$

As shown by Jacques [20], Jacques and Katzman [21], and Katzman [22], it sometimes is easier to compute the graded Betti numbers of edge ideals via the Eagon-Reiner reformulation of Hochster's formula because it is easier to deal with the reduced homology of the simplicial complexes $\mathrm{Link}_{\Delta^\vee} F$.

EXAMPLE 3.1.3. (Characteristic dependence of Betti numbers) From the conclusions of Theorems 3.1.1 and 3.1.2 it becomes clear that the characteristic of the field simply cannot be ignored since the characteristic may introduce some nonzero torsion. An example due to Reisner can be found in the introduction. Katzman [22] shows that even if $\mathcal{I}(G)$ is an edge ideal, the graded Betti numbers are not independent of $\mathrm{char}(k)$. Katzman gave the example of a graph G with edge ideal

$$\begin{aligned} \mathcal{I}(G) = & (x_1x_5, x_1x_6, x_1x_8, x_1x_{10}, x_2x_5, x_2x_6, x_2x_9, x_2x_{11}, x_3x_7, x_3x_8, x_3x_9, x_3x_{11}, \\ & x_4x_7, x_4x_8, x_4x_{10}, x_4x_{11}, x_5x_8, x_5x_9, x_6x_{10}, x_6x_{11}, x_7x_9, x_7x_{10}, x_8x_{11}) \end{aligned}$$

whose graded Betti numbers are different if $\mathrm{char}(k) = 2$ than those of the ideal when $\mathrm{char}(k) = 0$. The first example of an edge ideal whose Betti numbers depends upon the characteristic was first given by Hibi and is based upon Reisner's example. See Section 5 for some questions about this topic.

3.2. Graded Betti numbers of edge ideals. Recall that when G is a simple graph we have $\mathcal{I}(G) = I_\Delta$, where $\Delta = \Delta(G^c)$ is the clique complex of the complementary graph G^c . This observation allows us to use Hochster's and Eagon-Reiner's formulas to compute the graded Betti numbers of $\mathcal{I}(G)$ (a similar formula is found in [22]).

THEOREM 3.2.1. *Let G be a simple graph with edge ideal $\mathcal{I}(G)$. Then*

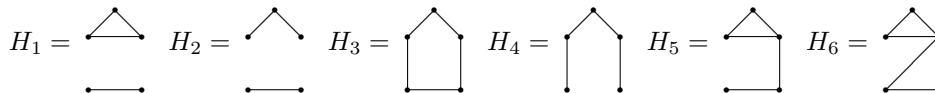
$$\beta_{i,j}(\mathcal{I}(G)) = \sum_{S \subseteq V_G, |S|=j} \dim_k \tilde{H}_{j-i-2}(\Delta(G_S^c), k) \text{ for all } i, j \geq 0.$$

PROOF. We sketch out the main idea of the proof. Let $\Delta = \Delta(G^c)$ be the simplicial complex defined by $\mathcal{I}(G)$. It follows from Proposition 3.1.1 that

$$\beta_{i,j}(\mathcal{I}(G)) = \beta_{i,j}(I_\Delta) = \sum_{m \in M_j, m \text{ is square-free}} \dim_k \tilde{H}_{j-i-2}(|m|, k)$$

where M_j consists of all the monomials of degree j in R . Since $\deg m = j$ and m is square-free, the variables that divide m give a subset $S \subseteq V_G$ of size j . Let G_S denote the induced subgraph of G on this vertex set S , and let G_S^c denote its complement. To finish the proof, it is enough to note that $|m|$, the restriction of $\Delta(G^c)$ to S , and $\Delta(G_S^c)$ are the same simplicial complex. \square

EXAMPLE 3.2.2. We illustrate how to apply the above theorem to compute $\beta_{2,5}(\mathcal{I}(G))$. Below are all the graphs H on 5 vertices with $\dim_k \tilde{H}_1(\Delta(H^c), k) > 0$:



For $i = 2, \dots, 6$, $\dim_k \tilde{H}_1(\Delta(H_i^c), k) = 1$, and $\dim_k \tilde{H}_1(\Delta(H_1^c), k) = 2$ (these numbers are independent of k). If we let $h_i(\Gamma)$ denote the number of induced subgraphs of the graph Γ isomorphic to H_i , then by the Theorem 3.2.1 we have

$$\beta_{2,5}(\mathcal{I}(G)) = 2h_1(G) + h_2(G) + h_3(G) + h_4(G) + h_5(G) + h_6(G).$$

As evident by the previous example, the formula of Theorem 3.2.1 is difficult to apply since one has to compute the dimensions of all the homology groups $\tilde{H}_{j-i-2}(\Delta(G_S^c), k)$ as S varies over all subsets of V_G of size j . It is also not immediately clear how this formula relates to combinatorial data, like degrees of the vertices or the number of cliques, associated to G . However, in special cases, we can still tease out interesting conclusions (as we will see below).

We now show how Theorem 3.2.1 (and other tools) can be used to give exact formulas for some of the graded Betti numbers of $\mathcal{I}(G)$ in terms of G . We begin by observing that we can restrict our search for graded Betti numbers to particular ranges.

THEOREM 3.2.3. *Let G be a simple graph with edge ideal $\mathcal{I}(G)$. If $\beta_{i,j}(\mathcal{I}(G)) \neq 0$, then $i + 2 \leq j \leq 2(i + 1)$.*

PROOF. Since $\mathcal{I}(G)$ is generated by quadrics, $\beta_{i,j}(\mathcal{I}(G)) = 0$ if $j < i + 2$. By using the Taylor resolution it can also be seen that $\beta_{i,j}(\mathcal{I}(G)) = 0$ for all $j > 2(i + 1)$ (see [22] for details). \square

For the extremal values of j , that is, $j = i + 2$ or $j = 2(i + 1)$, we can compute $\beta_{i,j}(\mathcal{I}(G))$ in terms of data from G for each i . Observe that these numbers are independent of $\text{char}(k)$ since they only depend the graph G .

THEOREM 3.2.4. *Let G be a simple graph with edge ideal $\mathcal{I}(G)$. Then for all $i \geq 0$*

$$\begin{aligned} \beta_{i,i+2}(\mathcal{I}(G)) &= \sum_{S \subseteq V_G, |S|=i+2} (\#\text{comp}(G_S^c) - 1) \text{ and} \\ \beta_{i,2(i+1)}(\mathcal{I}(G)) &= \left| \left\{ H \mid \begin{array}{l} H \text{ is a induced subgraph of } G \\ \text{consisting of } i+1 \text{ disjoint edges} \end{array} \right\} \right|. \end{aligned}$$

PROOF. The formula for $\beta_{i,i+2}(\mathcal{I}(G))$ is given in [26, Proposition 2.1]; it is a consequence of evaluating the formula of Theorem 3.2.1 at $j = i + 2$ and using the fact that $\dim_k \tilde{H}_0(\Gamma, k) + 1$ is the number of connected components of Γ . The formula for $\beta_{i,2(i+1)}(\mathcal{I}(G))$ comes from [22, Lemma 2.2] and relies on the Taylor resolution. \square

REMARK 3.2.5. From Theorem 3.2.4 the length of the linear strand is given by

$$\ell = \max\{i \mid \text{there exists } S \subseteq V_G \text{ with } |S| = i + 2 \text{ and } \#\text{comp}(G_S^c) > 1\}.$$

Theorem 3.2.4 gives us a means to compute the graded Betti numbers in the linear strand of $\mathcal{I}(G)$. However, one is required to sum over all subgraphs of a certain size which limits the usefulness of the result.

Roth and the second author [26] showed that in many cases one can find equivalent (and more easy to calculate) formulas for the graded Betti numbers in the linear strand. Their results are based upon the following decomposition for the formula for $\beta_{i,i+2}(\mathcal{I}(G))$:

$$(3.1) \quad \beta_{i,i+2}(\mathcal{I}(G)) = \sum_{\substack{S \subseteq V_G, \\ |S|=i+2, \\ G_S^c \text{ contains an} \\ \text{isolated vertex}}} (\#\text{comp}(G_S^c) - 1) + \sum_{\substack{S \subseteq V_G, \\ |S|=i+2, \\ G_S^c \text{ contains no} \\ \text{isolated vertices}}} (\#\text{comp}(G_S^c) - 1).$$

Recall that $k_i(G)$ is the number of induced subgraphs of G isomorphic to \mathcal{K}_i .

THEOREM 3.2.6. [26, Proposition 2.4] *Let G be a simple graph with edge ideal $\mathcal{I}(G)$. If G has no minimal 4-cycles, then*

$$\beta_{i,i+2}(\mathcal{I}(G)) = \sum_{v \in V_G} \binom{\deg v}{i+1} - k_{i+2}(G) \text{ for all } i \geq 0.$$

Furthermore, the above formula holds for all simple graphs G if $i = 0$ or 1 .

PROOF. The proof has two steps. The first step is to show that the second sum in (3.1) is 0. So, suppose that $S \subseteq V_G$ is such that G_S^c has no isolated vertex. We claim that G_S^c cannot have two connected components. By checking all possible graphs on 2 or 3 vertices, this is easy to see if $|S| = 2, 3$. If $|S| \geq 4$ and if G_S^c has at least two connected components, then there must be at least two edges with each edge in a different connected component. Let S' be the set of four vertices incident to these two edges. Then $G_{S'}$ is a minimal 4-cycle, contradicting our

hypothesis. Thus, if G_S^c has no isolated vertex, then G_S^c is connected, and thus makes no contribution to $\beta_{i,|S|}(\mathcal{I}(G))$.

The second step is to evaluate the first sum. We first observe that if G_S^c has an isolated vertex, it can have at most one connected component consisting of one or more edges. If G_S^c had two or more connected components having an edge, then by argument similar to the one given above, this would imply that G has a minimal 4-cycle. Therefore, to count $\#\text{comp}(G_S^c) - 1$ we can simply count the number of isolated vertices in G_S^c , the “ -1 ” term being taken care of by the component which is not a vertex; we thus overcount by one whenever G_S^c consists completely of isolated vertices.

For any vertex v , the number of subsets S of size $i + 2$ containing v such that v is an isolated vertex in G_S^c is $\binom{\deg v}{i+1}$. To take care of the overcount, note that G_S^c consists of isolated vertices exactly when G_S is a complete graph on $i + 2$ vertices. Subtracting the number of times this happens gives the formula above.

To compute $\beta_{i,i+2}(\mathcal{I}(G))$ when $i = 0$ or 1 , we need to count the number of connected components of G_S^c when $|S| = 2$ or 3 . But for any simple graph on two or three vertices, at most one connected component can be larger than a vertex. The proof is now the same as the one given above. \square

Since a forest has no cycles (and hence, no induced subgraphs isomorphic to \mathcal{K}_j with $j \geq 3$) we obtain:

COROLLARY 3.2.7. [26, Corollary 2.6] *Let G be a forest with edge ideal $\mathcal{I}(G)$. Then $\beta_{0,2}(\mathcal{I}(G)) = |E_G|$, and*

$$\beta_{i,i+2}(\mathcal{I}(G)) = \sum_{v \in V_G} \binom{\deg v}{i+1} \text{ for all } i \geq 1.$$

EXAMPLE 3.2.8. (The numbers $\beta_{1,j}(\mathcal{I}(G))$) The above results allow us to completely describe $\beta_{1,j}(\mathcal{I}(G))$ for all j and all graphs G :

$$\beta_{1,j}(\mathcal{I}(G)) = \begin{cases} \sum_{v \in V_G} \binom{\deg v}{2} - k_3(G) & \text{if } j = 3 \\ c_4(G^c) & \text{if } j = 4 \\ 0 & \text{otherwise.} \end{cases}$$

The fact that $\beta_{1,j}(\mathcal{I}(G)) = 0$ if $j \neq 3, 4$ comes from Theorem 3.2.3. The formula for $\beta_{1,3}(\mathcal{I}(G))$ is just Theorem 3.2.6. By Theorem 3.2.4

$$\beta_{1,4}(\mathcal{I}(G)) = |\{H \mid H \text{ is an induced subgraph of } G \text{ with exactly 2 disjoint edges}\}|.$$

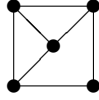
But any induced subgraph H that is exactly 2 disjoint edges corresponds to an induced cycle C_4 in G^c . So $\beta_{1,4}(\mathcal{I}(G))$ is the number of 4-cycles in G^c . The formula for $\beta_{1,3}(\mathcal{I}(G))$ was first proved in [4, Proposition 2.1].

By a careful analysis of the second sum in (3.1), Roth and the second author derived formulas for $\beta_{2,4}(\mathcal{I}(G))$ and $\beta_{3,5}(\mathcal{I}(G))$ for any simple graph G . The formula for $\beta_{2,4}(\mathcal{I}(G))$ verifies a conjecture found in [4, Conjecture 2.4].

THEOREM 3.2.9. [26, Proposition 2.8] *Let G be a simple graph with edge ideal $\mathcal{I}(G)$. Then*

$$\begin{aligned}\beta_{2,4}(\mathcal{I}(G)) &= \sum_{v \in V_G} \binom{\deg v}{3} - k_4(G) + k_{2,2}(G) \text{ and} \\ \beta_{3,5}(\mathcal{I}(G)) &= \sum_{v \in V_G} \binom{\deg v}{4} - k_5(G) + k_{2,3}(G) + w_4(G) + d(G)\end{aligned}$$

where $d(G)$ is the number of induced subgraphs of G isomorphic to the graph:



EXAMPLE 3.2.10. (The numbers $\beta_{2,j}(\mathcal{I}(G))$) By Theorem 3.2.3 the number $\beta_{2,j}(\mathcal{I}(G))$ is nonzero only if $j = 4, 5$, or 6 . The formula for $\beta_{2,4}(\mathcal{I}(G))$ can be found in the above theorem. We computed $\beta_{2,5}(\mathcal{I}(G))$ in Example 3.2.2. Finally, one can compute $\beta_{2,6}(\mathcal{I}(G))$ by using Theorem 3.2.4. Notice that the numbers $\beta_{2,j}(\mathcal{I}(G))$ are independent of $\text{char}(k)$ for all j .

When G has a minimal 4-cycle, we can compute upper and lower bounds on the graded Betti numbers in the linear strand.

THEOREM 3.2.11. [26, Proposition 3.1] *Let G be a simple graph with edge ideal $\mathcal{I}(G)$. Suppose that G has an induced 4-cycle. Then for all $i \geq 2$*

$$\beta_{i,i+2}(\mathcal{I}(G)) \geq \sum_{v \in V_G} \binom{\deg v}{i+1} - k_{i+2}(G) + k_{2,i}(G) + k_{3,i-1}(G) + \cdots + k_{\lfloor \frac{i+2}{2} \rfloor, \lceil \frac{i+2}{2} \rceil}(G).$$

PROOF. The first sum in (3.1) is bounded below by $\sum_{v \in V_G} \binom{\deg v}{i+1} - k_{i+2}(G)$, while the second sum is bounded below by $k_{2,i}(G) + k_{3,i-1}(G) + \cdots + k_{\lfloor \frac{i+2}{2} \rfloor, \lceil \frac{i+2}{2} \rceil}(G)$. \square

Recall that the elements of \mathbb{N}^n can be given a total ordering using the lexicographical order defined by $(a_1, \dots, a_n) > (b_1, \dots, b_n)$ if $a_1 = b_1, \dots, a_{i-1} = b_{i-1}$ but $a_i > b_i$. This induces an ordering on the monomials of R : $x_1^{a_1} \cdots x_n^{a_n} >_{\text{lex}} x_1^{b_1} \cdots x_n^{b_n}$ if $(a_1, \dots, a_n) > (b_1, \dots, b_n)$. A monomial ideal I is a *lex ideal* if for each $d \in \mathbb{N}$, a basis for I_d is the $\dim_k I_d$ largest monomials of degree d with respect to the lexicographical ordering.

THEOREM 3.2.12. [26, Proposition 3.2] *Let G be a simple graph with edge ideal $\mathcal{I}(G)$. If $\{m_1, \dots, m_{|E_G|}\}$ are the $|E_G|$ largest monomials of degree 2 in R with respect to the lexicographical ordering, then*

$$\beta_{i,i+2}(\mathcal{I}(G)) \leq \sum_{t=1}^{|E_G|} \binom{u_t - 1}{i}$$

where u_t is the largest index of a variable dividing m_t .

PROOF. One uses Eliahou-Kervaire's formula [3] for the growth of the graded Betti numbers of stable (and hence lex) ideals to bound $\beta_{i,i+2}(\mathcal{I}(G))$. \square

We now turn our attention to the global behavior of the resolutions. The algebraic invariants and properties in which we shall be interested include the regularity, the linear strand, and property $N_{2,p}$.

The first such result is due to Zheng which describes the regularity of a forest. Zheng introduced a notion of two edges being disconnected; precisely, two edges u_1v_1 and u_2v_2 of a simple graph G are *disconnected* if (a) the two edges do not share a common vertex, and (b) $u_1u_2, u_1v_2, v_1u_2, v_1v_2$ are not edges of G . Note that a pair of disconnected edges can belong to the same connected component of G . Alternatively, if $d(x, y)$ denotes the distance between the vertices x and y , that is, the least length of a path from x to y , then Zheng's definition is equivalent to saying that two edges u_1v_1 and u_2v_2 are disconnected if $d(u_1, u_2), d(u_1, v_2), d(v_1, u_2)$ and $d(v_1, v_2)$ are all at least 2.

THEOREM 3.2.13. [34, Theorem 2.18] *Let G be a forest with edge ideal $\mathcal{I}(G)$. Then $\text{reg}(\mathcal{I}(G)) = j + 1$ where j is the maximal number of pairwise disconnected edges in G .*

EXAMPLE 3.2.14. Zheng [34, Remark 2.19] points out that if $G = C_5$, the 5-cycle, then $\mathcal{I}(G)$ is an example where $\text{reg}(\mathcal{I}(G)) \neq$ maximal number of pairwise disconnected edges $+1$. So, the conclusion of Theorem 3.2.13 does not hold in general.

Recall that a cycle C is a *minimal cycle* if C has length at least 4 and contains no chord. An ideal I is said to satisfy *property $N_{2,p}$* for some $p \geq 1$ if I is generated by quadratics and its minimal free resolution is linear up to the p th step, i.e., $\beta_{i,j}(I) = 0$ for all $0 \leq i \leq p-1$ and $j > i+2$. Eisenbud, Green, Hulek and Popescu [2] gave an interesting characterization for property $N_{2,p}$ for edge ideals in terms of the minimal cycles of G^c . We restate this result as follows:

THEOREM 3.2.15. [2, Theorem 2.1] *Let G be a simple graph with edge ideal $\mathcal{I}(G)$. Then $\mathcal{I}(G)$ satisfies property $N_{2,p}$ with $p > 1$ if and only if every minimal cycle in G^c has length $\geq p + 3$.*

PROOF. Hochster's formula (Theorem 3.1.1) is employed liberally throughout the proof of [2]. A careful study of the reduced simplicial homology groups $\tilde{H}_i(|m|, k)$, where $|m|$ denotes the restriction of $\Delta = \Delta(G^c)$ to the vertices corresponding to the variable dividing m , is required in the proof. We will sketch out an alternative combinatorial proof in the next section (see Corollary 4.3.8). \square

Since a chordal graph is a graph that has no minimal cycles, the following result of Fröberg [11] becomes a corollary of Theorem 3.2.15.

COROLLARY 3.2.16. [11, Theorem 1] *Let G be a graph with edge ideal $\mathcal{I}(G)$. Then $\mathcal{I}(G)$ has a linear resolution if and only if G^c is a chordal graph.*

EXAMPLE 3.2.17. (The resolution of $\mathcal{I}(\mathcal{K}_{a,b})$) Let $G = \mathcal{K}_{a,b}$ be a complete bipartite graph. We write the vertex set of G as $V_G = \{x_1, \dots, x_a, y_1, \dots, y_b\}$ so that $E_G = \{x_iy_j \mid 1 \leq i \leq a, 1 \leq j \leq b\}$. For all $a, b \geq 1$, the complement of G is the disjoint union of \mathcal{K}_a and \mathcal{K}_b . Since G^c has no induced cycles of length ≥ 4 , the resolution of $\mathcal{I}(G)$ is linear by Corollary 3.2.16.

Because $G^c = \mathcal{K}_a \cup \mathcal{K}_b$, for any $S \subseteq V_G$ with $|S| = i + 2$, we have

$$\#\text{comp}(G_S^c) = \begin{cases} 2 & \text{if } S \cap \{x_1, \dots, x_a\} \neq \emptyset \text{ and } S \cap \{y_1, \dots, y_b\} \neq \emptyset \\ 1 & \text{otherwise} \end{cases}.$$

By Theorem 3.2.4, to determine $\beta_{i,i+2}(\mathcal{I}(G))$ it therefore suffices to count the number of subsets $S \subseteq V_G$ with $|S| = i + 2$ and $\#\text{comp}(G_S^c) = 2$.

There are $\binom{a+b}{i+2}$ subsets of V_G that contain $i + 2$ distinct vertices. Furthermore, $\binom{a}{i+2}$ of these subsets must contain only vertices among $\{x_1, \dots, x_a\}$; similarly, $\binom{b}{i+2}$ of these subsets contain only vertices among $\{y_1, \dots, y_b\}$. It thus follows that

$$\beta_{i,i+2}(\mathcal{I}(\mathcal{K}_{a,b})) = \binom{a+b}{i+2} - \binom{a}{i+2} - \binom{b}{i+2} \text{ for all } i \geq 0$$

since the expression on the right hand side counts the number of subsets $S \subseteq V_G$ with $|S| = i + 2$ and S contains at least one x_i vertex and one y_j vertex.

By adapting this proof, one can find formulas for the graded Betti numbers of the edge ideals for the multipartite graph $\mathcal{K}_{d_1, \dots, d_n}$ (see also [20, Theorem 5.3.8]).

4. Splittable monomial ideals

In this section we present a new tool for investigating the graded Betti numbers of facet (and edge) ideals. This approach, which uses the notion of a splittable ideal introduced by Eliahou and Kervaire (see [3]), was first explored by the authors [14]. This notion is surprisingly strong in that a splittable monomial approach produces results from which previously known results can be deduced as corollaries. As well, this tool gives us a means to generalize results known about the resolutions of edge ideals to the more general situation of facet ideals.

4.1. Splittable ideals. For a monomial ideal I , let $\mathcal{G}(I)$ denote the set of minimal monomial generators of I ; this set is uniquely determined (cf. [23, Lemma 1.2]).

DEFINITION 4.1.1. A monomial ideal I is *splittable* if I is the sum of two nonzero monomial ideals J and K , that is, $I = J + K$, such that

- (1) $\mathcal{G}(I)$ is the disjoint union of $\mathcal{G}(J)$ and $\mathcal{G}(K)$.
- (2) there is a *splitting function*

$$\begin{aligned} \mathcal{G}(J \cap K) &\rightarrow \mathcal{G}(J) \times \mathcal{G}(K) \\ w &\mapsto (\phi(w), \psi(w)) \end{aligned}$$

satisfying

- (a) for all $w \in \mathcal{G}(J \cap K)$, $w = \text{lcm}(\phi(w), \psi(w))$.
- (b) for every subset $S \subset \mathcal{G}(J \cap K)$, both $\text{lcm}(\phi(S))$ and $\text{lcm}(\psi(S))$ strictly divide $\text{lcm}(S)$.

If J and K satisfy the above properties, then $I = J + K$ is a *splitting* of I .

When $I = J + K$ is a splitting of a monomial ideal I , then there is a relation between $\beta_{i,j}(I)$ and the graded Betti numbers of the “smaller” ideals.

THEOREM 4.1.2 (Eliahou-Kervaire [3], Fatabbi [9]). *Suppose that I is a splittable monomial ideal with splitting $I = J + K$. Then for all $i, j \geq 0$,*

$$\beta_{i,j}(I) = \beta_{i,j}(J) + \beta_{i,j}(K) + \beta_{i-1,j}(J \cap K).$$

This theorem suggests an approach to the study of the numbers $\beta_{i,j}(\mathcal{I}(\Delta))$. Precisely, one wishes to find splittings of the ideal $\mathcal{I}(\Delta) = J + K$ such that the ideals J, K , and $J \cap K$ are related to facet ideals of subcomplexes of Δ . Theorem 4.1.2 then provides a recursive-type formula for the numbers $\beta_{i,j}(\mathcal{I}(\Delta))$. This is the general strategy of [14] and the results of this approach are described below. Note that the formulas will not be recursive in general since we may not be able to split the new facet ideals arising from J, K , and $J \cap K$.

4.2. Splitting edges. Let G be a simple graph with edge ideal $\mathcal{I}(G)$ and $e = uv \in E_G$. If we set

$$J = (uv) \text{ and } K = \mathcal{I}(G \setminus e),$$

then $\mathcal{I}(G) = J + K$. In general this may not be a splitting of $\mathcal{I}(G)$ because the second condition of Definition 4.1.1 may not hold. Thus, an edge $e = uv$ with the property that $\mathcal{I}(G) = (uv) + \mathcal{I}(G \setminus e)$ is a splitting is a special type of edge. We give such an edge e the following name:

DEFINITION 4.2.1. An edge $e = uv$ is a *splitting edge* of G if $J = (uv)$ and $K = \mathcal{I}(G \setminus e)$ give a splitting of $\mathcal{I}(G)$.

The following theorem characterizes all splitting edges in a simple graph. Recall that $N(u)$ denotes the set of distinct neighbors of the vertex u .

THEOREM 4.2.2. [14, Theorem 3.4] *An edge $e = uv$ is a splitting edge of G if and only if $N(u) \subseteq (N(v) \cup \{v\})$ or $N(v) \subseteq (N(u) \cup \{u\})$.*

PROOF. We shall sketch the main ideas of the proof. Let $N(u) \setminus \{v\} = \{u_1, \dots, u_n\}$ and $N(v) \setminus \{u\} = \{v_1, \dots, v_m\}$. Let $H = G \setminus (N(u) \cup N(v))$. It can be seen that

$$\begin{aligned} \mathcal{G}(J \cap K) = & \{uvu_i \mid u_i \notin (N(u) \cap N(v))\} \cup \{uvv_j \mid v_j \notin (N(u) \cap N(v))\} \cup \\ & \{uvz \mid z \in (N(u) \cap N(v))\} \cup \{uvm \mid m \in \mathcal{I}(H)\}. \end{aligned}$$

To prove the “if” direction, we observe that if $N(u) \subseteq (N(v) \cup \{v\})$ then

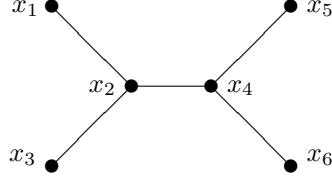
$$\mathcal{G}(J \cap K) = \{uvv_j \mid j = 1, \dots, m\} \cup \{uvm \mid m \in \mathcal{I}(H)\}.$$

This allows us to construct a splitting function $\mathcal{G}(J \cap K) \rightarrow \mathcal{G}(J) \times \mathcal{G}(K)$ as follows

$$w \mapsto (\phi(w), \varphi(w)) = \begin{cases} (uv, vv_j) & \text{if } w = uvv_j \\ (uv, m) & \text{if } w = uvm. \end{cases}$$

The “only if” direction is proved by contrapositive. Assume that $N(u) \not\subseteq (N(v) \cup \{v\})$ and $N(v) \not\subseteq (N(u) \cup \{u\})$. Then there exist vertices $x, y \in V_G$ such that $ux, vy \in E_G$ and $uy, vx \notin E_G$. Now, by using Definition 4.1.1 we can show that there cannot exist a splitting function $\mathcal{G}(J \cap K) \rightarrow \mathcal{G}(J) \times \mathcal{G}(K)$. \square

EXAMPLE 4.2.3. (Splitting edges) Consider the following graph G :



The edge x_1x_2 is a splitting edge of G but x_2x_4 is not.

Once one has identified splitting edges, one can apply the following formula.

THEOREM 4.2.4. [14, Theorem 3.7] *Let $e = uv$ be a splitting edge of G , and set $H = G \setminus (N(u) \cup N(v))$. Set $n = |N(u) \cup N(v)| - 2$. Then for all $i \geq 1$ and all $j \geq 0$*

$$\beta_{i,j}(\mathcal{I}(G)) = \beta_{i,j}(\mathcal{I}(G \setminus e)) + \sum_{l=0}^i \binom{n}{l} \beta_{i-1-l, j-2-l}(\mathcal{I}(H))$$

where $\beta_{-1,0}(\mathcal{I}(H)) = 1$ and $\beta_{-1,j}(\mathcal{I}(H)) = 0$ if $j > 0$.

PROOF. By Theorem 4.2.2 we must have $N(u) \subseteq (N(v) \cup \{v\})$ or $N(v) \subseteq (N(u) \cup \{u\})$. Suppose $N(u) \subseteq (N(v) \cup \{v\})$ and let $N(v) \setminus \{u\} = \{v_1, \dots, v_n\}$. As before, observe that

$$J \cap K = uv((v_1, \dots, v_n) + \mathcal{I}(H)).$$

The statement now follows by combining Theorem 4.1.2 and the observation that the resolution of $(v_1, \dots, v_n) + \mathcal{I}(H)$ can be derived from the tensor product of the resolutions of $R/(v_1, \dots, v_n)$ and $R/\mathcal{I}(H)$ (since H does not contain any vertices from among $\{v_1, \dots, v_n\}$). \square

Theorem 4.2.4 allows us to recover most of the known results in the literature about the graded Betti numbers of an edge ideal of a forest with fuller generality. We begin by using our formula to give a recursive formula for the graded Betti numbers of $\mathcal{I}(G)$ when G is a forest. This formula was first proved in [20, 21] via different means. In fact, our result is slightly more general since it applies to any leaf of a forest, while [20, 21] required that a special leaf be removed. Recall that by a leaf of G , we are referring to an edge with a vertex of degree 1.

COROLLARY 4.2.5. *Let $e = uv$ be any leaf of a forest G . If $\deg v = n$ and $N(v) = \{u, v_1, \dots, v_{n-1}\}$, then for $i \geq 1$ and $j \geq 0$*

$$\beta_{i,j}(\mathcal{I}(G)) = \beta_{i,j}(\mathcal{I}(T)) + \sum_{l=0}^i \binom{n-1}{l} \beta_{i-1-l, j-2-l}(\mathcal{I}(H))$$

where $T = G \setminus e = G \setminus \{u\}$ and $H = G \setminus \{u, v, v_1, \dots, v_{n-1}\}$. Here $\beta_{-1,0}(\mathcal{I}(H)) = 1$ and $\beta_{-1,j}(\mathcal{I}(H)) = 0$ if $j > 0$.

PROOF. The hypotheses imply that $\deg u = 1$. Since $N(u) \subseteq (N(v) \cup \{v\})$, uv is a splitting edge. Now apply Theorem 4.2.4. The formula is recursive since T and H are forests and so they each have a leaf, that is, a splitting edge. \square

REMARK 4.2.6. Our formula for $\beta_{i,j}(\mathcal{I}(G))$ is expressed in a different form when compared to the one found in [20, 21]. However, using the fact that $\beta_{i-1,j}(I) = \beta_{i,j}(R/I)$ for all $i \geq 1$ and $j \geq 0$ for any homogeneous ideal I of R we can recover the formula of [20, 21].

Theorem 4.2.4 can also be used to relate algebraic invariants, such as the regularity and the projective dimension, of an edge ideal of a graph to that of subgraphs.

COROLLARY 4.2.7. [14, Corollary 3.8] *Let $e = uv$ be a splitting edge of a graph G , and let $H = G \setminus (N(u) \cup N(v))$. Let $n = |N(u) \cup N(v)| - 2$. Then we have*

- (1) $\text{reg}(\mathcal{I}(G)) = \max\{2, \text{reg}(\mathcal{I}(G \setminus e)), \text{reg}(\mathcal{I}(H)) + 1\}$.
- (2) $\text{pd}(\mathcal{I}(G)) = \max\{\text{pd}(\mathcal{I}(G \setminus e)), \text{pd}(\mathcal{I}(H)) + n + 1\}$.

Notice that Corollary 4.2.7(2) generalizes [21, Theorem 4.8] which proved the formula in the case that G was a forest.

REMARK 4.2.8. Besides recovering the main theorem of [21], Theorem 4.2.4 enables us to recover Theorem 3.2.13 first proved by Zheng [34]. Precisely, we can produce a new combinatorial proof of Theorem 3.2.13 by using induction on the number of edges in the graph and combining Theorem 4.2.4 and Corollary 4.2.7 (see [14, Corollary 3.13] the details).

4.3. Splitting vertices. Let G be a simple graph, and let v be a vertex of G with $N(v) = \{v_1, \dots, v_d\}$. This section complements the results of the previous section by determining when $\mathcal{I}(G) = J + K$ with $J = (vv_1, \dots, vv_d)$ and $K = \mathcal{I}(G \setminus \{v\})$ is a splitting of $\mathcal{I}(G)$.

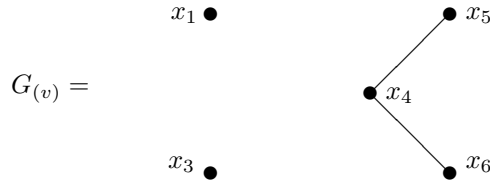
LEMMA 4.3.1. *With the notation as above, set*

$$\begin{aligned} G_i &:= G \setminus (N(v) \cup N(v_i)) \text{ for } i = 1, \dots, d, \text{ and} \\ G_{(v)} &:= G_{\{v_1, \dots, v_d\}} \cup \{e \in E_G \mid e \text{ incident to } v_1, \dots, v_d, \text{ but not } v\}. \end{aligned}$$

Then

$$J \cap K = v\mathcal{I}(G_{(v)}) + vv_1\mathcal{I}(G_1) + vv_2\mathcal{I}(G_2) + \dots + vv_d\mathcal{I}(G_d).$$

EXAMPLE 4.3.2. Consider the same graph G as in Example 4.2.3. Take $v = x_2$. Then $N(v) = \{v_1 = x_1, v_2 = x_3, v_3 = x_4\}$. In this case, $G_1 = G_2$ is the graph with two isolated vertices x_5 and x_6 , G_3 is the empty graph, and $G_{(v)}$ is the graph



Observe that if $v \in V_G$ is such that $\deg v = 0$, then $\beta_{i,j}(\mathcal{I}(G)) = \beta_{i,j}(\mathcal{I}(G \setminus \{v\}))$ for all $i, j \geq 0$ since $\mathcal{I}(G) = \mathcal{I}(G \setminus \{v\})$. It therefore suffices to compute the graded Betti numbers of the edge ideals of each connected component of G that contains one or more edges. If $\deg v = d > 0$ and if $G \setminus \{v\}$ consists of isolated vertices, then $G = \mathcal{K}_{1,d}$, the complete bipartite graph of size $1, d$. In this situation, the graded Betti numbers of $\mathcal{I}(G)$ are completely known as seen in Example 2.3.3. We now give a name to a vertex $v \in V_G$ that does not fall into either of the above two cases.

DEFINITION 4.3.3. A vertex $v \in V_G$ is a *splitting vertex* if $\deg v = d > 0$ and $G \setminus \{v\}$ is not the graph of isolated vertices.

Our choice of name is suitable in light of the following theorem.

THEOREM 4.3.4. [14, Theorem 4.2] *Let v be a splitting vertex of G with $N(v) = \{v_1, \dots, v_d\}$. Then $\mathcal{I}(G)$ is a splittable monomial ideal with a splitting given by $J = (vv_1, \dots, vv_d)$ and $K = \mathcal{I}(G \setminus \{v\})$.*

PROOF. By Lemma 4.3.1 we have an explicit description for the generators of $\mathcal{G}(J \cap K)$. A splitting function $\mathcal{G}(J \cap K) \rightarrow \mathcal{G}(J) \times \mathcal{G}(K)$ is then given by $w \mapsto (\phi(w), \varphi(w))$ where

$$\phi(w) = \begin{cases} vv_i & \text{if } w = vv_iv_j \in \mathcal{G}(v\mathcal{I}(G_{(v)})) \text{ and } i < j \\ vv_i & \text{if } w = vv_iy \in \mathcal{G}(v\mathcal{I}(G_{(v)})) \text{ and } y \notin N(v) \\ vv_i & \text{if } w = vv_im \in \mathcal{G}(vv_i\mathcal{I}(G_i)) \end{cases}$$

and

$$\varphi(w) = \begin{cases} v_iv_j & \text{if } w = vv_iv_j \in \mathcal{G}(v\mathcal{I}(G_{(v)})) \text{ and } i < j \\ v_iy & \text{if } w = vv_iy \in \mathcal{G}(v\mathcal{I}(G_{(v)})) \text{ and } y \notin N(v) \\ m & \text{if } w = vv_im \in \mathcal{G}(vv_i\mathcal{I}(G_i)). \end{cases}$$

□

Applying Theorem 4.1.2 we obtain:

THEOREM 4.3.5. [14, Theorem 4.4] *Let v be a splitting vertex of G with $N(v) = \{v_1, \dots, v_d\}$. Let $G_{(v)}$ and G_i for $i = 1, \dots, d$ be defined as in Lemma 4.3.1. Then*

$$\beta_{i,j}(\mathcal{I}(G)) = \beta_{i,j}(\mathcal{I}(\mathcal{K}_{1,d})) + \beta_{i,j}(\mathcal{I}(G \setminus \{v\})) + \beta_{i-1,j}(L)$$

where $L = v\mathcal{I}(G_{(v)}) + vv_1\mathcal{I}(G_1) + \dots + vv_d\mathcal{I}(G_d)$ and $\mathcal{K}_{1,d}$ is the complete bipartite graph of size $1, d$.

PROOF. One combines Lemma 4.3.1, Theorem 4.3.4 and Theorem 4.1.2. □

Theorem 4.3.5 allows us to relate algebraic invariants, such as the regularity and the projective dimension, of an edge ideal of a graph to that of subgraphs.

COROLLARY 4.3.6. [14, Corollary 4.5] *Let $v \in V_G$ be any vertex of a simple graph G , and suppose $\deg v = d$. Then*

- (1) $\text{reg}(\mathcal{I}(G)) \geq \max\{2, \text{reg}(\mathcal{I}(G \setminus \{v\}))\}$.
- (2) $\text{pd}(\mathcal{I}(G)) \geq \max\{d - 1, \text{pd}(\mathcal{I}(G \setminus \{v\}))\}$.

Jacques [20, Proposition 2.1.4] first proved Corollary 4.3.6 (2) in the case that v is a *terminal vertex*, i.e. when $\deg v \leq 1$.

Unlike Theorem 4.2.4, we cannot extract a recursive formula from Theorem 4.3.5 since the ideal $L = v\mathcal{I}(G_{(v)}) + vv_1\mathcal{I}(G_1) + \dots + vv_d\mathcal{I}(G_d)$ is not the edge ideal of a graph. However, a recursive formula for the graded Betti numbers in the linear strand can be deduced from Theorem 4.3.5:

COROLLARY 4.3.7. *Let v be a splitting vertex of a graph G . Then for all $i \geq 0$,*

$$\beta_{i,i+2}(\mathcal{I}(G)) = \beta_{i,i+2}(\mathcal{I}(\mathcal{K}_{1,d})) + \beta_{i,i+2}(\mathcal{I}(G \setminus \{v\})) + \beta_{i-1,i+1}(\mathcal{I}(G_{(v)})).$$

PROOF. By evaluating the formula of Theorem 4.3.5 at $j = i + 2$ we get

$$\beta_{i,i+2}(\mathcal{I}(G)) = \beta_{i,i+2}(\mathcal{I}(\mathcal{K}_{1,d})) + \beta_{i,i+2}(\mathcal{I}(G \setminus \{v\})) + \beta_{i-1,i+2}(L).$$

Since $vv_1\mathcal{I}(G_1) + \dots + vv_d\mathcal{I}(G_d)$ is generated by monomials of degree 4 and $v\mathcal{I}(G_{(v)})$ is generated by monomials of degree 3, we have $\beta_{i-1,i+2}(L) = \beta_{i-1,i+2}(v\mathcal{I}(G_{(v)})) = \beta_{i-1,i+1}(\mathcal{I}(G_{(v)}))$. \square

Theorem 4.3.5 also enables us to give new combinatorial proofs for many interesting results. For example, the characterization of property $N_{2,p}$ of Eisenbud et. al. for quadratic square-free monomial ideals can be proved without the use of Hochster's formula. We have included a sketch of the new proof (see [14, Corollary 4.7] for complete details).

COROLLARY 4.3.8. [2, Theorem 2.1] *Let G be a simple graph with edge ideal $\mathcal{I}(G)$. Then $\mathcal{I}(G)$ satisfies property $N_{2,p}$ with $p > 1$ if and only if every minimal cycle in G^c has length $\geq p + 3$.*

PROOF. One can prove this statement by induction on $|V_G|$, the number of vertices. By Theorem 4.3.5 we have

$$\beta_{i,j}(\mathcal{I}(G)) = \beta_{i,j}(\mathcal{I}(\mathcal{K}_{1,d})) + \beta_{i,j}(\mathcal{I}(G \setminus \{v\})) + \beta_{i-1,j}(L).$$

So, $\mathcal{I}(G)$ satisfies property $N_{2,p}$ if and only if $\mathcal{I}(\mathcal{K}_{1,d})$ and $\mathcal{I}(G \setminus \{v\})$ satisfy property $N_{2,p}$ and L satisfies property $N_{3,p-1}$. By Example 2.3.3, the ideal $\mathcal{I}(\mathcal{K}_{1,d})$ always has a linear resolution, so it satisfies property $N_{2,p}$. The induction hypothesis allows us to show $\mathcal{I}(G \setminus \{v\})$ satisfies property $N_{2,p}$. Finally, L satisfies property $N_{3,p-1}$ if and only if $L = v\mathcal{I}(G_{(v)})$ and $\mathcal{I}(G_{(v)})$ satisfies property $N_{2,p-1}$. The heart of the proof is to then verify that $L = v\mathcal{I}(G_{(v)})$ and $\mathcal{I}(G_{(v)})$ satisfies property $N_{2,p-1}$ if and only if every minimal cycle of G^c containing v has length $\geq p + 3$. \square

REMARK 4.3.9. Fröberg's result about edge ideals with linear resolutions (Theorem 3.2.16), and the formula of Theorem 3.2.6 first proved by Roth and the second author are also examples of corollaries of Theorem 4.3.5. See [14] for the details of these proofs.

4.4. Splitting facets. In this section we will show that the notion of splittable ideals can also be used quite profitably to study resolutions of facet ideals in general. Throughout this section Δ will denote a simplicial complex on a vertex set V_Δ .

DEFINITION 4.4.1. Let Δ be a simplicial complex, and let F be a facet of Δ . The *connected component* of F in Δ , denoted by $\text{conn}_\Delta(F)$, is the connected component of Δ containing F . If $\text{conn}_\Delta(F) \setminus F = \langle G_1, \dots, G_p \rangle$, then we define the *reduced connected component* of F in Δ , denoted by $\overline{\text{conn}}_\Delta(F)$, to be the simplicial complex whose facets are given by $G_1 \setminus F, \dots, G_p \setminus F$, where if there exist G_i and G_j such that $\emptyset \neq G_i \setminus F \subseteq G_j \setminus F$, then we shall disregard the bigger facet $G_j \setminus F$ in $\overline{\text{conn}}_\Delta(F)$.

EXAMPLE 4.4.2. Consider the simplicial complex Δ with the facet set $\mathcal{F}(\Delta) = \{\{1, 2, 3\}, \{1, 3, 4\}, \{1, 4, 5\}, \{1, 5, 6\}\}$. Let $F = \{1, 5, 6\}$. Then $\text{conn}_\Delta(F) = \Delta$ and $\overline{\text{conn}}_\Delta(F)$ is the simplicial complex with the facet set $\{\{2, 3\}, \{4\}\}$. Note that $\{1, 3, 4\} \setminus F = \{3, 4\}$ contains $\{4\} = \{1, 4, 5\} \setminus F$ so we disregard the bigger set (which is $\{3, 4\}$) in obtaining $\overline{\text{conn}}_\Delta(F)$.

For a facet F of a simplicial complex Δ , we denote by $\Delta' = \Delta \setminus F$ the simplicial complex obtained by removing F from the facet set of Δ . Let

$$J = (x^F) \text{ and } K = \mathcal{I}(\Delta').$$

Note that $\mathcal{G}(\mathcal{I}(\Delta))$ is the disjoint union of $\mathcal{G}(J)$ and $\mathcal{G}(K)$. We are interested in finding F such that $\mathcal{I}(\Delta) = J + K$ gives a splitting of $\mathcal{I}(\Delta)$.

DEFINITION 4.4.3. With the above notation, we call F a *splitting facet* of Δ if $\mathcal{I}(\Delta) = J + K$ is a splitting of $\mathcal{I}(\Delta)$.

The following result gives a recursive-type formula for the graded Betti numbers of the facet ideal of a simplicial complex in terms of the Betti numbers of facet ideals of subcomplexes. This result also generalizes Theorem 4.2.4.

THEOREM 4.4.4. [14, Theorem 5.5] *Let F be a splitting facet of a simplicial complex Δ . Then for all $i \geq 1$ and $j \geq 0$*

$$\beta_{i,j}(\mathcal{I}(\Delta)) = \beta_{i,j}(\mathcal{I}(\Delta')) + \sum_{l_1=0}^i \sum_{l_2=0}^{j-|F|} \beta_{l_1-1,l_2}(\mathcal{I}(\overline{\text{conn}}_{\Delta}(F))) \beta_{i-l_1-1,j-|F|-l_2}(\mathcal{I}(\Omega))$$

where $\Delta' = \Delta \setminus F$ and $\Omega = \Delta \setminus \text{conn}_{\Delta}(F)$. Here $\beta_{-1,0}(I) = 1$ and $\beta_{-1,j}(I) = 0$ if $j > 0$ for $I = \mathcal{I}(\overline{\text{conn}}_{\Delta}(F))$ and $\mathcal{I}(\Omega)$.

PROOF. It can be shown that $J \cap K = x^F(\mathcal{I}(\overline{\text{conn}}_{\Delta}(F)) + \mathcal{I}(\Omega))$. Observe that Ω and $\overline{\text{conn}}_{\Delta}(F)$, by definition, do not share any common vertices. Thus, the minimal free resolution of $\mathcal{I}(\overline{\text{conn}}_{\Delta}(F)) + \mathcal{I}(\Omega)$ can be derived from the tensor product of the resolutions of $\mathcal{I}(\overline{\text{conn}}_{\Delta}(F))$ and $\mathcal{I}(\Omega)$. The result now follows by applying Theorem 4.1.2. \square

We will now show that our formula in Theorem 4.4.4 is recursive when Δ is a simplicial forest. To do so, we first show that a leaf of Δ is a splitting facet. Recall that if F is a leaf of Δ , then F must have a vertex that does not belong to any other facet of the simplicial complex (see [5, Remark 2.3]).

THEOREM 4.4.5. [14, Theorem 5.6] *If F is a leaf of Δ , then F is a splitting facet of Δ .*

PROOF. The proof uses a similar line of reasoning as that of the “if” direction of Theorem 4.2.2. An explicit description for the generators of $\mathcal{G}(J \cap K)$ can be given and a splitting function $s : \mathcal{G}(J \cap K) \rightarrow \mathcal{G}(J) \times \mathcal{G}(K)$ is constructed in the most natural way. The fact that F contains a vertex x that does not belong to any other facets of Δ guarantees that the function s does indeed satisfy all the conditions of Definition 4.1.1. \square

Recall that a forest is a simplicial complex with the property that every nonempty connected subcomplex has a leaf. Since $\text{conn}_{\Delta}(F)$, $\Delta \setminus F$ and $\Delta \setminus \text{conn}_{\Delta}(F)$ are subcomplexes of Δ , it follows directly from the definition that if Δ is a forest then so are $\text{conn}_{\Delta}(F)$, $\Delta \setminus F$ and $\Delta \setminus \text{conn}_{\Delta}(F)$. Thus, to show that our formula in Theorem 4.4.4 is recursive when Δ is a simplicial forest, we need to show that $\overline{\text{conn}}_{\Delta}(F)$ is also a forest; this is the content of the next lemma.

LEMMA 4.4.6. [14, Lemma 5.7] *Let F be a facet of a forest Δ . Then $\overline{\text{conn}}_{\Delta}(F)$ is a forest.*

Our recursive formula for simplicial trees generalizes Corollary 4.2.5 to higher dimensions.

THEOREM 4.4.7. [14, Theorem 5.8] *Let F be a leaf of a simplicial forest Δ , and let $\Delta' = \Delta \setminus F$ and $\Omega = \Delta \setminus \text{conn}_\Delta(F)$. Then Δ', Ω , and $\overline{\text{conn}}_\Delta(F)$ are also simplicial forests and for all $i \geq 1$ and $j \geq 0$*

$$\beta_{i,j}(\mathcal{I}(\Delta)) = \beta_{i,j}(\mathcal{I}(\Delta')) + \sum_{l_1=0}^i \sum_{l_2=0}^{j-|F|} \beta_{l_1-1,l_2}(\mathcal{I}(\overline{\text{conn}}_\Delta(F))) \beta_{i-l_1-1,j-|F|-l_2}(\mathcal{I}(\Omega))$$

where $\beta_{-1,0}(I) = 1$ and $\beta_{-1,j}(I) = 0$ for $j > 0$ if $I = \mathcal{I}(\overline{\text{conn}}_\Delta(F))$ or $\mathcal{I}(\Omega)$.

Recall that a simplicial complex Δ is said to be a pure $(d-1)$ -dimensional simplicial complex if $\dim F = d-1$, i.e., $|F| = d$, for any facet F of Δ . For a face G of dimension $d-2$ of a pure $(d-1)$ -dimensional simplicial complex Δ we define the *degree* of G , written $\deg_\Delta(G)$, to be the cardinality of the set $\{F \in \mathcal{F}(\Delta) \mid G \subseteq F\}$. Let $\mathcal{A}(\Delta)$ denote the set of $(d-2)$ -dimensional faces of Δ . The following result gives a formula for the graded Betti numbers in the linear strand of the facet ideal of a pure simplicial forest.

THEOREM 4.4.8. [14, Theorem 5.9] *Let Δ be a pure $(d-1)$ -dimensional forest (for some $d \geq 2$). Then*

$$\beta_{i,i+d}(\mathcal{I}(\Delta)) = \begin{cases} |\mathcal{F}(\Delta)| & \text{if } i = 0 \\ \sum_{G \in \mathcal{A}(\Delta)} \binom{\deg_\Delta(G)}{i+1} & \text{if } i \geq 1. \end{cases}$$

PROOF. The proof uses induction on the number of facets of Δ and makes use of Theorem 4.4.7. \square

Theorem 4.4.8 was first proved by Zheng [34] under the extra condition that Δ is connected in codimension 1. By using the notion of splittable ideals, this hypothesis can be removed. When $d = 2$, then Δ is a forest in the standard sense, and we recover Corollary 3.2.7. We can therefore view Theorem 4.4.8 as a partial generalization of Theorem 3.2.6.

5. Open questions

As noted in the introduction, one of our goals in writing this survey is to promote further research on the resolutions of square-free monomial ideals from a facet ideal point-of-view. We end this paper with some natural questions whose answers we would be interested in knowing.

5.1. Building a dictionary. One of the themes stressed in this paper is how the combinatorial data of either a graph G or a simplicial complex Δ appears in the minimal graded free resolution of $\mathcal{I}(G)$ or $\mathcal{I}(\Delta)$. Although we have shown that many graded Betti numbers can be described directly from the combinatorial data, there is still much we do not know. We therefore pose the general question:

QUESTION 5.1.1. Let $\mathcal{I}(\Delta)$ be the facet ideal of a simplicial complex. For which i and j is there a formula for the graded Betti number $\beta_{i,j}(\mathcal{I}(\Delta))$ in terms of combinatorial data of Δ ?

This question is probably too imprecise; one should not expect a simple answer to this question because some of the numbers $\beta_{i,j}(\mathcal{I}(\Delta))$ will depend upon the characteristic of the field (as seen in Reisner's example in the introduction). However, an interesting place to start is to see what structure Δ must have to force $\beta_{i,j}(\mathcal{I}(\Delta))$ to be zero.

Among the graded Betti numbers of a facet ideal, those in the linear strand are of particular interest. We raise the following question:

QUESTION 5.1.2. Let Δ be a pure $(d - 1)$ -dimensional simplicial complex. Is there a formula for $\beta_{i,i+d}(\mathcal{I}(\Delta))$ which describes the linear strand of the resolution of $\mathcal{I}(\Delta)$ similar to that of Theorem 4.4.8 and Corollary 3.2.6?

When the simplicial complex is of dimension one (i.e., it is a graph) Question 5.1.2 has been addressed positively by Roth and the second author (see Corollary 3.2.6) for edge ideals of graphs having no minimal 4-cycles. It is natural to ask the question for graphs which contain minimal 4-cycles.

QUESTION 5.1.3. If G has minimal 4-cycles, what is a formula for $\beta_{i,i+2}(\mathcal{I}(G))$?

Finally, when $\mathcal{I}(G)$ is an edge ideal, we know $\beta_{0,j}(\mathcal{I}(G))$ and $\beta_{1,j}(\mathcal{I}(G))$ and $\beta_{2,j}(\mathcal{I}(G))$ can be computed directly from the graph (see Examples 2.3.2, 3.2.8, and 3.2.10, respectively) for all j . Hence, the first place to search for new formulas for $\beta_{i,j}(\mathcal{I}(G))$ is in the case that $i = 3$. Note that one could take the approach used in Example 3.2.2 by compiling a list of all graphs H with $\dim_k \tilde{H}_i(\Delta(H), k) > 0$. More compact formulas, however, would be preferred. Indeed, it would be nice to find an alternative formula for $\beta_{2,5}(\mathcal{I}(G))$ that avoided having to identifying large numbers of induced subgraphs.

Of course, once we have built a reasonably sized dictionary, we want to use the tools of commutative algebra to answer questions about graph theory. For example, Roth and the second author [26] showed that if one uses the Bigatti-Hullet-Pardue theorem about the growth of graded Betti numbers of lexicographical ideals, one can obtain a crude bound on the number of triangles in a graph. Do other such results await us?

5.2. Characteristic-independence. These questions are inspired by Katzman's paper [22]. From Hochster's formula (see Theorem 3.1.1) it follows that the graded Betti numbers of a monomial ideal may depend upon the field k . Reisner's example (see Section 4 of [22]) of the triangulation of the real projective plane is a classical example of how the graded Betti numbers depend upon $\text{char}(k)$. Highlighted below are some of Katzman's results on how the numbers $\beta_{i,j}(\mathcal{I}(G))$ of an edge ideal $\mathcal{I}(G)$ depend upon $\text{char}(k)$.

THEOREM 5.2.1. *Let $\mathcal{I}(G)$ be the edge ideal of a simple graph G . Then*

- (i) $\beta_{i,j}(\mathcal{I}(G))$ is independent of $\text{char}(k)$ for all $i \leq 5$.
- (ii) $\beta_{i,j}(\mathcal{I}(G))$ is independent of $\text{char}(k)$ if $|V_G| \leq 10$.
- (iii) there exists exactly 4 non-isomorphic graphs G with $|V_G| = 11$ such that the numbers $\beta_{i,j}(\mathcal{I}(G))$ depend upon $\text{char}(k)$. In each case the Betti number $\beta_{i,j}(\mathcal{I}(G))$ that depends upon $\text{char}(k)$ has $i = 7$ or 8, and $\text{char}(k) = 2$.

Theorem 5.2.1 (i) extends an earlier result of Terai and Hibi [30] that $\beta_{2,j}(\mathcal{I}(G))$ and $\beta_{3,j}(\mathcal{I}(G))$ are independent of $\text{char}(k)$. The above theorem does not tell us

whether the numbers $\beta_{6,j}(\mathcal{I}(G))$ depend upon $\text{char}(k)$. By Theorem 3.2.3 we know that $\beta_{6,j}(\mathcal{I}(G)) \neq 0$ only if $j = 8, \dots, 14$. Katzman was able to show that $\beta_{6,j}(\mathcal{I}(G))$ does not depend upon the characteristic if $j \neq 12$, but left open the question when $j = 12$. This brings us to our first question.

QUESTION 5.2.2. Does the graded Betti number $\beta_{6,12}(\mathcal{I}(G))$ depend upon $\text{char}(k)$?

Theorem 5.2.1 (iii) says that one must consider graphs G with at least 12 vertices to find an example. Related to this question, we could ask:

QUESTION 5.2.3. Can we identify graphs G (or simplicial complexes Δ) through some combinatorial means (e.g., G or Δ has a subgraph or a subcomplex of a particular form) with the property that the graded Betti numbers of their edge ideals depend on $\text{char}(k)$?

An answer to this more general question might then provide an easy answer to our first question. Note that in the paper of Katzman [22] all of the examples have the property that the numbers $\beta_{i,j}(\mathcal{I}(G))$ only change if $\text{char}(k) = 2$. One is naturally lead to ask if the graded Betti numbers only change if $\text{char}(k) = 2$. As explained to us by Katzman, the answer to this question is no since for any prime p , one can always find a graph G so that the graded Betti numbers of $\mathcal{I}(G)$ are different in $\text{char}(k) = p$ and $\text{char}(k) = 0$. To find such a graph, begin with any simplicial complex whose homology depends upon the characteristic that you desire. Then construct the barycentric subdivision of the simplicial complex where the new vertices are the old nonempty faces, and the new faces are chains of old nonempty faces. Construct a graph H whose vertices are the old faces and whose edges are pairs of incomparable faces. Then the homology of the simplicial complex is the same as its barycentric subdivision. Furthermore, the barycentric subdivision is the clique complex associated to H . Because $\mathcal{I}_{\Delta(H)} = \mathcal{I}(H^c)$, the graded Betti numbers of $\mathcal{I}(H^c)$ change if the characteristic is p .

5.3. Other algebraic invariants and properties. Besides the graded Betti numbers, we are also interested in other algebraic invariants and properties of edge ideals and facet ideals which are related to Betti numbers. For example, the regularity, which measures the width of the resolution; the projective dimension, which measure the length of the resolution; and property $N_{d,p}$, which captures how long the resolution must have linear syzygies.

Inspired by Corollary 3.2.13, we raise the following question.

QUESTION 5.3.1. Let Δ be a simplicial complex. Is there a formula that relates $\text{reg}(\mathcal{I}(\Delta))$ to combinatorial data, e.g., the number of subcomplexes of a particular type, of Δ ?

To address Question 5.3.1, one may want to start with the edge ideal of a graph which is not necessarily a tree and generalize Corollary 3.2.13. Another direction is to generalize Corollary 3.2.13 to higher dimension, i.e., to find formula for $\text{reg}(\mathcal{I}(\Delta))$ when Δ is a pure simplicial tree.

Since the method of mathematical induction has proved to be significant in obtaining many of our results, it would be interesting to relate algebraic invariants and properties of edge ideals (or facet ideals) of graphs (or simplicial complexes)

to that of subgraphs (or subcomplexes). We propose to seek for generalizations of Corollaries 4.2.7 and 4.3.6.

QUESTION 5.3.2. Let Δ be a simplicial complex. Let F be a facet of Δ and let v be a vertex of Δ .

- (1) Is there a formula which relates $\text{reg}(\mathcal{I}(\Delta))$ and $\text{pd}(\mathcal{I}(\Delta))$ to $\text{reg}(\mathcal{I}(\Delta \setminus F))$ and $\text{pd}(\mathcal{I}(\Delta \setminus F))$?
- (2) Is there a formula which relates $\text{reg}(\mathcal{I}(\Delta))$ and $\text{pd}(\mathcal{I}(\Delta))$ to $\text{reg}(\mathcal{I}(\Delta \setminus \{v\}))$ and $\text{pd}(\mathcal{I}(\Delta \setminus \{v\}))$?

Here, by $\Delta \setminus \{v\}$ we mean the simplicial complex one obtains by removing from Δ all facets that contain v .

In connection to Green's famous conjecture on canonical curves, the property that an ideal has linear syzygies up to a given step has sparked much research. One of these is the characterization for property $N_{2,p}$ of edge ideals due to Eisenbud, Green, Hulek and Popescu (see Corollary 3.2.15). We would like to find a similar characterization in higher dimension.

QUESTION 5.3.3. Let Δ be a pure $(d - 1)$ -dimensional simplicial complex. Is there a necessary and sufficient condition, based upon combinatorial data of Δ , for $\mathcal{I}(\Delta)$ to satisfy property $N_{d,p}$ for $p > 1$, i.e., $\mathcal{I}(\Delta)$ is generated in degree d (which is obvious) and has linear syzygies up to the p -th step?

5.4. Splittable monomial ideals. In the previous section we demonstrated the usefulness of the notion of splittable ideals to study the graded Betti numbers of edge and facet ideals. Our remaining questions are interested in extending some of these ideas.

In Section 4 we considered two natural splittings of an edge ideal in terms of two natural graph operations, namely, removing an edge and removing a vertex. We can ask if there are any other ways to split the generators of an edge ideal.

QUESTION 5.4.1. If $\mathcal{I}(G)$ is the edge ideal of a graph G , are there other splittings $\mathcal{I}(G) = J + K$ that give us information on the graded Betti numbers of $\mathcal{I}(G)$ in terms of that of subgraphs of G ?

We have classified all splitting edges for edge ideals. We have also shown that for facet ideals, the facets corresponding to leaves are splitting facets. However, we have left open the question if there are any other splitting facets. So one is lead to ask:

QUESTION 5.4.2. Is there a classification of splitting facets of a simplicial complex?

An affirmative answer to Question 5.4.2 was recently obtained in the two authors' ongoing joint work with J. Sidman. It is our hope that this classification of splitting facets will lead to interesting consequences, for example, a formula calculating the regularity of a simplicial forest similar to that of Theorem 3.2.13.

While we introduced a notion of a splitting vertex for graphs, we have not identified an analog of this concept for simplicial complexes. So we can ask:

QUESTION 5.4.3. What is the correct generalization of a splitting vertex in the context of simplicial complexes?

We saw in [14] that if v is a non-isolated vertex of a graph G such that $K = \mathcal{I}(G \setminus \{v\}) \neq (0)$, then $\mathcal{I}(G) = J + K$ is a splitting of $\mathcal{I}(G)$ (here, $J = (\{vx \mid vx \in E_G\})$). The next example shows that the same phenomenon is not true for simplicial complexes in general. Thus, the question of characterizing splitting vertices of simplicial complexes is more subtle than that for graphs.

EXAMPLE 5.4.4. Consider the simplicial complex Δ with the facet set $\{F_1 = \{0, 1, 2\}, F_2 = \{0, 3, 4\}, F_3 = \{0, 5, 6\}, G_1 = \{1, 2, 3\}, G_2 = \{3, 4, 5\}, G_3 = \{5, 6, 1\}\}$. Consider the vertex 0 of Δ . Then F_1, F_2, F_3 are facets of Δ containing 0. Let $J = (x^{F_1}, x^{F_2}, x^{F_3})$ and $K = (x^{G_1}, x^{G_2}, x^{G_3})$. We claim that $\mathcal{I}(\Delta) = J + K$ is not a splitting for $\mathcal{I}(\Delta)$. Indeed, suppose there exists a splitting function $s = (\phi, \varphi) : \mathcal{G}(J \cap K) \rightarrow \mathcal{G}(J) \times \mathcal{G}(K)$ satisfying the two conditions of Definition 4.1.1. Let $L_i = F_i \cup G_i$ for $i = 1, 2, 3$. Then $S = \{x^{L_1}, x^{L_2}, x^{L_3}\}$ is a subset of $\mathcal{G}(J \cap K)$. Thus, by definition $\text{lcm}(\phi(S))$ must strictly divide $\text{lcm}(S)$. Moreover, it is easy to see that $\phi(x^{L_i}) = x^{F_i}$ for $i = 1, 2, 3$. However, we now have $\text{lcm}(\phi(S)) = x^{F_1 \cup F_2 \cup F_3} = x^{\{0, 1, 2, 3, 4, 5, 6\}} = \text{lcm}(S)$, a contradiction. This shows that 0 cannot be a splitting vertex of Δ .

As we saw in [14], the idea of a splitting vertex is used to derive a recursive formula for the graded Betti numbers in the linear strand of edge ideals. This also enables us to give a combinatorial proof for a result of Eisenbud et. al. which characterizes property $N_{2,p}$ for edge ideals. Thus, an answer to Question 5.4.3 might therefore lead one to a characterization of property $N_{d,p}$ for the facet ideal of a pure $(d - 1)$ -dimensional simplicial complex.

Acknowledgments

We would like to thank all the organizers involved with the Midwest Algebra, Geometry and their Interactions Conference (MAGIC) for a wonderful conference, and for their encouragement to write this survey. We would also especially like to thank both Jessica Sidman and Moty Katzman for reading an earlier draft of this paper and for providing invaluable suggestions and improvements. The first author acknowledges the Louisiana Board of Regents Enhancement Award and the second author acknowledges the funding received by NSERC while working on this project.

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