Rational Surfaces
from an algebraic perspective

by

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To My Parents,

For Their Endless Love and Support!

Statement of Originality

I hereby declare that the materials and results in this thesis, unless accompanied by specific references, are original and have not been published elsewhere.

Kingston, July 15th, 2000

Huy Tài Hà

Abstract

This thesis attempts to investigate various embeddings of rational surfaces. Most of the attention is paid to those of the blowup of $\mathbb{P}^2$ at a set of points.

In Chapter 1, the notions of a matrix and its ideal of 2×2 minors are generalized to higher dimensional. It is proved that the ideal of 2×2 minors of a generic box-shaped matrix is prime and perfect. It is shown that such an ideal gives the algebraic realization for the Segre embedding of the product of several projective spaces, $\mathbb{P}(V_1) \times \ldots \times \mathbb{P}(V_n) \hookrightarrow \mathbb{P}(V_1 \otimes \ldots \otimes V_n)$. As consequences, the homogeneous coordinate ring of the above Segre embedding is a Cohen-Macaulay ring, and a Kozsul algebra.

Chapter 2 establishes the notion of the blowup of $\mathbb{P}^n$ at a subscheme and gives a general setup for the study in Chapter 3. It is proved that the blowup of $\mathbb{P}^n$ along a subscheme is the Bi-Proj of certain Rees algebras. Chapter 3 addresses the study of the blowup of $\mathbb{P}^2$ at a set of $\binom{d+1}{2}$ points which are in generic position, embedded in projective spaces by linear systems of plane curves going through these points. An explicit description of a set of defining equations for these projective embeddings of this blowup surface is given.

Chapter 4 looks at the embeddings of blowup surfaces in product spaces, or equivalently, the Rees algebra of certain codimension two perfect ideals. The defining ideal $I_{\mathbb{X}} = \oplus_{t \geq \alpha} I_t$ of a set of points $\mathbb{X}$ in $\mathbb{P}^2$ is considered. Section 4.1 studies a set of defining equations for the Rees algebra of the ideal generated by $I_{\alpha+1}$ for a general choice of the points in $\mathbb{X}$. Section 4.2 revisits the problem addressed in Chapter 3 but in a more general situation, where the number of points is arbitrary. A method of deriving a generating set for the defining ideal of certain projective embeddings of the blowup of $\mathbb{P}^2$ along a set of arbitrary number of points is demonstrated provided the points are general. In Section 4.3, for an arbitrary set of points $\mathbb{X}$ in $\mathbb{P}^2$ ($I_{\mathbb{X}} = \oplus_{t \geq \alpha} I_t$ is the defining ideal of $\mathbb{X}$), it is proved that when t is big enough, the Rees algebras $\mathcal{R}(I_t)$ is always Cohen-Macaulay and defined by quadratics.

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Finally, I dedicate this thesis to my parents, for their endless love and support.

Contents

CHAPTER 0

Introduction

"The essence of mathematics is its freedom."
G. Cantor (1845 - 1918)

Since the work of Hironaka ([**Hi**]), it has been understood that blowing up is an essential tool in the theory of resolution of singularities. Blowing up the projective plane at a subscheme results in a *rational surface*. In the class of rational surfaces, those obtained by blowing up $\mathbb{P}^2$ at a set of points were, perhaps, paid the most considerable attention. The inspiring work of Mumford ([**Mum**]) followed by the significant theorems and conjectures of Green ([**Gr**]) were among the first important works which suggested that, in order to study these blowup surfaces, one might start by embedding them in different projective spaces and study those embeddings in details. This direction was initiated by Geramita and Gimigliano ([**Gi-1**], [**Gi-2**], [**G-G**]), and pursued by many authors in the last ten years ([**Gi-Lo**], [**G-G-P**], [**G-G-H**], [**Hol**], [**Hol-1**], [**Hol-2**]). To study a variety, one normally studies its Cohen-Macaulayness, its defining equations and, more generally, the higher order syzygies among these equations. It has been realized that many classical results can be phrased as properties of the minimal free resolution of certain projective embeddings of a variety. A well known example is the non-singular cubic surface in $\mathbb{P}^3$; this surface is obtained by blowing up $\mathbb{P}^2$ at a set of 6 points, and embedding the resulting surface into $\mathbb{P}^3$ using the very ample divisor corresponding to the linear system of plane cubics through these 6 points.

The aim of this thesis is to continue the same line of study and to further investigate various embeddings of the blowup of $\mathbb{P}^2$ at a set of points. The thesis is structured as follows. In the Introduction we give an outline to the problems this thesis is concerned with, a brief literature review for each problem, the approaches we will be investigating, and the main results regarding these problems. In Chapter 1, we introduce the notions of a *box-shaped matrix* and its *ideal of* 2×2 *minors*, which give an algebraic realization of the Segre embedding of the product of several projective spaces. These ideas also provide a new tool in the study of projective embeddings of blowup surfaces, which is carried out in Chapter 3. Chapter 2 establishes the notion of the blowup of $\mathbb{P}^n$ at a subscheme. We attempt to fill the gap in the connection between two ways of defining this blowup: the classical definition and the modern realization that has been used in the last fifteen years. It is proved that the blowup of $\mathbb{P}^n$ at a subscheme, embedded into appropriate product spaces, is, in fact, the closure of the graph of certain rational maps which are defined using the linear systems of hypersurfaces containing this subscheme. Chapter 2 gives the motivation and the setting for the study in Chapter 3. Chapter 3 continues the study of [**Gi-1**], [**Gi-2**], [**G-G**], [**Gi-Lo**], [**Hol**], [**Hol-1**] and [**Hol-2**], in investigating the blowup of $\mathbb{P}^2$ at certain subsets of $\binom{d+1}{2}$ points, embedded in projective spaces by linear systems of plane curves going through these points. Sets of defining equations for some embeddings of these blowup surfaces are described. In Chapter 4, we push the study in Chapter 3 a step further by looking at the blowup of $\mathbb{P}^2$ at a set of points, embedded into product spaces. This is equivalent to studying the Rees algebras of certain codimension two perfect ideals. The asymptotic behaviour of the Rees algebras of the ideals generated by homogeneous pieces of the defining ideal of a set of points in $\mathbb{P}^2$ is considered. The results on the Rees algebra are also coupled with the study on diagonal subalgebras of a bi-graded algebra of [**STV**] and [**CHTV**] to completely answer the same question raised in Chapter 3 in a more general situation, when the number of points being considered is arbitrary.

Throughout this thesis, $\mathfrak{k}$ will denote the ground field. For simplicity, we assume that $\mathfrak{k}$ is algebraically closed and of characteristic 0. We also let $\mathbb{P}^n = \mathbb{P}^n_{\mathfrak{k}}$ denote the n-dimensional projective space over $\mathfrak{k}$, for any positive integer n. Moreover, if F is a homogeneous polynomial over $r+1$ variables, and P is a point in $\mathbb{P}^r$, we say that P satisfies (or the coordinates of P satisfy) F if F evaluated at the coordinates of P equals to zero. Most of the terms and terminologies being used in this thesis, unless otherwise defined, follow those of [**Ei-2**] and [**Hart**].

0.1. Box-shaped matrices and their ideals of 2×2 minors

Ideals of minors of a matrix have been thoroughly studied over many decades. They play a significant role in the study of projective varieties. It had been a major classical problem to show that the ideal of $t \times t$ minors of a generic matrix is a prime and perfect ideal. The proof for a general value of t is due to Eagon and Hochster from their important work in [**H-E**]. The minimal free resolution of these ideals was then found by Lascoux followed by many others (cf. [**La**], [**P-W**]).

The notions of a *box-shaped matrix* and its *ideal of* 2×2 *minors* are generalizations, respectively, of a matrix and its ideal of 2×2 minors. Higher order minors seem to lack geometric significance. In Chapter 1, we give definitions of box-shaped matrices and their ideals of 2×2 minors, then focus our attention on box-shaped matrices of indeterminates. Our main result is the following theorem.

Theorem 0.1 (Theorem 1.5)**.** *If $\mathcal{A}$ is a box-shaped matrix of indeterminates, then $I_2(\mathcal{A})$ is a prime ideal in $\mathfrak{k}[\mathcal{A}]$ (here, $I_2(\mathcal{A})$ is the ideal of 2×2 minors of $\mathcal{A}$ and $\mathfrak{k}[\mathcal{A}]$ is the ring obtained by adjoining the elements of $\mathcal{A}$ to the field $\mathfrak{k}$).*

Suppose now that $V_1, \ldots, V_n$ are vector spaces of dimensions $r_1, \ldots, r_n$, respectively. The primeness of $I_2(\mathcal{A})$ for a generic box-shaped matrix $\mathcal{A}$ of size $r_1 \times \ldots \times r_n$, coupled with previous work of Grone ([**Grone**]), shows that the 2×2 minors of $\mathcal{A}$ are the defining equations for the Segre embedding

$$\mathbb{P}(V_1) \times \ldots \times \mathbb{P}(V_n) \hookrightarrow \mathbb{P}(V_1 \otimes \ldots \otimes V_n).$$

This geometric realization of the ideal of 2×2 minors of a generic box-shaped matrix enables us to study its perfection, Hilbert function and a Gröbner basis. Section 1.3 carries out this study. We prove the following results.

Proposition 0.2 (Proposition 1.7)**.** *The Hilbert function of $I_2(\mathcal{A})$ is*

$$\mathbf{H}(I_2(\mathcal{A}), t) = \binom{\prod_{i=1}^n r_i + t - 1}{t} - \prod_{i=1}^n \binom{r_i + t - 1}{t} \ \forall t \geq 0.$$

Theorem 0.3 (Theorem 1.8)**.** *If $\mathcal{A}$ is an n-dimensional generic box-shaped matrix of size $r_1 \times \ldots \times r_n$, then $I_2(\mathcal{A})$ is a perfect ideal of grade $\prod_{i=1}^n r_i - \sum_{i=1}^n r_i + (n-1)$.*

Theorem 0.4 (Theorem 1.9)**.** *Suppose $\mathcal{A} = (x_{i_1 \ldots i_n})$ is a generic box-shaped matrix of size $r_1 \times \ldots \times r_n$. Then, under the degree reverse lexicographic monomial ordering on $R_{\mathcal{A}} = \mathfrak{k}[\mathcal{A}]$, in which the variables $x_{i_1 \ldots i_n}$ are ordered by lexicographic ordering on their indices (assuming that $1 < 2 < \ldots < n$), the 2×2 minors of $\mathcal{A}$ form a Gröbner basis for $I_2(\mathcal{A})$.*

The Gröbner basis found in Theorem 1.9 gives rise to the following interesting corollary regarding the koszul property of the homogeneous coordinate ring of the Segre embedding.

Corollary 0.4.1 (Corollary 1.9.1)**.** *Suppose $V_1, \ldots, V_n$ are vector spaces of dimensions $r_1, \ldots, r_n$. Then, the homogeneous coordinate ring of the Segre embedding*

$$\mathbb{P}(V_1) \times \ldots \times \mathbb{P}(V_n) \hookrightarrow \mathbb{P}(V_1 \otimes \ldots \otimes V_n)$$

is a koszul algebra.

In the last section of Chapter 1, we briefly pay attention to a particular class of box-shaped matrices, namely those of dimension 3. These box-shaped matrices give a new tool in describing the defining equations for certain projective embeddings of blowup surfaces, which is discussed in Chapter 3.

0.2. The blowup of $\mathbb{P}^n$ at a subscheme

Suppose $\mathbb{X}$ is a set of points in $\mathbb{P}^n$ which is defined by the ideal $I_{\mathbb{X}} \subseteq \mathfrak{k}[x_0, \ldots, x_n]$. Let $\mathcal{I}_{\mathbb{X}}$ be the ideal sheaf of $\mathbb{X}$, i.e. $\mathcal{I}_{\mathbb{X}} = \tilde{I_{\mathbb{X}}}$. The blowup of $\mathbb{P}^n$ at $\mathbb{X}$ is defined to be the Proj of the sheaf of $\mathcal{O}_{\mathbb{P}^n}$-algebras $\oplus_{t\geq 0}\mathcal{I}_{\mathbb{X}}^t$.

In the last fifteen years, another way of looking at the blowup of $\mathbb{P}^n$ along $\mathbb{X}$ has been investigated (cf. [**Gi-1**], [**Gi-2**], [**G-G**], [**G-G-P**], [**G-G-H**], [**Gi-Lo**], [**Hol**], [**Hol-1**], [**Hol-2**]). We briefly illustrate the idea as follows. Suppose $I_{\mathbb{X}} = \oplus_{\gamma\geq\alpha} I_\gamma$ is the homogeneous decomposition of $I_{\mathbb{X}}$ (α is the least degree of a non-zero homogeneous form in $I_{\mathbb{X}}$). For each $\gamma \geq \alpha$, we can use I_γ to define the following rational map

$$\varphi_\gamma : \mathbb{P}^n \dashrightarrow \mathbb{P}^{m_\gamma}$$

by sending each $P \in \mathbb{P}^n \backslash Z(I_\gamma)$ to the point $[F_{\gamma 0}(P) : \ldots : F_{\gamma m_\gamma}(P)] \in \mathbb{P}^{m_\gamma}$, where $Z(I_\gamma)$ is the zero set of the ideal generated by I_γ, and $\{F_{\gamma 0}, \ldots, F_{\gamma m_\gamma}\}$ is a system of generators for the $\mathfrak{k}$-vector space I_γ. Let Γ_γ and Λ_γ be the graph and the image of this rational map, and let $\overline{\Gamma_\gamma}$ and $\overline{\Lambda_\gamma}$ be their closures inside $\mathbb{P}^n \times \mathbb{P}^{m_\gamma}$ and $\mathbb{P}^{m_\gamma}$, respectively. It seems to be part of the folklore that when γ is at least as large as the degrees of all the generators of a minimal system of generators for I, $\overline{\Gamma_\gamma}$ is the blowup of $\mathbb{P}^n$ along $\mathbb{X}$, and when I_γ corresponds to a very ample divisor on $\overline{\Gamma_\gamma}$, $\overline{\Lambda_\gamma}$ gives a projective embedding of the blowup of $\mathbb{P}^n$ along $\mathbb{X}$ into the projective space $\mathbb{P}^{m_\gamma}$.

There is, however, no literature that I can find on the connection between the two ways of looking at the blowup of $\mathbb{P}^n$ along $\mathbb{X}$ as presented. Chapter 2 tries to fill this gap.

Section 2.1 investigates the blowup of $\mathbb{P}^n$ at an arbitrary subscheme. Suppose V is a subscheme of $\mathbb{P}^n$ defined by the ideal $I = \oplus_{\gamma\geq\alpha} I_\gamma \subseteq \mathfrak{k}[x_0, \ldots, x_n]$. With the same setting and $\varphi_\gamma, \Gamma_\gamma, \Lambda_\gamma, \overline{\Gamma_\gamma}$ and $\overline{\Lambda_\gamma}$ similarly defined, it is proved that $\overline{\Gamma_\gamma}$ is indeed the blowup of $\mathbb{P}^n$ along V, embedded into an appropriate product space, when γ is at least as large as the degrees of all the generators of a minimal system of generators for

I. This comes from the following theorem and the fact that the Rees algebra $\mathcal{R}(I_\gamma)$ of the ideal generated by I_γ (see in the next section for the definition of the Rees algebra of an ideal) is the bi-graded coordinate ring of $\overline{\Gamma_\gamma}$ in $\mathbb{P}^n \times \mathbb{P}^{m_\gamma}$ (Proposition 2.4). The readers are recommended to see the Proposition-Definition at the end of Section 2.1.

Theorem 0.5 (Theorem 2.3)**.** *Suppose V is a subscheme of $\mathbb{P}^n$ given by the ideal I. Then, the blowup of $\mathbb{P}^n$ along V is the scheme Bi-Proj $\mathcal{R}(I)$, where $\mathcal{R}(I)$ is the Rees algebra of I and inherits a natural bi-gradation from that of $R[t]$ ($R = \mathfrak{k}[x_0, \ldots, x_n]$ is the coordinate ring of $\mathbb{P}^n$).*

In Section 2.2, the study in Section 2.1 is restricted to the class of the blowups of $\mathbb{P}^n$ along a set of points. We confirm the folklore knowledge as mentioned.

0.3. Projective embeddings of blowup surfaces

Suppose $\mathbb{X} = \{P_1, \ldots, P_s\} \subseteq \mathbb{P}^2$ is a set of s distinct points, $I_{\mathbb{X}} = \wp_1 \cap \ldots \cap \wp_s \subseteq \mathfrak{k}[w_1, w_2, w_3]$ its defining ideal. To the set of points $\mathbb{X}$, we also associate two invariants, the least degree of a non-zero homogeneous form in $I_{\mathbb{X}}$, denoted by $\alpha = \alpha(I_{\mathbb{X}})$, and the least integer at which the difference function of the Hilbert function $H_{\mathbb{X}}$ equals 0, denoted by $\sigma = \sigma(I_{\mathbb{X}})$ (where the Hilbert function of $\mathbb{X}$ is defined by $H_{\mathbb{X}}(t) = \dim_k(\mathfrak{k}[w_1, w_2, w_3]/I_{\mathbb{X}})_t)$. It is known (cf. [**D-G-M**]) that:

(1) $H_{\mathbb{X}}(t) \leq H_{\mathbb{X}}(t+1)$ for all t.
(2) $H_{\mathbb{X}}(t) \leq \min\{s, \binom{t+2}{2}\}$ for all t.
(3) If $H_{\mathbb{X}}(t_0) = s$ then $H_{\mathbb{X}}(t) = s$ for all $t \geq t_0$.
(4) There exists t such that $H_{\mathbb{X}}(t) = s$.

This leads naturally to the following definition.

Definition. The set $\mathbb{X}$ is said to be in *generic position* if the Hilbert function of $\mathbb{X}$ is $H_{\mathbb{X}}(t) = \min\{s, \binom{t+2}{2}\}$ for all t (i.e. $H_{\mathbb{X}}(t)$ always attains its maximum possible value).

Similar to the study in Chapter 2 (restricted to $\mathbb{P}^2$), suppose now that $I_{\mathbb{X}} = \oplus_{\gamma \geq \alpha} I_\gamma$ is the homogeneous decomposition of $I_{\mathbb{X}}$, and $\mathbb{P}^2(\mathbb{X})$ is the blowup of $\mathbb{P}^2$ centered at $\mathbb{X}$. For each $\gamma \geq \alpha$, we again use I_γ to define a rational map

$$\varphi_\gamma : \mathbb{P}^2 \dashrightarrow \mathbb{P}^{m_\gamma}$$

similar to that in Section 0.2. We also let Γ_γ and Λ_γ be the graph and the image of φ_γ, and let $\overline{\Gamma_\gamma}$ and $\overline{\Lambda_\gamma}$ be their closures in $\mathbb{P}^2 \times \mathbb{P}^{m_\gamma}$ and $\mathbb{P}^{m_\gamma}$, respectively. As a consequence of the results in Chapter 2, it is known that when γ is bigger than or equal to the degrees of the generators of a minimal system of generators for $I_{\mathbb{X}}$, $\overline{\Gamma_\gamma}$ is the blowup $\mathbb{P}^2(\mathbb{X})$ embedded in $\mathbb{P}^2 \times \mathbb{P}^{m_\gamma}$. In this situation, the linear systems in $\mathbb{P}^2$ corresponding to I_γ are discussed. These linear systems are in one-to-one correspondence with a certain class of divisors on $\overline{\Gamma_\gamma}$. We denote the divisor which corresponds to I_γ by $|I_\gamma|$ (see Chapter 2). In the same chapter, we also show that when the divisor $|I_\gamma|$ is very ample, $\overline{\Lambda_\gamma}$ is a projective embedding of $\mathbb{P}^2(\mathbb{X})$ in $\mathbb{P}^{m_\gamma}$ (in fact, all our discussions in Chapter 2 is for $\mathbb{P}^n$ for any integer n).

In the last fifteen years, much effort has been spent on the problem of finding systems of defining equations for $\overline{\Lambda_\gamma}$ for various values of γ when $\mathbb{X}$ is in generic position (cf. [**Gi-1**], [**Gi-2**], [**G-G**], [**Gi-Lo**], [**Hol**], [**Hol-1**], [**Hol-2**]).

Before giving a brief literature review on this problem, we recall a result which plays an important role in the sequel. Let $E_1, \ldots, E_s$ be the classes of the exceptional divisors corresponding to the blowup of $\mathbb{P}^2$ at the points $P_1, \ldots, P_s$, respectively. If E_0 is the pull back to $\mathbb{P}^2(\mathbb{X})$ of the class of a general line in $\mathbb{P}^2$, then it is well known that Pic $(\mathbb{P}^2(\mathbb{X})) \simeq \mathbb{Z}^{s+1} = < E_0, E_1, \ldots, E_s >$. The following theorem is due to Coppens.

Theorem 0.6 (cf. [**G-G-P**], Theorem 2.1)**.** *For each $t \geq \sigma = \sigma(I_{\mathbb{X}})$, let $D_t = tE_0 - E_1 - \ldots - E_s$. Then*

(1) *D_t is very ample on $\mathbb{P}^2(\mathbb{X})$ for all $t \geq \sigma + 1$.*
(2) *D_σ is very ample on $\mathbb{P}^2(\mathbb{X}) \Leftrightarrow$ for any line $\mathcal{L}$ of $\mathbb{P}^2$, $\deg(\mathcal{L} \cap \mathbb{X}) < \sigma$.*

This theorem is actually proved for the blowup of $\mathbb{P}^n$ at a set of fat points. A stronger version of Theorem 0.6 for the blowup of $\mathbb{P}^2$ was, in fact, first proved in [**D-G**, Theorem 3.1]. Also, with our current notation, the divisor $|I_\gamma|$ is the same as $D_\gamma = \gamma E_0 - E_1 - \ldots - E_s$ for each $\gamma \geq \alpha$ (see Chapter 2).

A great deal of work has concentrated on an important special case, when the number of points being considered is $s = \binom{d+1}{2}$ for some positive integer d and those points do not lie on a curve of degree $d-1$ (cf. [**Gi-1**], [**Gi-2**], [**G-G**]). In this case,

$$I_{\mathbb{X}} = I_d \oplus I_{d+1} \oplus I_{d+2} \oplus \ldots$$

is generated by I_d, and $\sigma(I_{\mathbb{X}}) = d$ (see [**G-M**]). In this situation, it follows from Theorem 0.6 that I_t corresponds to a very ample divisor for all $t \geq d+1$ (i.e. $|I_t|$ is very ample for all $t \geq d+1$). If in addition, there are no d points of $\mathbb{X}$ lying on a line, then I_d also corresponds to a very ample divisor. Under this assumption, Gimigliano studied the embedding $\overline{\Lambda_d}$ of $\mathbb{P}^2(\mathbb{X})$, which results in a White surface ([**Gi-1**] and [**Gi-2**]). White surfaces had also been studied in the classical literature ([**Whi**] and [**Room**]). Gimigliano proved the following result.

Theorem 0.7 (Gimigliano, 1987)**.** *Suppose $\mathbb{X} = \{P_1, \ldots, P_s\} \subseteq \mathbb{P}^2$ is a set of s distinct points which are in generic position (s arbitrary). Then*

(1) *For each $t \geq \alpha = \alpha(I_{\mathbb{X}})$, if there are no t points in $\mathbb{X}$ lying on a line, then the homogeneous coordinate ring of $\overline{\Lambda_t}$ is Cohen-Macaulay for any t.*
(2) *If $s = \binom{d+1}{2}$ for some integer d, and there are no d points in $\mathbb{X}$ lying on a line, then the defining ideal of $\overline{\Lambda_d}$ is generated by the maximal minors of a $3 \times d$ matrix of linear forms, and has the same Betti numbers as that of the*

ideal of maximal minors of a generic $3 \times d$ *matrix (which is given by the Eagon-Northcott complex).*

The embedding $\overline{\Lambda_{d+1}}$ of $\mathbb{P}^2(\mathbb{X})$, which is a Room surface, has been studied in detail by Geramita and Gimigliano ([**G-G**]). We summarize their results in the following theorem.

Theorem 0.8 (Geramita-Gimigliano, 1991)**.** *Suppose* $\mathbb{X} = \{P_1, \ldots, P_s\} \subseteq \mathbb{P}^2$ *is a set of* s *distinct points which are in generic position (*s *arbitrary). Then*

(1) *Let* d *be the least integer such that* $\binom{d+1}{2} \geq s$*. Then, the defining ideal of* $\overline{\Lambda_t}$ *for all* $t \geq d+1$ *is minimally generated by quadratic equations.*
(2) *If* $s = \binom{d+1}{2}$ *for some integer* d*, then the defining ideal of* $\overline{\Lambda_{d+1}}$ *is generated by the* 2×2 *minors of a* $3 \times (d+1)$ *matrix of linear forms, and has the same Betti numbers as that of the ideal of* 2×2 *minors of a generic* $3 \times (d+1)$ *matrix.*

When s is not a binomial coefficient number, say $s = \binom{d+1}{2} + k$ $(1 \leq k \leq d)$, $I_{\mathbb{X}} = \oplus_{t \geq d} I_t$ is generated by I_d and I_{d+1}, and $\sigma(I_{\mathbb{X}}) = d+1$ (see [**G-M**]). Under the assumption that the points in $\mathbb{X}$ are general enough, I_{d+1} corresponds to a very ample divisor, and the embedding of $\mathbb{P}^2(\mathbb{X})$ given by this divisor $|I_{d+1}|$, $\overline{\Lambda_{d+1}}$, was studied by Gimigliano and Lorenzini in [**Gi-Lo**]. They showed the following result.

Theorem 0.9 (Gimigliano-Lorenzini, 1993)**.** *Suppose* $\mathbb{X} = \{P_1, \ldots, P_s\} \subseteq \mathbb{P}^2$ *is a set of* s *distinct points, where* $s = \binom{d+1}{2} + k$ *(*$1 \leq k \leq d$*) for some integer* d*. Then, for a general choice of the points in* $\mathbb{X}$, $\overline{\Lambda_{d+1}}$ *has a Cohen-Macaulay homogeneous coordinate ring, and is generated by the* 3×3 *minors of a* $k \times 3$ *matrix* B *of linear forms, the* 2×2 *minors of a* $3 \times (d-k+1)$ *matrix* X *of indeterminates, and the entries of* $B.X$*.*

Part of this thesis, Chapter 3, is devoted to a further study of the defining ideals of $\overline{\Lambda_t}$ for all t. We look at the case when $s = \binom{d+1}{2}$ for some integer d, and continue the study of [**Gi-1**], [**Gi-2**], and [**G-G**]. We give an explicit description for the defining equations of $\overline{\Lambda_t}$ for all $t \geq d+1$. Our main result is the following theorem.

Theorem 0.10 (Theorem 3.6)**.** *Let $\mathbb{X}$ be a set of $s = \binom{d+1}{2}$ distinct points in $\mathbb{P}^2$ which are in generic position, and $I_{\mathbb{X}} = \oplus_{t \geq d} I_t$ the homogeneous decomposition of the defining ideal $I_{\mathbb{X}}$ of $\mathbb{X}$. Suppose $t = d+n$ $(n \geq 1)$. Then, the embedding $\overline{\Lambda_t}$ of the blowup of $\mathbb{P}^2$ along $\mathbb{X}$ is defined by $\binom{n+1}{2}d$ linear forms and the 2×2 minors of a box-shaped matrix of linear forms.*

0.4. Rees algebras of codimension two perfect ideals

Suppose A is a commutative ring with identity, $I \subseteq A$ an ideal of A. The Rees algebra of I with respect to A is defined to be the subring $A[It]$ of the ring $A[t]$. We denote the Rees algebra of I by $\mathcal{R}_A(I)$, or simply $\mathcal{R}(I)$ when there is no confusion about which ring is being discussed.

The Rees algebra of an ideal is a classical object that has been studied for many decades. Our interest in Rees algebras comes from the fact that they are algebraic realizations of certain blowup surfaces embedded into product spaces (see Chapter 2). To be more precise, again, let $\mathbb{X} = \{P_1, \ldots, P_s\} \subseteq \mathbb{P}^2$ be a set of s distinct points, $I_{\mathbb{X}} = \oplus_{t \geq \alpha} I_t$ its defining ideal, and $\mathbb{P}^2(\mathbb{X})$ the blowup of $\mathbb{P}^2$ along $\mathbb{X}$. As before, for each $t \geq \alpha$, we consider the rational map φ_t, its graph and image Γ_t and Λ_t, and their closures $\overline{\Gamma_t}$ and $\overline{\Lambda_t}$. It is proved in Chapter 2 that the Rees algebra of the ideal generated by I_t is the bi-graded coordinate ring of $\overline{\Gamma_t}$, so it gives an algebraic realization of the blowup $\mathbb{P}^2(\mathbb{X})$, embedded in an appropriate product space (for $t \gg 0$). Moreover, by the work of [**STV**] and then followed by [**CHTV**], it is known that certain properties of the Rees algebra of $I_{\mathbb{X}}$ can be transformed to properties of the projective embeddings $\overline{\Lambda_t}$ of $\mathbb{P}^2(\mathbb{X})$ for any value of t. Thus, to study projective embeddings of $\mathbb{P}^2(\mathbb{X})$, one naturally considers studying the Rees algebra of

$I_{\mathbb{X}}$ and the Rees algebras of the ideals generated by I_t for various values of t (which we shall denote by $\mathcal{R}(I_t)$). Chapter 4 addresses this study.

The first important result on the Rees algebras of codimension two perfect ideals in a polynomial ring is due to Morey and Ulrich. We rephrase their result as follows.

Theorem 0.11 ([**M-U**], Theorem 1.3)**.** *Let $R = \mathfrak{k}[x_1, \ldots, x_d]$ be a polynomial ring over an infinite field, let I be a codimension two perfect ideal of R with a linear presentation matrix. Assume that the number of generators of I is more than d, and that I satisfies condition G_d. Then $\mathcal{R}(I)$ is Cohen-Macaulay and is generated by the maximal minors of a matrix of linear forms.*

Here, the condition G_s, s an integer, means that the minimal number of generators of I_p is less than or equal to the dimension of R_p for every prime ideal $p \supseteq I$ such that $\dim R_p \leq s - 1$.

When reduced to the class of defining ideals for a generic set of points in $\mathbb{P}^2$, the restriction on the presentation matrix in Theorem 0.11 requires the number of points to be a binomial coefficient number. When the number of points is arbitrary, or the set of points are not in generic position, not much is known about the Rees algebra of its defining ideal. In our approach to the problem, inspired by works of Mumford ([**Mum**]) and Green ([**Gr**]), we look at an arbitrary set of points in $\mathbb{P}^2$ and study how the Rees algebras of the ideals generated by the homogeneous pieces of its defining ideal behave asymptotically.

In Section 4.1, we start by looking at the case when the points in $\mathbb{X}$ are in generic position. We prove the following results.

Theorem 0.12 (Theorem 4.3)**.** *Let $I = \oplus_{t \geq d} I_t$ be the defining ideal of $s = \binom{d+1}{2}$ points in $\mathbb{P}^2$ which are in generic position. Then, the defining equations for the Rees algebra $\mathcal{R}(I_{d+1})$ of the ideal generated by I_{d+1} are the 2×2 minors of a $3 \times (d+2)$*

matrix of linear forms. Moreover, $\mathcal{R}(I_{d+1})$ *is Cohen-Macaulay, and has the same Betti numbers as that of the ideal of* 2×2 *minors of a generic* $3 \times (d+2)$ *matrix.*

Theorem 0.13 (Theorem 4.7). *Let* $I = \oplus_{t \geq d} I_t$ *be the defining ideal for a set of* $s = \binom{d+1}{2} + k$ $(1 \leq k \leq d)$ *points in* $\mathbb{P}^2$*. Then, for a general choice of the points, the Rees algebra* $\mathcal{R}(I_{d+1})$ *of the ideal generated by* I_{d+1} *is Cohen-Macaulay and defined by the* 3×3 *minors of a* $k \times 3$ *matrix* B *of linear forms, the* 2×2 *minors of a* $3 \times (d-k+2)$ *matrix* X *of indeterminates and the entries of the product matrix* $B.X$*.*

We also note that these results can be extended to a larger class of codimension two perfect ideals of any polynomial ring (Theorems 4.4 and 4.8). These results provide the necessary information to completely answer questions on the defining equations of projective embeddings of $\mathbb{P}^2(\mathbb{X})$, as we discuss in Section 4.2. In Section 4.2, for a set $\mathbb{X}$ of s general points in $\mathbb{P}^2$ (s an arbitrary number, $s = \binom{d+1}{2} + k,\ 0 \leq k \leq d$), we couple our results in Theorems 4.4 and 4.8 with the study of [**STV**] and [**CHTV**] to demonstrate a method of deriving a system of defining equations for the embedding $\overline{\Lambda_t}$ of the blowup $\mathbb{P}^2(\mathbb{X})$ for all $t \geq d+1$ (the readers are refered back to Section 0.3 for the discussion on projective embeddings of the blowup surface $\mathbb{P}^2(\mathbb{X})$).

Theorems 0.12 and 0.13 also give motivation toward the study of asymptotic behaviour of the Rees algebras $\mathcal{R}(I_t)$ as t gets large, for the defining ideal $I_{\mathbb{X}} = \oplus_{t \geq \alpha} I_t$ of an arbitrary set of points $\mathbb{X} \subseteq \mathbb{P}^2$. This study is carried on in Section 4.3. The main result of Section 4.3 is the following theorem.

Theorem 0.14 (Theorem 4.13). *Suppose* $\mathbb{X}$ *is an arbitrary set of points in* $\mathbb{P}^2$*, and* $I_{\mathbb{X}} = \oplus_{t \geq \alpha} I_t$ *is its defining ideal. Then, there exists an integer* d_0 *such that for all* $t \geq d_0$*, the Rees algebra* $\mathcal{R}(I_t)$ *of the ideal generated by* I_t *is Cohen-Macaulay and defined by quadratics.*

In Section 4.3, we also introduce the notion of being *arithmetic Cohen-Macaulay* (a.CM) for a subscheme of the product scheme $\mathbb{P}^n \times \mathbb{P}^m$. We give a cohomological characterization of this property.

Theorem 0.15 (Theorem 4.11)**.** *Suppose $V \subseteq \mathbb{P}^n \times \mathbb{P}^m$ is a proper closed subscheme of dimension d of $\mathbb{P}^n \times \mathbb{P}^m$. Then,*

(1) *If V is a.CM, then $H^i(\mathcal{I}_V(a,b)) = 0$ for all $a, b \in \mathbb{Z}$ and $1 \leq i \leq d$, where $\mathcal{I}_V$ is the ideal sheaf of V in $\mathbb{P}^n \times \mathbb{P}^m$.*

(2) *Suppose $d \neq n, m$, and $H^i(\mathcal{I}_V(a,b)) = 0$ for all $a, b \in \mathbb{Z}$ and $1 \leq i \leq d$. If in addition, $H^{d+1}(\mathcal{I}_V(a,b)) = 0$ for all $a, b \geq 0$, and for every $j > 0$,*

$$R^j\pi_{1*}(\mathcal{O}_V(p,q)) = 0 \ \forall p \in \mathbb{Z}, q \geq 0,$$

and

$$R^j\pi_{2*}(\mathcal{O}_V(p,q)) = 0 \ \forall q \in \mathbb{Z}, p \geq 0,$$

then V is a.CM.

Here, π_1 and π_2 are the two projection maps $\mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^n$ and $\mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^m$ restricted to V.

CHAPTER 1

Box-shaped matrices and their ideals of 2×2 minors

In this chapter, we generalize the notions of a matrix and its ideal of 2×2 minors to that of a *box-shaped matrix* and its *ideal of* 2×2 *minors* . We concentrate mainly on box-shaped matrices whose entries are algebraically independent over the ground field $\mathfrak{k}$. Such box-shaped matrices are called *generic box-shaped matrices* or *box-shaped matrices of indeterminates.* The primeness of the ideal of 2×2 minors of a generic box-shaped matrix is proved in Section 1.2. Section 1.3 studies its perfection, its Hilbert function and gives a Gröbner basis with respect to the degree reverse lexicographic monomial ordering. In Section 1.4, we investigate a particular class of box-shaped matrices, those of dimension 3. It is proved that the ideal of 2×2 minors of certain box-shaped matrices, which are close to being generic, is also prime.

The techniques we use in this chapter are inspired by those of [**Sha**] in his study of ideals of 2×2 minors of a matrix. Most of our lemmas are generalizations to higher dimensions of those given in [**Sha**]. While some of the proofs follow the same line as their 2-dimensional version, most require more arguments.

1.1. Box-shaped matrices

Let R be a commutative ring that contains a field $\mathfrak{k}$, which is algebraically closed and of characteristic 0. An n-dimensional array of elements in R ($n \geq 2$)

$$\mathcal{A} = (a_{i_1 \dots i_n})_{1 \leq i_j \leq r_j,\ \forall j=1,\dots,n},$$

can be realized as the box

$$\mathbf{B} = \{(i_1, \ldots, i_n) | 1 \leq i_j \leq r_j, \ \forall j\},$$

in which each integral point $(i_1, \ldots, i_n)$ is assigned the value $a_{i_1 \ldots i_n} \in R$.

Definition. An n-dimensional array $\mathcal{A}$, with its box-shaped realization $\mathbf{B}$, is called an *n-dimensional box-shaped matrix* of size $r_1 \times \ldots \times r_n$.

We associate to each box-shaped matrix $\mathcal{A}$ of elements in R a ring $R_{\mathcal{A}} = \mathfrak{k}[\mathcal{A}]$, the subring of R obtained by adjoining the elements of $\mathcal{A}$ to the field $\mathfrak{k}$.

Definition. Suppose $\mathcal{A}$ is an n-dimensional box-shaped matrix of size $r_1 \times \ldots \times r_n$ of elements in R. For each $l = 1, 2, \ldots, n$, we call

$$a_{i_1 \ldots i_l \ldots i_n} a_{j_1 \ldots j_l \ldots j_n} - a_{i_1 \ldots i_{l-1} j_l i_{l+1} \ldots i_n} a_{j_1 \ldots j_{l-1} i_l j_{l+1} \ldots j_n} \in R_{\mathcal{A}},$$

(where $(i_1, \ldots, i_n)$ and $(j_1, \ldots, j_n)$ are any two integral points in $\mathbf{B}$), a 2×2 *minor about the l-th coordinate* of $\mathcal{A}$. A 2×2 *minor* of $\mathcal{A}$ is a 2×2 minor about at least one of its coordinates. We let $I_2(\mathcal{A})$ be the ideal of $R_{\mathcal{A}}$ generated by all the 2×2 minors of $\mathcal{A}$, and call it the *ideal of* 2×2 *minors* of the box-shaped matrix $\mathcal{A}$.

Most of our study in this chapter is done on a special class of box-shaped matrices, those whose entries are algebraically independent over $\mathfrak{k}$.

Definition. A box-shaped matrix of elements in R whose entries are algebraically independent over the ground field $\mathfrak{k}$ is call a *generic box-shaped matrix* or a *box-shaped matrix of indeterminates.*

From now on, unless stated otherwise, we focus our attention on box-shaped matrices of indeterminates. Suppose $\mathcal{A} = (x_{i_1 \ldots i_n})_{(i_1, \ldots, i_n) \in \mathbf{B}}$ is an n-dimensional generic box-shaped matrix of size $r_1 \times \ldots \times r_n$ with its box-shaped realization $\mathbf{B}$. For each $l = 1, \ldots, n$, let

$$\mathcal{A}_l = (x_{i_1 \ldots i_n})_{(i_1, \ldots, i_n) \in \mathbf{B} \text{ and } i_l < r_l},$$

and denote by $I_2(\mathcal{A}_l)$ its ideal of 2×2 minors (in the ring obtained by adjoining the elements of $\mathcal{A}_l$ to $\mathfrak{k}$). For each $l = 1, \ldots, n$, we also let

$$\mathbf{B}_l = \{(i_1, \ldots, i_n) \in \mathbf{B} | i_l = r_l\},$$

and

$$I_l = < I_2(\mathcal{A}_l), \{x_{i_1 \ldots i_n} | (i_1, \ldots, i_n) \in \mathbf{B}_l\} > \subseteq R_{\mathcal{A}}.$$

Throughout this chapter, to any box-shaped matrix $\mathcal{A}$, we always associate box-shaped matrices $\mathcal{A}_l$, boxes $\mathbf{B}_l$ and all the ideals I_l defined as above. The first crucial property of box-shaped matrices of indeterminates comes in the following lemma.

Lemma 1.1. *Suppose $\mathcal{A} = (x_{i_1 \ldots i_n})_{(i_1, \ldots, i_n) \in \mathbf{B}}$ is a box-shaped matrix of indeterminates in R. Then,*

(a) *For any $l \neq s \in \{1, \ldots, n\}$, we have*

$$I_l \cap I_s = < I_2(\mathcal{A}), \{x_{i_1 \ldots i_n} | (i_1, \ldots, i_n) \in \mathbf{B}_l \cap \mathbf{B}_s\} > .$$

(b) *For any distinct elements $l_1, l_2, \ldots, l_t$ of $\{1, 2, \ldots, n\}$ $(2 \leq t \leq n)$, we have*

$$\cap_{j=1}^t I_j = < I_2(\mathcal{A}), \{x_{i_1 \ldots i_n} | (i_1, \ldots, i_n) \in \cap_{j=1}^t \mathbf{B}_j\} > .$$

PROOF. **(a)** For convenience, we denote by LHS and RHS the left hand side and the right hand side of the presented equality, respectively. It is clear that $RHS \subseteq LHS$. We need to show the opposite direction. Let $F \in LHS$. Since $F \in I_l$, we can write $F = F' + F''$, where

$$F' \in I_2(\mathcal{A}_l), \text{ and } F'' = \sum_{(i_1, \ldots, i_n) \in \mathbf{B}_l} F_{i_1 \ldots i_n} x_{i_1 \ldots i_n}.$$

It suffices to show that $F'' \in RHS$. F'' certainly belongs to I_s. Now, for $(i_1, \ldots, i_n) \in \mathbf{B}_l$, we write $F_{i_1 \ldots i_n}$ in the form

$$F_{i_1 \ldots i_n} = \sum_{(j_1, \ldots, j_n) \in \mathbf{B}_s} G_{i_1 \ldots i_n, j_1 \ldots j_n} x_{j_1 \ldots j_n} + G_{i_1 \ldots i_n},$$

where $G_{i_1\dots i_n}$ is independent of the indeterminates $\{x_{j_1\dots j_n}|(j_1,\dots,j_n)\in \mathbf{B}_s\}$. Then $F''=G+G'$, where

$$G=\sum_{(i_1,\dots,i_n)\in\mathbf{B}_l} G_{i_1\dots i_n}x_{i_1\dots i_n},$$

and

$$\begin{aligned} G' &= \sum_{(i_1,\dots,i_n)\in\mathbf{B}_l,(j_1,\dots,j_n)\in\mathbf{B}_s} G_{i_1\dots i_n,j_1\dots j_n}x_{i_1\dots i_n}x_{j_1\dots j_n} \\ &= \sum_{\substack{(i_1,\dots,i_n)\in\mathbf{B}_l,\\(j_1,\dots,j_n)\in\mathbf{B}_s}} \Big(G_{i_1\dots i_n,j_1\dots j_n}X_{i_1\dots i_n,j_1\dots j_n}+x_{i_1\dots i_{s-1}j_s i_{s+1}\dots i_n}T_{i_1\dots i_n,j_1\dots j_n}\Big), \end{aligned}$$

where

$$X_{i_1\dots i_n,j_1\dots j_n}=x_{i_1\dots i_n}x_{j_1\dots j_n}-x_{i_1\dots i_{s-1}j_s i_{s+1}\dots i_n}x_{j_1\dots j_{s-1}i_s j_{s+1}\dots j_n},$$

and $T_{i_1\dots i_n,j_1\dots j_n}\in R_{\mathcal{A}}$. Clearly, $X_{i_1\dots i_n,j_1\dots j_n}$ is a 2×2 minor about the s-th coordinate of $\mathcal{A}$, and since the sum is taken on $(i_1,\dots,i_n)\in\mathbf{B}_l$ and $(j_1,\dots,j_n)\in\mathbf{B}_s$, the point $(i_1,\dots,i_{s-1},j_s,i_{s+1},\dots,i_n)$ belongs to $\mathbf{B}_l\cap\mathbf{B}_s$. Thus, $G'\in RHS$.

It remains to show that $G\in RHS$. Again, we have $G\in I_s$, so we can write:

$$G=H+\sum_{(j_1,\dots,j_n)\in\mathbf{B}_s} H_{j_1\dots j_n}x_{j_1\dots j_n},$$

where $H\in I_2(\mathcal{A}_s)$. We may also assume that H and $H_{j_1\dots j_n}$, where $(j_1,\dots,j_n)\in\mathbf{B}_l\cap\mathbf{B}_s$, are independent of the indeterminates

$$\{x_{j_1\dots j_n}|(j_1,\dots,j_n)\in\mathbf{B}_s\backslash(\mathbf{B}_l\cap\mathbf{B}_s)\}.$$

Then

$$G-H-\sum_{(j_1,\dots,j_n)\in\mathbf{B}_l\cap\mathbf{B}_s} H_{j_1\dots j_n}x_{j_1\dots j_n}=\sum_{(j_1,\dots,j_n)\in\mathbf{B}_s\backslash(\mathbf{B}_l\cap\mathbf{B}_s)} H_{j_1\dots j_n}x_{j_1\dots j_n}.$$

The left hand side of the above equality is independent of all the indeterminates

$$\{x_{j_1\dots j_n}|(j_1,\dots,j_n)\in\mathbf{B}_s\backslash(\mathbf{B}_l\cap\mathbf{B}_s)\}.$$

Thus, both sides must be zero. This implies that

$$G=H+\sum_{(j_1,\dots,j_n)\in\mathbf{B}_l\cap\mathbf{B}_s} H_{j_1\dots j_n}x_{j_1\dots j_n}\subseteq RHS.$$

We have proved $LHS \subseteq RHS$. Thus, the given equality follows.

(b) We will use induction on t. For $t = 2$ the equality is proved in part (a). Suppose $t > 2$, and the equality is true for $t - 1$. We then have

$$\cap_{j=1}^{t-1} I_j = < I_2(\mathcal{A}), \{x_{i_1 \dots i_n} | (i_1, \dots, i_n) \in \cap_{j=1}^{t-1} \mathbf{B}_j\} > .$$

It remains to prove

$$< I_2(\mathcal{A}), \{x_{i_1 \dots i_n} | (i_1, \dots, i_n) \in \cap_{j=1}^{t} \mathbf{B}_j\} > =$$
$$= \ < I_2(\mathcal{A}), \{x_{i_1 \dots i_n} | (i_1, \dots, i_n) \in \cap_{j=1}^{t-1} \mathbf{B}_j\} > \cap \ I_t.$$

We can proceed as in the proof of part (a) to show that the equality above is indeed true. Hence, the presented equality is true for all $2 \leq t \leq n$. □

In particular, we obtain the following corollary.

Corollary 1.1.1. $\cap_{l=1}^{n} I_l = < I_2(\mathcal{A}), x_{r_1 \dots r_n} >$.

1.2. The prime-ideal theorem

Henceforth we shall assume that our ring R is a domain. The primeness of $I_2(\mathcal{A})$ for a generic box-shaped matrix $\mathcal{A}$ comes as a consequence of a series of lemmas.

Lemma 1.2. *Suppose $F(\dots, x_{i_1 \dots i_n}, \dots)$ is an element of $R_{\mathcal{A}} = \mathfrak{k}[\mathcal{A}]$. If, for some $x_{i_1 \dots i_n}$ of $\mathcal{A}$, there exists a positive integer λ such that $x_{i_1 \dots i_n}^{\lambda} F \in I_2(\mathcal{A})$, then, for any $x_{j_1 \dots j_n}$ of $\mathcal{A}$ there exists a non-negative integer ν such that $x_{j_1 \dots j_n}^{\nu} F \in I_2(\mathcal{A})$.*

Proof. Denote by Z the multiplicatively closed subset of $R_{\mathcal{A}}$ consisting of all non-negative powers of $x_{j_1 \dots j_n}$, and let R_Z be the localization of $R_{\mathcal{A}}$ at the set Z. Let $\phi : R_{\mathcal{A}} \to R_Z$ be the ring homomorphism defined by $\phi(c) = c$ for all $c \in \mathfrak{k}$, and

$$\phi(x_{i_1 \dots i_n}) = \frac{x_{i_1 j_2 \dots j_n} x_{j_1 i_2 j_3 \dots j_n} \dots x_{j_1 \dots j_{n-1} i_n}}{x_{j_1 \dots j_n}^{n-1}} \text{ for all } x_{i_1 \dots i_n} \in \mathcal{A}.$$

Obviously, ϕ is a well-defined map. It is easy to verify that $\phi(a) = 0$ for any 2×2 minor a of $\mathcal{A}$. Thus, $\phi(I_2(\mathcal{A})) = 0$. Moreover,

$$x^{\lambda}_{i_1 \dots i_n} F(\dots, x_{i_1 \dots i_n}, \dots) \in I_2(\mathcal{A}).$$

Therefore, in R_Z,

$$\left(\frac{x_{i_1 j_2 \dots j_n} x_{j_1 i_2 j_3 \dots j_n} \dots x_{j_1 \dots j_{n-1} i_n}}{x^{n-1}_{j_1 \dots j_n}} \right)^{\lambda} F\Bigg(\dots, \phi(x_{i_1 \dots i_n}), \dots \Bigg) = 0.$$

Since $R_{\mathcal{A}}$ is a domain, so is R_Z. Hence,

$$F\Bigg(\dots, \phi(x_{i_1 \dots i_n}), \dots \Bigg) = 0 \text{ in } R_Z.$$

Now, using binomial expansions, we can write

$$F(\dots, x_{i_1 \dots i_n}, \dots) = F\Bigg(\dots, \frac{x_{i_1 j_2 \dots j_n} x_{j_1 i_2 j_3 \dots j_n} \dots x_{j_1 \dots j_{n-1} i_n}}{x^{n-1}_{j_1 \dots j_n}}, \dots \Bigg) + K,$$

where K belongs to the ideal of R_Z generated by elements of the form

$$x_{i_1 \dots i_n} - \frac{x_{i_1 j_2 \dots j_n} x_{j_1 i_2 j_3 \dots j_n} \dots x_{j_1 \dots j_{n-1} i_n}}{x^{n-1}_{j_1 \dots j_n}}.$$

The generators of this R_Z-ideal can be rewritten as

$$x_{i_1 \dots i_n} x^{n-1}_{j_1 \dots j_n} - x_{i_1 j_2 \dots j_n} x_{j_1 i_2 j_3 \dots j_n} \dots x_{j_1 \dots j_{n-1} i_n}.$$

We shall prove that K belongs to the ideal of R_Z generated by $I_2(\mathcal{A})$, or equivalently, we prove that these generators, considered as elements of $R_{\mathcal{A}}$, belong to $I_2(\mathcal{A})$. Indeed, using induction on n, modulo $I_2(\mathcal{A})$, we have

$$\begin{aligned} K_n &= x_{i_1 \dots i_n} x^{n-1}_{j_1 \dots j_n} - x_{i_1 j_2 \dots j_n} x_{j_1 i_2 j_3 \dots j_n} \dots x_{j_1 \dots j_{n-1} i_n} \\ &= x_{i_1 j_2 \dots j_n} x_{j_1 i_2 \dots i_n} x^{n-2}_{j_1 \dots j_n} + (x_{i_1 \dots i_n} x_{j_1 \dots j_n} - x_{i_1 j_2 \dots j_n} x_{j_1 i_2 \dots i_n}) x^{n-2}_{j_1 \dots j_n} \\ &\quad - x_{i_1 j_2 \dots j_n} x_{j_1 i_2 j_3 \dots j_n} \dots x_{j_1 \dots j_{n-1} i_n} \\ &\equiv x_{i_1 j_2 \dots j_n} x_{j_1 i_2 \dots i_n} x^{n-2}_{j_1 \dots j_n} - x_{i_1 j_2 \dots j_n} x_{j_1 i_2 j_3 \dots j_n} \dots x_{j_1 \dots j_{n-1} i_n} \\ &= x_{i_1 j_2 \dots j_n} K_{n-1}, \end{aligned}$$

where

$$K_{n-1} = x_{j_1 i_2 \dots i_n} x^{n-2}_{j_1 \dots j_n} - x_{j_1 i_2 j_3 \dots j_n} \dots x_{j_1 \dots j_{n-1} i_n}.$$

Since every indeterminate appearing in the expression K_{n-1} has j_1 in its first index, we can view K_{n-1} as just the same expression as K_n but given by the $(n-1)$-dimensional box-shaped matrix $\mathcal{A}' = (x_{i_1 \ldots i_n})_{i_1 = j_1}$. By induction hypothesis, K_{n-1} then belongs to $I_2(\mathcal{A}') \subseteq I_2(\mathcal{A})$. And hence, $K_n \in I_2(\mathcal{A})$.

We have just proved that K belongs to the ideal of R_Z generated by $I_2(\mathcal{A})$. Equivalently, $F(\ldots, x_{i_1 \ldots i_n}, \ldots)$ belongs to the ideal of R_Z generated by $I_2(\mathcal{A})$. Therefore, there exists a ν such that

$$x^{\nu}_{j_1 \ldots j_n} F(\ldots, x_{i_1 \ldots i_n}, \ldots) \in I_2(\mathcal{A}) \text{ in } R_{\mathcal{A}}.$$

The lemma is proved. □

Lemma 1.3. *Suppose $l \in \{1, 2, \ldots, n\}$. Suppose also that $F \in R_{\mathcal{A}} = \mathfrak{k}[\mathcal{A}]$ is a polynomial independent of the indeterminates $x_{i_1 \ldots i_n}$ for all $(i_1, \ldots, i_n) \in \mathbf{B}_l$ such that $I_2(\mathcal{A}_l) : F = I_2(\mathcal{A}_l)$. Then $I_l : F = I_l$.*

Proof. Let $FG \in I_l$. We need to show that $G \in I_l$. Write $G = G_1 + G_2$, where $G_1 \in < \{x_{i_1 \ldots i_n} | (i_1, \ldots, i_n) \in \mathbf{B}_l\} >$ and G_2 is independent of the indeterminates $x_{i_1 \ldots i_n}$ for $(i_1, \ldots, i_n) \in \mathbf{B}_l$. Then $G_1 \in I_l$, so it remains to show that $G_2 \in I_l$. We now have $FG_2 = FG - FG_1 \in I_l$. Moreover, since both F and G_2 are independent of the indeterminates $x_{i_1 \ldots i_n}$ for all $(i_1, \ldots, i_n) \in \mathbf{B}_l$, FG_2 is also independent of those indeterminates, and so $FG_2 \in I_2(\mathcal{A}_l)$. This implies $G_2 \in I_2(\mathcal{A}_l) \subseteq I_l$, and the lemma is proved. □

Lemma 1.4. *Let $F \in R_{\mathcal{A}} = \mathfrak{k}[\mathcal{A}]$ and suppose that $x^{\lambda}_{1 \ldots 1} F \in I_2(\mathcal{A})$ for some positive integer λ. Then $F \in I_2(\mathcal{A})$. In other words, $I_2(\mathcal{A}) : x^{\lambda}_{1 \ldots 1} = I_2(\mathcal{A})$.*

Proof. We use induction on n. When $n = 2$, the result follows from that of [**Sha**]. Suppose $n > 2$, $\mathcal{A}$ is an n-dimensional box-shaped matrix of indeterminates, and the lemma is true for any box-shaped matrix of lower dimension. We now use induction on $r_1 + \ldots + r_n$. We may assume that $r_i \geq 2$ for all $i = 1, \ldots, n$ (since otherwise, $\mathcal{A}$ collapses to an $(n-1)$-dimensional box-shaped matrix, and the result follows from the induction hypothesis), and the lemma is true for any n-dimensional box-shaped matrix with smaller value of $r_1 + \ldots + r_n$.

If F is of degree zero, then $x^{\lambda}_{1\ldots1}F$ belongs to the ideal of 2×2 minors of a box-shaped matrix obtained from $\mathcal{A}$ by letting all the indeterminates $x_{i_1\ldots i_n}$, for $(i_1, \ldots, i_n) \neq (1, \ldots, 1)$, be zero. Yet, this ideal is zero, so $F = 0 \in I_2(\mathcal{A})$. We may use induction again, assuming that the degree of F is bigger than zero, and the lemma holds for polynomials whose degree is smaller than that of F.

Now, clearly $x^{\lambda}_{1\ldots1}F \in I_2(\mathcal{A}) \subseteq \cap^{n}_{j=1}I_j$, so in particular, $x^{\lambda}_{1\ldots1}F \in I_j$ for all j. Moreover, by the induction hypothesis, we have $I_2(\mathcal{A}_j) : x^{\lambda}_{1\ldots1} = I_2(\mathcal{A}_j)$. Thus, by Lemma 1.3, $I_j : x^{\lambda}_{1\ldots1} = I_j$. This implies that $F \in \cap^{n}_{j=1}I_j = < I_2(\mathcal{A}), x_{r_1\ldots r_n} >$ (Corollary 1.1.1). Write $F = F_1 + x_{r_1\ldots r_n}F_2$, where $F_1 \in I_2(\mathcal{A})$. Since $I_2(\mathcal{A})$ is homogeneous, we may assume that the degree of F_2 is smaller than that of F. We have $x^{\lambda}_{1\ldots1}F = x^{\lambda}_{1\ldots1}F_1 + x^{\lambda}_{1\ldots1}x_{r_1\ldots r_n}F_2 \in I_2(\mathcal{A})$. Thus, $x_{r_1\ldots r_n}x^{\lambda}_{1\ldots1}F_2 \in I_2(\mathcal{A})$. By Lemma 1.2, there is a non-negative integer ν such that $x^{\lambda+\nu}_{1\ldots1}F_2 = x^{\nu}_{1\ldots1}x^{\lambda}_{1\ldots1}F_2 \in I_2(\mathcal{A})$. By our induction hypothesis on the degree of F, we have $F_2 \in I_2(\mathcal{A})$. Hence, $F \in I_2(\mathcal{A})$ as required. □

The primeness of the ideal of 2×2 minors of a box-shaped matrix in the generic case is stated as follows.

Theorem 1.5. *If $\mathcal{A}$ is a box-shaped matrix of indeterminates, then $I_2(\mathcal{A})$ is a prime ideal in $\mathfrak{k}[\mathcal{A}]$.*

Proof. Suppose that $F(\ldots, x_{i_1\ldots i_n}, \ldots)G(\ldots, x_{i_1\ldots i_n}, \ldots) \in I_2(\mathcal{A})$, where $F, G \in R_{\mathcal{A}} = \mathfrak{k}[\mathcal{A}]$. Let Z be the multiplicatively closed subset of $R_{\mathcal{A}}$ consisting of all non-negative powers of $x_{1\ldots1}$, and let R_Z be the localization of $R_{\mathcal{A}}$ at Z. Similar to what was done in Lemma 1.2, we define a map

$$\varphi : R_{\mathcal{A}} \to R_Z,$$

by sending $\mathfrak{k}$ to $\mathfrak{k}$, and sending $x_{i_1\ldots i_n}$ to $\frac{x_{i_1 1\ldots 1}x_{1 i_2 1\ldots 1}\ldots x_{1\ldots 1 i_n}}{x_{1\ldots 1}^{n-1}}$ for all $x_{i_1\ldots i_n} \in \mathcal{A}$. It is easy to verify that $\varphi(a) = 0$ for any 2×2 minors a of $\mathcal{A}$. Thus, $\varphi(I_2(\mathcal{A})) = 0$. Moreover, $F(\ldots, x_{i_1\ldots i_n}, \ldots)G(\ldots, x_{i_1\ldots i_n}, \ldots) \in I_2(\mathcal{A})$. Hence, in R_Z,

$$F(\ldots, \varphi(x_{i_1\ldots i_n}), \ldots)\; G(\ldots, \varphi(x_{i_1\ldots i_n}), \ldots) = 0.$$

Since $R_{\mathcal{A}}$ is a domain, so is R_Z. Thus, at least one of the two factors has to be zero. Suppose

$$F(\ldots, \frac{x_{i_1 1\ldots 1}x_{1 i_2 1\ldots 1}\ldots x_{1\ldots 1 i_n}}{x_{1\ldots 1}^{n-1}}, \ldots) = 0.$$

Now, as in the proof of Lemma 1.2, we deduce that there exists a ν such that $x_{1\ldots 1}^{\nu}F(\ldots, x_{i_1\ldots i_n}, \ldots) \in I_2(\mathcal{A})$ in $R_{\mathcal{A}}$. Hence, by Lemma 1.4, $F \in I_2(A)$, and this completes the proof. □

1.3. Segre embedding, Cohen-Macaulayness and Koszul property

Suppose $V_1, V_2, \ldots, V_n$ are vector spaces of dimensions $r_1, r_2, \ldots, r_n$, respectively. Recall the following definition.

Definition. A tensor $z \in V_1 \otimes \ldots \otimes V_n$ is said to be *decomposable* if there exist $v_j \in V_j$ for all $j = 1, \ldots, n$, such that $z = v_1 \otimes \ldots \otimes v_n$.

Now, let $\{e_{j1}, \ldots, e_{jr_j}\}$ be a basis for V_j for all $j = 1, \ldots, n$. Then a basis of $V_1 \otimes \ldots \otimes V_n$ is given by

$$\{\epsilon_{i_1\ldots i_n} = e_{1i_1} \otimes \ldots \otimes e_{ni_n} | 1 \leq i_j \leq r_j \;\forall j = 1, \ldots, n\}.$$

A tensor $z \in V_1 \otimes \ldots \otimes V_n$ is represented by

$$z = \sum_{i_1\ldots i_n} y_{i_1\ldots i_n}\epsilon_{i_1\ldots i_n},$$

and a vector $v_j \in V_j$ is given by

$$v_j = \sum_{k=1}^{r_j} u_{jk}e_{jk}.$$

Thus, to have $z = v_1 \otimes \ldots \otimes v_n$, is the same as to have

$$y_{i_1 \ldots i_n} = u_{1i_1} \ldots u_{ni_n}, \text{ for all } i_1 \ldots i_n.$$

These are the equations describing the image of the following Segre embedding (cf. [**Hart**, Exercise I.2.14] when $n = 2$):

$$\mathbb{P}(V_1) \times \ldots \times \mathbb{P}(V_n) \hookrightarrow \mathbb{P}(V_1 \otimes \ldots \otimes V_n).$$

Hence, a tensor $z \in V_1 \otimes \ldots \otimes V_n$ is decomposable if and only if its corresponding point in $\mathbb{P}(V_1 \otimes \ldots \otimes V_n)$ is in the image of the above Segre embedding.

The geometric realization of the ideal of 2×2 minors of a generic matrix $\mathcal{A}$ comes from the work of Grone ([**Grone**]), which we rephrase in the following proposition.

Proposition 1.6 (Grone, 1977)**.** *Suppose $\mathcal{A}$ is a generic box-shaped matrix of size $r_1 \times \ldots \times r_n$, and $V_1, \ldots, V_n$ are vector spaces of dimension $r_1, \ldots, r_n$, respectively. Then $I_2(\mathcal{A})$ gives a set of equations that describe the decomposable tensors in $V_1 \otimes \ldots \otimes V_n$.*

Since the Segre embedding of the product of several projective spaces is a closed immersion, Grone's result gives an immediate corollary, which demonstrates the geometric realization of $I_2(\mathcal{A})$.

Corollary 1.6.1. *If $\mathcal{A}$ is an n dimensional generic box-shaped matrix of size $r_1 \times \ldots \times r_n$, then $I_2(\mathcal{A})$ gives the defining ideal of the Segre embedding*

$$\mathbb{P}(V_1) \times \ldots \times \mathbb{P}(V_n) \hookrightarrow \mathbb{P}(V_1 \otimes \ldots \otimes V_n).$$

where $V_1, \ldots, V_n$ are vector spaces of dimensions $r_1, \ldots, r_n$, respectively.

Proof. The result follows from the fact that $I_2(\mathcal{A})$ is a prime ideal. □

From this, we can calculate the Hilbert function of the ideal of 2×2 minors of a generic box-shaped matrix as follows.

Proposition 1.7. *The Hilbert function of $I_2(\mathcal{A})$ is*

$$\mathbf{H}(I_2(\mathcal{A}), t) = \binom{\prod_{i=1}^n r_i + t - 1}{t} - \prod_{i=1}^n \binom{r_i + t - 1}{t} \ \forall t \geq 0.$$

Proof. It is easy to see that all homogeneous polynomials of degree t on $\mathbb{P}^{\prod r_i - 1}$ restricted to the image of $\mathbb{P}^{r_1-1} \times \ldots \times \mathbb{P}^{r_n-1}$ gives all multi-homogeneous polynomials of degree $(t, \ldots, t)$ in $\mathbb{P}^{r_1-1} \times \ldots \times \mathbb{P}^{r_n-1}$ (cf. [**Harr**]). Thus the Hilbert function of the homogeneous coordinate ring of the Segre embedding is $\prod_{i=1}^n \binom{r_i+t-1}{t}$. The proposition now follows. □

Remark: It is clear that any Segre embedding is Hilbertian, i.e. its Hilbert function and its Hilbert polynomial are the same (in non-negative degrees).

We now recall the notion of a perfect ideal in a polynomial ring.

Definition. Suppose A is a polynomial ring, $A = \mathfrak{k}[x_1, \ldots, x_m]$, a proper ideal $I \subseteq A$ is said to be *perfect* if the quotient ring A/I is Cohen-Macaulay.

The geometric realization of $I_2(\mathcal{A})$ and Propostion 1.7 give us the perfection of $I_2(\mathcal{A})$. The result is stated as follows.

Theorem 1.8. *If $\mathcal{A}$ is an n-dimensional generic box-shaped matrix of size $r_1 \times \ldots \times r_n$, then $I_2(\mathcal{A})$ is a perfect ideal of grade $\prod_{i=1}^n r_i - \sum_{i=1}^n r_i + (n-1)$.*

Proof. We let $R_i = \mathfrak{k}[y_{i,1}, \ldots, y_{i,r_i}]$ be the homogeneous coordinate ring of $\mathbb{P}^{r_i-1}$ for all i. Clearly, R_i is Cohen-Macaulay for all i. By results of [**S-V**, page 378] and Proposition 1.7, it follows by induction on n that the Segre product $\underline{\otimes}_{i=1}^n R_i$ is a Cohen-Macaulay ring. Furthermore, this ring is exactly the coordinate ring of the Segre embedding $\mathbb{P}^{r_1-1} \times \ldots \times \mathbb{P}^{r_n-1} \hookrightarrow \mathbb{P}^{\prod r_i - 1}$. Thus, since $I_2(\mathcal{A})$ is the defining ideal of this Segre embedding, i.e. $\underline{\otimes}_{i=1}^n R_i \simeq \mathfrak{k}[\mathcal{A}]/I_2(\mathcal{A})$, we have $I_2(\mathcal{A})$ is a perfect ideal. The grade of $I_2(\mathcal{A})$ comes from the codimension of the Segre embedding, which is exactly $\prod_{i=1}^n r_i - \sum_{i=1}^n r_i + (n-1)$. The theorem is proved. □

Remark: The perfection of $I_2(\mathcal{A})$ also comes from a more general result of Hochster ([**Ho-1**, Theorem 1]).

We have an immediate corollary.

Corollary 1.8.1. *Suppose $V_1, \ldots, V_n$ are vector spaces of dimensions $r_1, \ldots, r_n$. Then, the homogeneous coordinate ring of the Segre embedding*

$$\mathbb{P}(V_1) \times \ldots \times \mathbb{P}(V_n) \hookrightarrow \mathbb{P}(V_1 \otimes \ldots \otimes V_n)$$

is always Cohen-Macaulay.

The following theorem gives a Gröbner basis for $I_2(\mathcal{A})$.

Theorem 1.9. *Suppose $\mathcal{A} = (x_{i_1 \ldots i_n})$ is a generic box-shaped matrix of size $r_1 \times \ldots \times r_n$. Then, under the degree reverse lexicographic monomial ordering on $R_{\mathcal{A}} = \mathfrak{k}[\mathcal{A}]$, in which the variables $x_{i_1 \ldots i_n}$ are ordered by lexicographic ordering on their indices (assuming that $1 < 2 < \ldots < n$), the 2×2 minors of $\mathcal{A}$ form a Gröbner basis for $I_2(\mathcal{A})$.*

Proof. Let $\leq_{\text{lex}}$ be the lexicographic ordering on $\mathbb{N}^n$. We order the variables of $R_{\mathcal{A}}$ by

$$x_{i_1 \ldots i_n} \leq x_{j_1 \ldots j_n} \Leftrightarrow (i_1, \ldots, i_n) \leq_{\text{lex}} (j_1, \ldots, j_n),$$

and use degree reverse lexicographic ordering on the monomials of $R_{\mathcal{A}}$. We shall prove that under this monomial ordering, the 2×2 minors of $\mathcal{A}$ form a Gröbner basis for $I_2(\mathcal{A})$.

Let $\mathcal{G}$ be the collection of all 2×2 minors of $\mathcal{A}$. It suffices to show that the leading terms of $\mathcal{G}$ generate the leading term ideal of $I_2(\mathcal{A})$. By contradiction, suppose $F \in I_2(\mathcal{A})$, and T, the leading term of F, is not generated by the leading terms of $\mathcal{G}$. Clearly, from the nature of $I_2(\mathcal{A})$, T is a monomial with at least 2 different indeterminates. We consider a new partial ordering on the indeterminates of $R_{\mathcal{A}}$, defined by

$$x_{i_1 \ldots i_n} \preceq x_{j_1 \ldots j_n} \Leftrightarrow i_l \leq j_l \ \forall l = 1, \ldots, n.$$

Suppose $x_{i_1\ldots i_n}$ and $x_{j_1\ldots j_n}$ are any two different indeterminates present in T. Without loss of generality, assume that $x_{i_1\ldots i_n} < x_{j_1\ldots j_n}$, i.e. there exists a positive integer u such that $i_l = j_l$ for all $l = 1,\ldots,u-1$, and $i_u < j_u$. It is easy to see that if $x_{i_1\ldots i_n} \not\preceq x_{j_1\ldots j_n}$ then there exists another integer $v > u$ such that $i_v > j_v$. In this case,

$$x_{i_1\ldots i_{v-1}j_v i_{v+1}\ldots i_n} < x_{i_1\ldots i_n}, x_{j_1\ldots j_n} < x_{j_1\ldots j_{v-1}i_v j_{v+1}\ldots j_n}.$$

Thus, $x_{i_1\ldots i_n}x_{j_1\ldots j_n}$ is the leading term of

$$x_{i_1\ldots i_n}x_{j_1\ldots j_n} - x_{i_1\ldots i_{v-1}j_v i_{v+1}\ldots i_n}x_{j_1\ldots j_{v-1}i_v j_{v+1}\ldots j_n} \in \mathcal{G},$$

whence T is generated by the leading terms of $\mathcal{G}$, a contradiction. Hence, these two indeterminates must be comparable, i.e. $x_{i_1\ldots i_n} \preceq x_{j_1\ldots j_n}$. This is true for any two different indeterminates of T. Therefore, T can be rewritten as

$$T = x_{t_{11}\ldots t_{1n}}x_{t_{21}\ldots t_{2n}}\ldots x_{t_{p1}\ldots t_{pn}},$$

for some positive integer $p \geq 2$, where

$$x_{t_{11}\ldots t_{1n}} \preceq x_{t_{21}\ldots t_{2n}} \preceq \ldots \preceq x_{t_{p1}\ldots t_{pn}}.$$

Now, let $[y_{i,1} : \ldots : y_{i,r_i}]$ represent the homogeneous coordinates of $\mathbb{P}^{r_i-1}$ for all $i = 1,\ldots,n$. Since $I_2(\mathcal{A})$ is the defining ideal of the Segre embedding

$$\mathbb{P}^{r_1-1} \times \ldots \times \mathbb{P}^{r_n-1} \hookrightarrow \mathbb{P}^{\prod r_i - 1},$$

F vanishes when we substitute the indeterminate $x_{i_1\ldots i_n}$ by $\prod_{l=1}^{n} y_{l,i_l}$ for all $(i_1,\ldots,i_n)$. It is also clear that after this substitution, F becomes a polynomial on the variables $y_{i,j}$. This polynomial is zero for all values of the variables $y_{i,j}$, so it must be the zero polynomial (since the ground field $\mathfrak{k}$ is infinite). This implies that there must be a term T' of F ($T' \neq T$) which cancels T after the substitution. Suppose $x_{k_1\ldots k_n}$ is an indeterminate present in T'. Since T' cancels T after the substitution, for each $l = 1,\ldots,n$, $k_l \in \{t_{1l},\ldots,t_{pl}\}$. From the partial ordering on the indeterminates in T, it is now clear that $k_l \geq t_{1l}$ for all $l = 1,\ldots,n$, whence $x_{t_{11}\ldots t_{1n}} \leq x_{k_1\ldots k_n}$. If $x_{t_{11}\ldots t_{1n}} < x_{k_1\ldots k_n}$ for every indeterminate $x_{k_1\ldots k_n}$ in T', then $T < T'$, which is a contradiction since T is the leading term of F. Otherwise, suppose $x_{t_{11}\ldots t_{1n}}$ is contained in T', then by considering $T/x_{t_{11}\dots t_{1n}}$ and $T'/x_{t_{11}\dots t_{1n}}$, and continuing the process, we eventually would, again, get a contradiction.

The theorem is proved. □

Remark: From the proof above, it is easy to see that the 2×2 minors of $\mathcal{A}$ form a Gröbner basis for $I_2(\mathcal{A})$ under any monomial ordering on $R_{\mathcal{A}}$ that satisfies the condition that if $g = x_{i_1\dots i_n}x_{j_1\dots j_n} - x_{p_1\dots p_n}x_{q_1\dots q_n}$ is an element of $\mathcal{G}$, where $x_{p_1\dots p_n} \preceq x_{q_1\dots q_n}$, then $x_{i_1\dots i_n}x_{j_1\dots j_n}$ is the leading term of g. Degree reverse lexicographic monomial ordering is merely one of those monomial orderings that satisfies this condition. We choose this ordering since it is available in most computational algebra packages, such as CoCoA and Macaulay2.

This theorem gives rise to an interesting corollary.

Corollary 1.9.1. *Suppose $V_1, \dots, V_n$ are vector spaces of dimensions $r_1, \dots, r_n$. Then, the homogeneous coordinate ring of the Segre embedding*

$$\mathbb{P}(V_1) \times \dots \times \mathbb{P}(V_n) \hookrightarrow \mathbb{P}(V_1 \otimes \dots \otimes V_n)$$

is a Koszul algebra.

Proof. This follows from the fact that all 2×2 minors of $\mathcal{A}$ are quadratic forms. □

1.4. 3-dimensional box-shaped matrices

In this section, we briefly look at a particular class of box-shaped matrices, those of dimension 3. Besides the usual matrices, 3-dimensional box-shaped matrices are the easiest that can be visualized. To visualize all the 2×2 minors of a 3-dimensional box-shaped matrix, one only needs to take any two lines parallel to one of the axes, and looks at their intersection with any two planes parallel to the other two axes of our fixed system of coordinates. 3-dimensional box-shaped matrices not only describe the

Segre embedding of the product of 3 projective spaces, but also give a tool in studying certain blowup surfaces, as it will be discussed in Chapter 3. We first extend the notion of a box-shaped matrix of indeterminates to that of a *weak box-shaped matrix of indeterminates.*

Definition. Suppose $\mathcal{A} = (a_{ijk})_{(i,j,k)\in\mathbf{B}}$ is a box-shaped matrix of forms in a ring R. For each integer l let $\mathcal{A}_{(x,l)}$ be the matrix given by the collection $\{a_{ijk}|(i,j,k) \in \mathbf{B}, i = l\}$. We call $\mathcal{A}_{(x,l)}$ an *x-section* of the box-shaped matrix $\mathcal{A}$. The y-sections and z-sections of $\mathcal{A}$ are defined similarly.

Definition. A box-shaped matrix $\mathcal{A} = (x_{ijk})_{(i,j,k)\in\mathbf{B}}$, with box-shaped realization $\mathbf{B}$, of forms in a domain R is called a *weak box-shaped matrix of indeterminates* if

(a) All the entries in $\mathcal{A}$ are indeterminates of R (allowing repetition).
(b) $< I_2(\mathcal{A}), x_{r_1r_2r_3} > = \cap_{l=1}^{3} I_l$ where the ideals I_l are defined as that of a general n-dimensional box-shaped matrix.
(c) There exists an integral point $(i,j,k) \in B$ such that when we set all indeterminates other than x_{ijk} of $\mathfrak{k}[\mathcal{A}]$ to zero, the ideal $I_2(\mathcal{A})$ is the zero ideal.
(d) The ideals of 2×2 minors of sections $\mathcal{A}_{(x,i)}$, $\mathcal{A}_{(y,j)}$ and $\mathcal{A}_{(z,k)}$ are prime ideals.

Example: Consider $S = \mathfrak{k}[x_1, x_2, x_3, y_1, y_2, y_3]$. Let $\mathbf{B}$ be the $2{\times}2{\times}2$ box $\{(i,j,k) \mid 0 \leq i,j,k \leq 1\}$, and let $\mathcal{A} = (a_{ijk})_{(i,j,k)\in\mathbf{B}}$ be the box-shaped matrix given by $a_{000} = x_1, a_{100} = x_2 = a_{010}, a_{110} = x_3, a_{001} = y_1, a_{101} = y_2 = a_{011}$ and $a_{111} = y_3$. Then $\mathcal{A}$ is a weak box-shaped matrix of indeterminates, but not a box-shaped matrix of indeterminates.

With this bigger class of box-shaped matrices, the primeness of their ideals of 2×2 minors still holds.

Proposition 1.10. *$I_2(\mathcal{A})$ is a prime ideal in $\mathfrak{k}[\mathcal{A}]$ for any weak box-shaped matrix of indeterminates $\mathcal{A}$.*

PROOF. First, we can always re-arrange the indices such that (i, j, k) becomes $(1, 1, 1)$. The proof now follows along the same lines as the proof of Theorem 1.5. $\square$

CHAPTER 2

Blowup of $\mathbb{P}^n$ at a subscheme

The blowup of $\mathbb{P}^n$ at a subscheme is a classical notion which is formally defined as the Proj of a sheaf of graded $\mathcal{O}_{\mathbb{P}^n}$-algebras ([**Hart**]). In the last fifteen years, in studying the blowup of $\mathbb{P}^n$ along a set of points, many authors have considered this blowup, embedded into projective spaces, as the closure of the image of certain rational maps (cf. [**Gi-1**], [**Gi-2**], [**G-G**], [**Gi-Lo**], [**G-G-P**], [**Hol**], [**Hol-1**], [**Hol-2**]). It seems to be part of the folklore that this way of looking at the blowup of $\mathbb{P}^n$ at a set of points agrees with the classical definition of the blowup. Unfortunately, nowhere in the literature could I find a proof of the equivalence of these two ways of looking at the blowup. The aim of this chapter is to fill this gap, i.e. to establish an equivalence between the two definitions as mentioned.

In Section 2.1, a realization of the blowup of $\mathbb{P}^n$ at a subscheme is discussed. We prove that the blowup of $\mathbb{P}^n$ at a subscheme, embedded into appropriate product spaces, is the Bi-Proj of certain Rees algebras. In fact, we show that the blowup of $\mathbb{P}^n$ at a subscheme, embedded into product spaces, is the closure of the graph of certain rational maps.

Section 2.2 studies a special class of rational n-folds, the class of the blowups of $\mathbb{P}^n$ at a set of points, $\mathbb{X}$. We investigate a particular class of very ample divisors on these blowups, namely those corresponding to linear systems of hypersurfaces containing $\mathbb{X}$. This class of very ample divisors gives a class of projective embeddings of the blowup of $\mathbb{P}^n$ at $\mathbb{X}$. When $n = 2$ these embeddings are the objects of study in Chapter 3 and in Section 4.2.

2.1. Bi-Proj and blowup of $\mathbb{P}^n$ at a subscheme

Generally, the blowup of a noetherian scheme along a subscheme is classically defined as follows.

Definition. Suppose X is a noetherian scheme, and V is a subscheme of X defined by the ideal sheaf $\mathcal{J}$. Let $\Im = \oplus_{t\geq 0}\mathcal{J}^t$ be the sheaf of graded $\mathcal{O}_X$-algebras where $\mathcal{J}^t$ is the t^{th} power of $\mathcal{J}$ and $\mathcal{J}^0 = \mathcal{O}_X$. The blowup of X along V is defined to be Proj $\Im$, together with the natural projection Proj $\Im \to X$. This is also defined to be the blowup of X along the ideal sheaf $\mathcal{J}$.

Example ([**Hart**, Example II.7.12.1]): Suppose X is $\mathbb{A}^n_{\mathfrak{k}}$, the n-dimensional affine space over $\mathfrak{k}$, and P is the origin. $X = \text{Spec}A$ where $A = \mathfrak{k}[x_1, \ldots, x_n]$, and P corresponds to the ideal $J = (x_1, \ldots, x_n)$. The blowup of X along P is Proj S, where $S = \oplus_{t\geq 0}J^t$. Let $\varphi : A[y_1, \ldots, y_n] \to S$ be the surjective map of graded rings given by sending y_i to the element $x_i \in J$, considered as an element of S in degree 1. Then, Proj S is isomorphic to the closed subscheme of Proj $A[y_1, \ldots, y_n] = \mathbb{P}^{n-1}_A$ defined by the kernel of φ, which is generated by the polynomials $\{x_iy_j - x_jy_i \mid i, j = 1, \ldots, n\}$. This coincides with the classical definition of blowing up an n-dimensional affine space at a point ([**Hart**, Chapter I]).

We shall now briefly say what a Bi-Proj of a bi-graded algebra is. For a more detailed discussion on Bi-Proj and its structure sheaf, we refer the readers to [**STV**], [**Hyry**] and [**Vid**].

Definition. Suppose $A = \oplus_{i,j\in\mathbb{Z}}A_{i,j}$ is a bi-graded $\mathfrak{k}$-algebra which is generated, as $\mathfrak{k}$-algebra, by $A_{0,1}$ and $A_{1,0}$. Let A_+ be the A-ideal $\oplus_{i,j\geq 1}A_{i,j}$. The Bi-Proj of A, as a set, is defined as follows.

$$\text{Bi-Proj } A = \{\wp \mid \wp \text{ a bi-homogeneous prime ideal of } A, \wp \not\supseteq A_+\}.$$

We refer the readers back to Chapter 0 for the definition of the Rees algebra of an ideal before proceeding with the next lemma.

Lemma 2.1. *Let V be a subscheme of $\mathbb{A}^n_{\mathfrak{k}}$ defined by the ideal $I \subseteq \mathfrak{k}[x_1, \ldots, x_n]$. Then, the blowup of $\mathbb{A}^n_{\mathfrak{k}}$ along V is the scheme Proj $\mathcal{R}(I)$.*

Proof. The proof follows easily from the discussion in the Example presented above. □

To proceed, suppose $\mathbb{P}^n = \mathbb{P}^n_{\mathfrak{k}}$ is the projective n-dimensional space over $\mathfrak{k}$ and $R = \mathfrak{k}[x_0, \ldots, x_n]$ is its coordinate ring. From now until the end of this section, we let V be a subscheme of $\mathbb{P}^n$ given by the homogeneous ideal $I \subseteq R$. Suppose also that $\mathcal{I}$ is the ideal sheaf of V (i.e. $\mathcal{I} = \tilde{I}$).

Suppose J is a proper homogeneous ideal of R. Then, as a subring of $R[t]$, the Rees algebra of J inherits the natural bi-gradation of $R[t]$, where $\deg(x_i) = (1, 0)$ and $\deg(t) = (0, 1)$. Thus, the Rees algebra $\mathcal{R}(J)$ of J has a natural bi-gradation associated to which there is a Bi-Proj structured scheme.

Lemma 2.2. *As sets, we have*

$$\{\wp \in \textit{Bi-Proj}\ \mathcal{R}(J) \mid x_i \notin \wp\} = \textit{Proj}\ \mathcal{R}(J)_{(x_i)},$$

where $\mathcal{R}(J)_{(x_i)}$ is the homogeneous localization of $\mathcal{R}(J)$ at the element x_i.

Proof. On one hand, for each $\wp \in$ Bi-Proj $\mathcal{R}(J)$ such that $x_i \notin \wp$, by homogeneous localization at x_i we get a proper homogeneous (since the degree in t does not change) prime ideal of $\mathcal{R}(J)_{(x_i)}$. On the other hand, for each $\wp' \in$ Proj $\mathcal{R}(J)_{(x_i)}$, by using x_i as an extra variable to homogenize $\wp'$ (with respect to the degree on $x_0, \ldots, x_n$), we get a bi-homogeneous prime ideal of $\mathcal{R}(J)$. Moreover, the composition of these two processes is clearly the identity. Thus, there is an one-to-one correspondence between $\{\wp \in$ Bi-Proj $\mathcal{R}(J) \mid x_i \notin \wp\}$ and Proj $\mathcal{R}(J)_{(x_i)}$. □

We have the first important result of this section is the following theorem.

Theorem 2.3. *Suppose V is a subscheme of $\mathbb{P}^n$ given by the ideal I. Then, the blowup of $\mathbb{P}^n$ along V is the scheme Bi-Proj $\mathcal{R}(I)$, where $\mathcal{R}(I)$ is the Rees algebra* *of I and inherits a natural bi-gradation from that of $R[t]$ ($R = \mathfrak{k}[x_0, \ldots, x_n]$ is the homogeneous coordinate ring of $\mathbb{P}^n$).*

Proof. Suppose $\mathcal{I}$ is the ideal sheaf of V inside $\mathbb{P}^n$. We first have

$$\mathbb{P}^n = \text{Proj } R = \cup_{i=0}^n \text{Spec} R_{(x_i)},$$

where $R_{(x_i)}$ is the homogeneous localization of R at x_i. Moreover, the ideal sheaf $\mathcal{I}$ is then given by the ideals $I_{(x_i)} \subseteq R_{(x_i)}$. Thus, it follows from Lemma 2.1 that the blowup of $\mathbb{P}^n$ along V is the natural gluing of the affine blowups Proj $R_{(x_i)}[I_{(x_i)}t]$. Since $R_{(x_i)}[I_{(x_i)}t] = \mathcal{R}(I)_{(x_i)}$, we can identify Proj $R_{(x_i)}[I_{(x_i)}t]$ with $\{\wp \in \text{Bi-Proj } \mathcal{R}(I) \mid x_i \notin \wp\}$ (Lemma 2.2). Hence, the gluing of Proj $R_{(x_i)}[I_{(x_i)}t]$ corresponds to

$$\text{Bi-Proj } \mathcal{R}(I) = \cup_{i=0}^n \{\wp \in \text{Bi-Proj } \mathcal{R}(I) \mid x_i \notin \wp\}$$

in a natural way. The theorem is proved. □

Now, suppose $I = \oplus_{\gamma \geq \alpha} I_\gamma \subseteq R = \mathfrak{k}[x_0, \ldots, x_n]$ is the homogeneous decomposition of the defining ideal of a subscheme V in $\mathbb{P}^n$. For each $\gamma \geq \alpha$, we can use I_γ to define a rational map

$$\varphi_\gamma : \mathbb{P}^n \dashrightarrow \mathbb{P}^{m_\gamma},$$

by sending each point $P \in \mathbb{P}^n \backslash Z(I_\gamma)$ to $[F_{\gamma 0}(P) : \ldots : F_{\gamma m_\gamma}(P)]$, where $Z(I_\gamma)$ is the zero set of the ideal defined by I_γ, and $\{F_{\gamma 0}, \ldots, F_{\gamma m_\gamma}\}$ is a system of generators for I_γ as a $\mathfrak{k}$-vector space. Let $\Gamma_\gamma \subseteq \mathbb{P}^n \times \mathbb{P}^{m_\gamma}$ and $\Lambda_\gamma \subseteq \mathbb{P}^{m_\gamma}$ be the graph and the image of this rational map φ_γ respectively, and let $\overline{\Gamma_\gamma}$ and $\overline{\Lambda_\gamma}$ be the closures of Γ_γ and Λ_γ. Moreover, denote by $\mathcal{R}(I_\gamma)$ the Rees aglebra of the ideal generated by I_γ.

Proposition 2.4. *The Rees algebra $\mathcal{R}(I_\gamma)$ is the bi-graded coordinate ring of $\overline{\Gamma_\gamma}$.*

Proof. Let $S = \mathfrak{k}[x_0, \ldots, x_n, y_0, \ldots, y_{m_\gamma}]$ be the coordinate ring of $\mathbb{P}^n \times \mathbb{P}^{m_\gamma}$. S can be viewed as a bi-graded $\mathfrak{k}$-algebra with $\deg(x_i) = (1, 0)$ and $\deg(y_j) = (0, 1)$. Consider the $\mathfrak{k}$-algebra homomorphism

$$\chi : S \to \mathcal{R}(I_\gamma)$$

given by sending x_i to x_i for all i, and y_j to $F_{\gamma j}t$ for all j. It is clear that χ is a surjective bi-graded homomorphism. Thus kerχ is a bi-homogeneous ideal of S. To prove that the Rees algebra $\mathcal{R}(I_\gamma)$ is the bi-graded coordinate ring of $\overline{\Gamma_\gamma}$, we need to prove that kerχ is the defining ideal of $\overline{\Gamma_\gamma}$. This is indeed true since

$$\begin{aligned}
& G \in \ker\chi, G \text{ is bi-homogeneous} \\
\Leftrightarrow \quad & G(x_0, \ldots, x_n, F_{\gamma 0}t, \ldots, F_{\gamma m_\gamma}t) = 0 \\
\Leftrightarrow \quad & G(x_0, \ldots, x_n, F_{\gamma 0}, \ldots, F_{\gamma m_\gamma}) = 0 \\
\Leftrightarrow \quad & G(x_0, \ldots, x_n, F_{\gamma 0}, \ldots, F_{\gamma m_\gamma}) = 0 \\
& \text{(for all } P = [x_0 : \ldots : x_n] \text{ that } F_{\gamma 0}(P), \ldots, F_{\gamma m_\gamma}(P) \text{ are not all zero)} \\
\Leftrightarrow \quad & G([P, Q]) = 0 \text{ for all } [P, Q] \in \Gamma_\gamma \\
\Leftrightarrow \quad & G \in \text{ defining ideal of } \Gamma_\gamma \\
\Leftrightarrow \quad & G \in \text{ defining ideal of } \overline{\Gamma_\gamma}.
\end{aligned}$$

□

Proposition 2.4 implies that $\overline{\Gamma_\gamma}$ with the structure sheaf coming from the bi-graded Rees algebra $\mathcal{R}(I_\gamma)$ is the blowup of $\mathbb{P}^n$ along the ideal sheaf associated to the ideal generated by I_γ. This gives rise to the following Proposition-Definition.

Proposition-Definition: Suppose V is a subscheme of $\mathbb{P}^n$ defined by the ideal $I = \oplus_{\gamma \geq \alpha} I_\gamma$. Suppose also that for each $\gamma \geq \alpha$, $\overline{\Gamma_\gamma}$ is the closure of the graph of the rational map given by I_γ (as above). Then, for γ bigger than or equal to the degrees of the generators of a minimal system of generators of I, the ideal generated by I_γ gives the same ideal sheaf as that given by I, so we define $\overline{\Gamma_\gamma}$ to be the *blowup of* $\mathbb{P}^n$ *along* V, embedded in a certain product space.

2.2. Rational n-folds and very ample divisors

Suppose $\mathbb{X} = \{P_1, \ldots, P_s\} \subseteq \mathbb{P}^n$ is a set of s distinct points (s arbitrary). Let $\mathbb{P}^n(\mathbb{X})$ be the blowup of $\mathbb{P}^n$ along $\mathbb{X}$. Suppose also that $I_\mathbb{X} = \oplus_{\gamma \geq \alpha} I_\gamma$ is the homogeneous decomposition of the defining ideal $I_{\mathbb{X}} \subseteq R = \mathfrak{k}[x_0, \ldots, x_n]$ of $\mathbb{X}$. As before, for each $\gamma \geq \alpha$, we use I_γ to define a rational map

$$\varphi_\gamma : \mathbb{P}^n \dashrightarrow \mathbb{P}^{m_\gamma}.$$

Again, let Γ_γ and Λ_γ be the graph and the image of this map in $\mathbb{P}^n \times \mathbb{P}^{m_\gamma}$ and $\mathbb{P}^{m_\gamma}$, respectively. We also let $\overline{\Gamma_\gamma}$ and $\overline{\Lambda_\gamma}$ be the closures of Γ_γ and Λ_γ, respectively.

Suppose now that γ is an integer which is bigger than or equal to the degrees of the generators of a minimal system of generators for $I_{\mathbb{X}}$. We have shown (and define) in the previous section that $\overline{\Gamma_\gamma}$ is the blowup of $\mathbb{P}^n$ along $\mathbb{X}$. By definition, $\overline{\Gamma_\gamma}$ is a rational n-fold (since it is birationally equivalent to $\mathbb{P}^n$). We shall discuss certain *very ample* divisors on $\overline{\Gamma_\gamma}$.

The $\mathfrak{k}$-vector space I_γ corresponds to the linear system of hypersurfaces in $\mathbb{P}^n$ of degree γ containing the points of $\mathbb{X}$. This is a subsystem of the complete system of hypersurfaces of degree γ in $\mathbb{P}^n$. As in [**Hart**, Chapter V], we denote the linear system corresponding to I_γ by $|\gamma H - P_1 - \ldots - P_s|$, where H represents the class of a general hyperplane in $\mathbb{P}^n$. Let $E_1, \ldots, E_s$ be the exceptional divisors corresponding to the blowups at $P_1, \ldots, P_s$, respectively. Also, let E_0 be the class of the pull back to $\overline{\Gamma_\gamma}$ of a general hyperplane in $\mathbb{P}^n$. Then, similar to what was said in [**Hart**], there is a natural one-to-one correspondence between the linear systems of the form $|\gamma H - P_1 - \ldots - P_s|$ on $\mathbb{P}^n$ and the complete linear systems of the form $|\gamma E_0 - E_1 - \ldots - E_s|$ on the blowup of $\mathbb{P}^n$ along $\mathbb{X}$ (i.e. $\overline{\Gamma_\gamma}$), and the two corresponding systems have the same dimensions. We denote the divisor $\gamma E_0 - E_1 - \ldots - E_s$ on $\overline{\Gamma_\gamma}$ by $|I_\gamma|$ (since it corresponds to I_γ).

Definition. A divisor D on a noetherian scheme X is called *very ample* if there is an immersion $i : X \to \mathbb{P}^m$ for some m such that $\mathcal{O}_X(D) \simeq i^*(\mathcal{O}_{\mathbb{P}^m}(1))$, where $\mathcal{O}_X(D)$ is the local free sheaf associated to D on X (see [**Hart**]).

It follows from [**Hart**, Theorem II.7.1] that a divisor D on a noetherian scheme X is very ample if and only if the rational map on X given by the global sections of $\mathcal{O}_X(D)$ is an immersion.

Going back to our discussion, let $\pi_1 : \mathbb{P}^n \times \mathbb{P}^{m_\gamma} \longrightarrow \mathbb{P}^n$ and $\pi_2 : \mathbb{P}^n \times \mathbb{P}^{m_\gamma} \longrightarrow \mathbb{P}^{m_\gamma}$ be the two natural projections on $\mathbb{P}^n \times \mathbb{P}^{m_\gamma}$. By abuse of notation, we use π_1 and π_2 also for the restrictions of those projections on $\overline{\Gamma_\gamma}$. The following results are not very difficult, but they provide an explaination on why there has been a lot of study on the problem of finding generating sets for the defining ideal of $\overline{\Lambda_\gamma}$ for various values of γ, and give the necessary setting for the study in Chapters 3 and 4.

Proposition 2.5. *For φ_γ, $\overline{\Gamma_\gamma}$ and π_2 defined as above (γ at least as large as the degrees of all the generators of a minimal system of generators for the defining ideal of $\mathbb{X}$),*

$$\overline{Im\ \varphi_\gamma} = \pi_2(\overline{\Gamma_\gamma}).$$

Proof. It is clear that the projection is a closed map, so $\pi_2(\overline{\Gamma_\gamma})$ is a closed subset of $\mathbb{P}^{m_\gamma}$.

Let $P = [\overline{y_0} : \ldots : \overline{y_{m_\gamma}}] \in \mathbb{P}^{m_\gamma}$ be any point of Im φ_γ, then there exists $\overline{x} = [\overline{x_0} : \ldots : \overline{x_n}] \in \mathbb{P}^n$ such that $P = \varphi_\gamma(\overline{x})$. Thus,

$$[\overline{x_0} : \ldots : \overline{x_n}, \overline{y_0} : \ldots : \overline{y_{m_\gamma}}] \in \Gamma_\gamma \subseteq \overline{\Gamma_\gamma}.$$

Hence, $P \in \pi_2(\overline{\Gamma_\gamma})$. This is true for any $P \in$ Im φ_γ, so Im $\varphi_\gamma \subseteq \pi_2(\overline{\Gamma_\gamma})$, i.e.

$$\overline{\text{Im}\ \varphi_\gamma} \subseteq \overline{\Gamma_\gamma}.$$

Now, suppose Z is a closed subset of $\mathbb{P}^{m_\gamma}$ such that Im $\varphi_\gamma \subseteq Z \subset \pi_2(\overline{\Gamma_\gamma})$. Then,

$$\Gamma_\gamma \subseteq \pi_2^{-1}(Z) \cap \overline{\Gamma_\gamma} \subseteq \pi_2^{-1}(\pi_2(\overline{\Gamma_\gamma})) \cap \overline{\Gamma_\gamma} = \overline{\Gamma_\gamma}.$$

Since $\pi_2^{-1}(Z) \cap \overline{\Gamma_\gamma}$ is a closed subset of $\overline{\Gamma_\gamma}$, this implies that $\pi_2^{-1}(Z) \cap \overline{\Gamma_\gamma} = \overline{\Gamma_\gamma} = \pi_2^{-1}(\pi_2(\overline{\Gamma_\gamma})) \cap \overline{\Gamma_\gamma}$, which contradicts the fact that $Z \subset \pi_2(\overline{\Gamma_\gamma})$. Thus, we must have

$$\overline{\text{Im}\ \varphi_\gamma} \supseteq \pi_2(\overline{\Gamma_\gamma}).$$

The proposition is proved. □

Proposition 2.6. *Suppose $\overline{\Gamma_\gamma}$ and $|I_\gamma|$ are as defined (γ at least as large as the degrees of all the generators of a minimal system of generators for the defining ideal of $\mathbb{X}$). If $|I_\gamma|$ is very ample on $\overline{\Gamma_\gamma}$, then π_2 is a homeomorphism on $\overline{\Gamma_\gamma}$.*

Proof. Suppose the divisor $|I_\gamma|$ is very ample on $\overline{\Gamma_\gamma}$. Let ϕ_γ be the rational map defined on $\overline{\Gamma_\gamma}$ by the global sections of the invertible sheaf $\mathcal{O}_{\overline{\Gamma_\gamma}}(|I_\gamma|)$ (associated to the divisor $|I_\gamma|$). Then, ϕ_γ is an immersion from $\overline{\Gamma_\gamma}$ into $\mathbb{P}^{m_\gamma}$ (since $|I_\gamma|$ and I_γ have the same dimensions - see [**Hart**]). It follows from [**Hart**, Theorem II.7.13] that on $\Gamma_\gamma = \overline{\Gamma_\gamma} \backslash (E_1 \cup \ldots \cup E_s)$ (where $E_1, \ldots, E_s$ are the exceptional divisors corresponding to the blowups at the points in $\mathbb{X}$, respectively),

$$\phi_\gamma = \varphi_\gamma \circ \pi_1 = \pi_2.$$

Thus, π_2 is an isomorphism on Γ_γ.

We also see that since $E_1, \ldots, E_s$ do not pairwise intersect (see [**Hart**]), π_2 is one-to-one over $E_1 \cup \ldots \cup E_s$. Therefore, π_2 is one-to-one over $\overline{\Gamma_\gamma}$. Moreover, π_2 is the projection on the second set of coordinates, so it is a continuous and closed map. Hence, π_2 is a homeomorphism on $\overline{\Gamma_\gamma}$. The proposition is proved. □

Remark: It seems to be a common belief that the condition $|I_\gamma|$ being very ample on $\overline{\Gamma_\gamma}$ is in fact equivalent to the condition that π_2 is an isomorphism on $\overline{\Gamma_\gamma}$. I am not yet able to prove this.

Propositions 2.5 and 2.6 ensure that when γ is bigger than or equal to the degrees of the generators of a minimal system of generators for $I_{\mathbb{X}}$, and $|I_\gamma|$ is a very ample divisor on $\overline{\Gamma_\gamma}$, $\overline{\Lambda_\gamma} = \pi_2(\overline{\Gamma_\gamma})$ is a projective embedding of the blowup $\mathbb{P}^n(\mathbb{X})$. When this happens, we also say that $\overline{\Lambda_\gamma}$ is the projective embedding of $\mathbb{P}^n(\mathbb{X})$ given by the linear system I_t (or given by the very ample divisor $|I_t|$). In Chapter 3 and Section 4.2, these projective embeddings are studied when $n = 2$.

CHAPTER 3

Projective embeddings of blowup surfaces

Let $\mathbb{X} = \{P_1, \ldots, P_s\}$ be a set of s distinct points in $\mathbb{P}^2$ which are in generic position, $I_{\mathbb{X}} = \oplus_{t \geq \alpha} I_t \subseteq R = \mathfrak{k}[w_1, w_2, w_3]$ its defining ideal, and $\mathbb{P}^2(\mathbb{X})$ the surface obtained by blowing up $\mathbb{P}^2$ along the points of $\mathbb{X}$. The bulk of this chapter is devoted to studying the problem of finding generating sets for the defining ideals of $\mathbb{P}^2(\mathbb{X})$, embedded in projective spaces by linear systems of curves going through the points of $\mathbb{X}$.

We push further works of Gimigliano ([**Gi-1**], [**Gi-2**]) and work of Geramita and Gimigliano ([**G-G**]) by, again, considering the case when $s = \binom{d+1}{2}$ for some integer d. In this case, we generalize Geramita and Gimigliano's argument on the Room surfaces and give an explicit description for the defining ideals of the embeddings of $\mathbb{P}^2(\mathbb{X})$ given by the linear systems I_t for all $t \geq d+1$. We will show that those defining ideals are the ideals of 2×2 minors of a 3-dimensional box-shaped matrix of linear forms. Since a 2×2 minor is quadratic, our result confirms that all the projective embeddings of $\mathbb{P}^2(\mathbb{X})$ given by the linear systems I_t ($t \geq d+1$) are indeed generated by quadratics, which agrees with the result of Geramita and Gimigliano (Theorem 0.8).

For the rest of this chapter, we assume that $\mathbb{X}$ is a set of $s = \binom{d+1}{2}$ points which are in generic position (for some integer d). It follows from [**G-M**] that $I_{\mathbb{X}}$ is generated in degree d, and that $\sigma(I_{\mathbb{X}}) = \alpha(I_{\mathbb{X}}) = d$. Suppose $t \geq d+1$. As in Chapters 0 and 2, we use I_t to define a rational map $\varphi_t : \mathbb{P}^2 \dashrightarrow \mathbb{P}^{m_t}$. We again let Γ_t and Λ_t be the graph and the image of φ_t, respectively, and let $\overline{\Gamma_t}$ and $\overline{\Lambda_t}$ be the closures of Γ_t and Λ_t in $\mathbb{P}^2 \times \mathbb{P}^{m_t}$ and $\mathbb{P}^{m_t}$, respectively. From Theorem 0.6, it is known that the divisor $|I_t|$ on $\mathbb{P}^2(\mathbb{X})$ is very ample (see Chapter 2 for the definition and expression of

$|I_t|$). Thus, $|I_t|$ gives a projective embedding $\overline{\Lambda_t}$ of the blowup surface $\mathbb{P}^2(\mathbb{X})$. In this chapter, we will give a set of defining equations for $\overline{\Lambda_t}$.

As we will see later, the results of [**STV**] (followed by [**CHTV**]) enable one to write down the equations for $\overline{\Lambda_t}$ knowing the defining equations for the Rees algebra $\mathcal{R}(I_{\mathbb{X}})$. This, coupled with the result of Morey and Ulrich (Theorem 0.11), makes it possible to write down the equations for $\overline{\Lambda_t}$ for any t. This method, however, has its disadvantages (as we will see in Section 4.2 in a more general context). Our approach to the problem is different from the mentioned method, and we are able to give an explicit description of a set of defining equations for $\overline{\Lambda_t}$ for all $t \geq d+1$.

We start with a simple observation.

Lemma 3.1. *Suppose S, R and T are Noetherian commutative rings with identity, and $\phi : S \to R$ and $\psi : R \to T$ are surjective ring homomorphisms. Suppose also that $f_1, f_2, \ldots, f_r$ are generators for ker $\phi \subseteq S$ and $g_1, g_2, \ldots, g_m$ are generators for ker $\psi \subseteq R$. Let p_j be a preimage of g_j for all j, then $f_1, \ldots, f_r, p_1, \ldots, p_m$ give a set of generators for ker $(\psi \circ \phi) \subseteq S$.*

Proof. Clearly, the f_i's and p_j's are all in ker $(\psi \circ \phi)$. Moreover, if $x \in$ ker $(\psi \circ \phi)$, then either $\phi(x) = 0$ or $\phi(x)$ is a linear combination of the g_j's. The result is now trivial. □

We now recall the notion of Catalecticant matrices. They will play an important role in the sequel. The definition we use is from [**Puc**].

Definition. Suppose p, q and r are positive integers. For each r-tuple of non-negative integers $J = (j_1, \ldots, j_r)$, we define the *weight* of J to be $|J| = j_1 + \ldots + j_r$. Suppose there is a set of symbols $\{y_J \mid |J| = p+q\}$ indexed by all the r-tuples of non-negative integers of weight $p+q$. The Catalecticant matrix of size $(p, q; r)$, denoted $\text{Cat}(p, q; r)$, over the given set of symbols is defined to be the matrix whose rows are indexed by r-tuples of non-negative integers of weight p, whose columns are indexed by r-tuples of non-negative integers of weight q, and whose (J_1, J_2) entry (J_1 and J_2 r-tuples of non-negative integers of weights p and q, respectively) is the symbol $y_{J_1+J_2}$.

We now return to the problem of finding the defining ideal for $\overline{\Lambda_t}$.

By the Hilbert-Burch theorem (cf. [**Bu**], [**Ei-1**], [**C-G-O**]), the generators of $I_{\mathbb{X}}$ are the $d \times d$ minors of a $d \times (d+1)$ matrix, say $\mathbf{L}$, of linear forms :

$$\mathbf{L} = (L_{ij}),\ L_{ij} \in R_1 \text{ for } i = 1, 2, \ldots, d \text{ and } j = 1, 2, \ldots, d+1.$$

In this notation,

$$I_{\mathbb{X}} = (F_1, \ldots, F_{d+1}),\ F_i = (-1)^{i+1}\det(\mathbf{L} \setminus i^{\text{th}} \text{ column}).$$

Let $t = d + n,\ (n \geq 1)$. For $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, we write w^α for $w_1^{\alpha_1} w_2^{\alpha_2} w_3^{\alpha_3}$, and denote $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3$. A system of generators of the vector space I_t is given by the $\binom{n+2}{2}(d+1)$ forms $w^\alpha F_j$, $j = 1, 2, \ldots, d+1$ and $|\alpha| = n$.

Consider the rational map

$$\varphi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^p,\ p = \binom{n+2}{2}(d+1) - 1,$$

given by $\varphi(P) = [w^\alpha F_j]$ (we order the α's by lexicographic ordering with $w_1 > w_2 > w_3$). $\overline{\Lambda_t}$ embedded in $\mathbb{P}^p$ is given by the closure of the image of φ.

Let $z_1 = w_1^n, z_2 = w_1^{n-1} w_2, \ldots, z_u = w_3^n$, where $u = \binom{n+2}{2}$ (again, we arrange the terms in lexicographic order). We use homogeneous coordinates $[x_{ij}]_{1 \leq i \leq u, 1 \leq j \leq d+1}$ in $\mathbb{P}^p$ such that

$$\varphi([\overline{w_1} : \overline{w_2} : \overline{w_3}]) = [\overline{x_{ij}}], \text{ where } \overline{x_{ij}} = \overline{z_i} F_j(\overline{w_1}, \overline{w_2}, \overline{w_3}). \tag{3.0.1}$$

Here, the $\overline{z_i}$ is z_i evaluated at $[\overline{w_1} : \overline{w_2} : \overline{w_3}]$ for all i.

The vector space dimension of I_t is $(n+1)d + \binom{n+2}{2}$, so there must be $\binom{n+1}{2}d$ dependence relations among the $w^\alpha F_j$'s. Those relations can be found as follows.

Let $\beta = (\beta_1, \beta_2, \beta_3)$ with $|\beta| = n - 1$. For each $l = 1, 2, \ldots, d$, we have

$$0 = \det \begin{pmatrix} w^\beta L_{l1} \; w^\beta L_{l2} \; \ldots \; w^\beta L_{l,d+1} \\ \mathbf{L} \end{pmatrix} = \sum_{j=1}^{d+1} L_{lj} w^\beta F_j.$$

Since $L_{lj} = \sum_{k=1}^{3} \lambda_{ljk} w_k$, so by grouping similar terms, we get

$$\sum_{|\alpha|=n, 1 \le j \le d+1} \mu_{l\alpha j} w^\alpha F_j = 0, \ \forall l = 1, 2, \ldots, d,$$

(the sum is taken over all α's such that $w^\alpha = w^\beta w_k$ for some k) where

$$\mu_{l\alpha j} = \sum_{w^\beta w_k = w^\alpha} \lambda_{ljk} \text{ for each } l, \alpha \text{ and } j.$$

These are the dependence relations on the $w^\alpha F_j$'s. In terms of z_i's, we can rewrite them as

$$\sum_{i,j} \mu_{lij} z_i F_j = 0, \ \forall l = 1, 2, \ldots, d.$$

These give rise to the following equations:

$$\sum_{1 \le i \le u, 1 \le j \le d+1} \mu_{lij} x_{ij} = 0, \ \forall l = 1, 2, \ldots, d. \tag{3.0.2}$$

There are d relations of the form (3.0.2) for each β, and the number of such β's is $\binom{n+1}{2}$. By abuse of notation, we denote the collection of these $\binom{n+1}{2} d$ relations by (3.0.2). The relations in (3.0.2) would be independent if we could show that the $\binom{n+1}{2} d \times \binom{n+2}{2}(d+1)$ matrix $\mathbf{E}$ of the coefficients μ_{lij} has maximal rank. As we now show, this is indeed the case. (Our proof uses an argument similar to that of Geramita and Gimigliano ([**G-G**]).

Lemma 3.2. $\mathbf{E}$ *has maximal rank.*

Proof. We assume, without loss of generality, that none of the points of $\mathbb{X}$ is $P = [0 : 0 : 1]$, and that the first minor of $\mathbf{L}$, F_1, does not vanish at P. Suppose

$$\mathbf{L} = A_1 w_1 + A_2 w_2 + A_3 w_3,$$

where the A_i's have entries in the ground field. This means that A_3 has maximal rank d (since $F_1(P) \neq 0$).

We arrange the β's in lexicographic order with $w_1 > w_2 > w_3$. For each β, the coefficient matrix of the d equations given by this β (see (3.0.2)) is $[\ldots A_1 \ldots A_2 \ldots A_3 \ 0]$ (there may be none or many 0's at the end), where the A_3 corresponds to variable x_{ij} for $j = 1, \ldots, d+1$ and i is the integer such that $z_i = w^\beta w_3$. Thus, $\mathbf{E}$ has the form :

$$\mathbf{E} = \begin{bmatrix} A_1 & \ldots & A_2 & \ldots & A_3 & 0 & & \\ \ldots & A_1 & \ldots & A_2 & \ldots & A_3 & 0 & \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & A_3 \end{bmatrix}$$

(The A_3 of the latter row is totally to the right of the A_3 of the former row, since if $\beta > \beta'$ then $w^\beta w_3 > w^{\beta'} w_3$). From each A_3, take d columns that give a matrix A_3' which has nonzero determinant. Putting them all together, we obtain a $\binom{n+1}{2} d \times \binom{n+1}{2} d$ matrix, which looks like the following:

$$\mathbf{E}' = \begin{bmatrix} A_3' & & & \\ & A_3' & & 0 \\ X & & \ddots & \\ & & & A_3' \end{bmatrix}.$$

This is a lower triangular matrix. Clearly, $\det \mathbf{E}' = (\det A_3')^{\binom{n+1}{2}} \neq 0$. Thus, the matrix $\mathbf{E}$ has maximal rank. $\square$

Obviously, on $\varphi(\mathbb{P}^2 \backslash \mathbb{X})$, the coordinates of the points satisfy the equations in (3.0.2). These are the equations coming from the dependence relations of the $w^\alpha F_j$'s that we are looking for.

Consider the matrix

$$M = \begin{bmatrix} x_{11} & x_{12} & \ldots & x_{1,d+1} \\ x_{21} & x_{22} & \ldots & x_{2,d+1} \\ \ldots & \ldots & \ldots & \ldots \\ x_{u1} & x_{u2} & \ldots & x_{u,d+1} \end{bmatrix}$$

It is easy to see that the points of $\varphi(\mathbb{P}^2 \backslash \mathbb{X})$ satisfy the equations obtained by setting all the 2×2 minors of M equal to zero. Denote the collection of these equations by (**).

Moreover, on $\varphi(\mathbb{P}^2\backslash\mathbb{X})$, each column of M has the form :

$$\begin{pmatrix} \overline{z_1}F_j \\ \overline{z_2}F_j \\ \dots \\ \overline{z_u}F_j \end{pmatrix}$$

where $\overline{z_1} = \overline{w_1}^n, \dots, \overline{z_u} = \overline{w_3}^n$ for some point $[\overline{w_1} : \overline{w_2} : \overline{w_3}] \in \mathbb{P}^2\backslash\mathbb{X}$. Clearly, the point $[\overline{z_1} : \dots : \overline{z_u}]$ is on the Veronese surface given by all the monomials of degree n in w_1, w_2 and w_3. Thus, the $\overline{z_i}$'s satisfy the defining equations of this Veronese surface, which are known to be the 2×2 minors of certain Catalecticant matrices ([**Puc**]). Therefore, on $\varphi(\mathbb{P}^2\backslash\mathbb{X})$, the coordinates $\overline{x_{1j}}, \overline{x_{2j}}, \dots, \overline{x_{uj}}$ satisfy the 2×2 minors of the Catalecticant matrix $\mathrm{Cat}(1, n-1; 3)$ of size $3 \times \binom{n+1}{2}$, for all $j = 1, 2, \dots, d+1$. Denote the collection of these equations by (***).

Let $\mathbf{V}$ be the algebraic set in $\mathbb{P}^p$ defined by all the equations in (3.0.2), (**) and (***).

Theorem 3.3. $\mathbf{V} = \overline{\Lambda_t}$ *as sets.*

Proof. Clearly, $\varphi(\mathbb{P}^2\backslash\mathbb{X}) \subseteq \mathbf{V}$. Since $\mathbf{V}$ is closed, $\overline{\Lambda_t}$ is integral (so $\overline{\Lambda_t}$ is irreducible, and $\overline{\Lambda_t} = \overline{\varphi(\mathbb{P}^2\backslash\mathbb{X})}$), we have

$$\overline{\Lambda_t} \subseteq \mathbf{V}.$$

We only need to show that

$$\mathbf{V} \subseteq \overline{\Lambda_t}.$$

From (3.0.1), we have (on $\varphi(\mathbb{P}^2\backslash\mathbb{X})$) :

$$\overline{x_{1j}}/\overline{z_1} = \overline{x_{2j}}/\overline{z_2} = \dots = \overline{x_{uj}}/\overline{z_u}, \quad \text{for all } j = 1, 2, \dots, d+1.$$

This can be rewritten as a number of systems of equations, one for each $i = 1, 2, \dots, u$

$$\begin{cases} \overline{x_{ij}}/\overline{z_i} & = & \overline{x_{1j}}/\overline{z_1} \\ \overline{x_{ij}}/\overline{z_i} & = & \overline{x_{2j}}/\overline{z_2} \\ & \dots & \\ \overline{x_{ij}}/\overline{z_i} & = & \overline{x_{uj}}/\overline{z_u} \end{cases} \quad \text{for } j = 1, 2, \dots, d+1.$$

Those relations give us, for each $i = 1, 2, \dots, u$:

$$(S_i) \left\{ \begin{array}{rcl} \overline{x_{ij} z_1} - \overline{x_{1j} z_i} & = & 0 \\ \overline{x_{ij} z_2} - \overline{x_{2j} z_i} & = & 0 \\ \dots & & \\ \overline{x_{ij} z_u} - \overline{x_{uj} z_i} & = & 0 \end{array} \right. \quad \text{for } j = 1, 2, \dots, d+1.$$

It is not hard to see that if the coordinates of $Q = [\overline{x_{ij}}] \in \mathbb{P}^p$ and $P = [\overline{z_i}] \in \mathbb{P}^{u-1}$ satisfy system (S_i) for some i, where $\overline{z_i} \neq 0$, then they satisfy systems (S_i) for all i.

Before going further, we prove a proposition similar to that of [**G-G**].

Proposition 3.4. *Let $Q = [\overline{x_{ij}}]$ be a point on $\mathbb{P}^p$, and suppose the coordinates of Q satisfy equations (**) (page 43). Then there exists a unique $P = [\overline{z_1} : \dots : \overline{z_u}] \in \mathbb{P}^{u-1}$ such that the homogeneous coordinates of P and Q satisfy the systems (S_i) for all i.*

Proof. Since the coordinates of Q satisfy equations (**), the matrix $M(Q)$ has rank 1, i.e. the rows of $M(Q)$ are all multiples of any nonzero row of $M(Q)$. Suppose the first row of $M(Q)$ is not identically zero (a similar argument works for other rows). Then there exist ν_i, for $i = 2, \dots, u$, such that

$$\overline{x_{ij}} = \nu_i \overline{x_{1j}}, \text{ for all } j = 1, 2, \dots, d+1.$$

We want $P \in \mathbb{P}^{u-1}$ such that the coordinates of P and Q satisfy the systems (S_i) for all i. We first consider P such that the coordinates of P and Q satisfy (S_1). This is the same as solving for $\overline{z_1}, \dots, \overline{z_u}$ in (S_1). The coefficients matrix becomes (projectively) a collection of :

$$N_j = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ -\nu_2 & 1 & 0 & \dots & 0 \\ -\nu_3 & 0 & 1 & \dots & 0 \\ \vdots & & & \ddots & \\ -\nu_u & 0 & 0 & \dots & 1 \end{pmatrix}, \text{ for } j = 1, 2, \dots, d+1.$$

Since N_j is independent of j and has rank exactly $u - 1$, the system (S_1) has exactly one projective solution. That gives a unique point $P \in \mathbb{P}^{u-1}$. Moreover, this unique

P clearly has non-zero $\overline{z_1}$, so the coordinates of P and Q in fact satisfy (S_i) for all i. Hence, P exists and is unique. $\diamond$

We now go back to the proof of the theorem. Having the proposition as above, if we can show that: for any points $P = [\overline{z_1} : \ldots : \overline{z_u}]$ and $Q = [\overline{x_{ij}}]$ such that the coordinates of Q satisfy the equations in (3.0.2), (**) and (***) (see pages 42 and 43), and the coordinates of P and Q satisfy the systems (S_i) for all i, that this implies Q must be in $\overline{\Lambda_t}$, then we will have $\mathbf{V} \subseteq \overline{\Lambda_t}$, and we will be done.

Suppose P and Q are such points. We can always assume that $\overline{z_1} \neq 0$. Consider the system of equations given by all the equations in (S_1) (if instead, $\overline{z_i} \neq 0$, then we look at the system (S_i)). As a system of linear equations in the variables (note the way we have rearranged the indices)

$$\{\overline{x_{ij}} | 1 \leq j \leq d+1, 1 \leq i \leq u\},$$

the coefficient matrix is :

$$A = \begin{pmatrix} B & & & \\ & B & & \\ & & \ddots & \\ & & & B \end{pmatrix},$$

where

$$B = \begin{pmatrix} 0 & 0 & 0 & \ldots & 0 \\ \overline{z_2} & -\overline{z_1} & 0 & \ldots & 0 \\ \overline{z_3} & 0 & -\overline{z_1} & \ldots & 0 \\ \vdots & & & \ddots & \\ \overline{z_u} & 0 & 0 & \ldots & -\overline{z_1} \end{pmatrix}.$$

Clearly, B has rank $u-1$, and has a non-trivial solution $[\overline{z_1} : \ldots : \overline{z_u}]$. Therefore, the solution to A must have the form :

$$[\overline{x_{ij}}] = [c_1\overline{z_1} : c_1\overline{z_2} : \ldots : c_1\overline{z_u} : c_2\overline{z_1} : \ldots : c_2\overline{z_u} : \ldots : c_{d+1}\overline{z_1} : \ldots : c_{d+1}\overline{z_u}],$$

where $c_1, \ldots, c_{d+1}$ are constants not all zero, and the indeterminates are ordered by $1 \leq j \leq d+1$ and $1 \leq i \leq u$.

Now, since the coordinates of Q also satisfy the equations in (***), which are the defining equations of Veronese surfaces, there exists a unique point

$$T = [\overline{w_1} : \overline{w_2} : \overline{w_3}] \in \mathbb{P}^2$$

such that $\overline{z_1} = \overline{w_1}^n, \overline{z_2} = \overline{w_1}^{n-1}\overline{w_2}, \ldots, \overline{z_u} = \overline{w_3}^n$. Thus,

$$Q = [c_1\overline{w_1}^n : \ldots : c_{d+1}\overline{w_3}^n]. \tag{3.0.3}$$

Lastly, the coordinates of Q satisfy the $\binom{n+1}{2}d$ equations in (3.0.2), so

$$\mathbf{L}(T)\begin{bmatrix} c_1 \\ c_2 \\ \cdots \\ c_{d+1} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \cdots \\ 0 \end{bmatrix}.$$

If $T \notin \mathbb{X}$, then $\mathbf{L}(T)$ has rank exactly d. Thus,

$$\begin{bmatrix} c_1 \\ c_2 \\ \cdots \\ c_{d+1} \end{bmatrix} = \rho \begin{bmatrix} F_1(T) \\ F_2(T) \\ \cdots \\ F_{d+1}(T) \end{bmatrix}. \tag{3.0.4}$$

This implies that $Q \in \overline{\Lambda_t}$.

If $T \in \mathbb{X}$, then $\mathbf{L}(T)$ has rank exactly $d-1$, so there is a 2-dimensional solution space, and these resulting Q's lie on a line of $\mathbf{V}$, which is one of the exceptional lines of $\overline{\Lambda_t}$.

Hence, we always have $Q \in \overline{\Lambda_t}$. We have proved that $\mathbf{V} = \overline{\Lambda_t}$ as sets. $\square$

To continue our study, we let $S = \mathfrak{k}[x_{ij}]$ be the homogeneous coordinate ring of $\mathbb{P}^p$. Suppose $\mathbf{C}$ is the Catalecticant matrix $\mathrm{Cat}(1, n-1; 3)$ over the set of indeterminates $\{z_i\}_{0 \le i \le \binom{n+2}{2}}$. $\mathbf{C}$ is of size $3 \times \binom{n+1}{2}$. Consider the box $\mathbf{B}$ of size $(d+1) \times 3 \times \binom{n+1}{2}$. Let $\mathcal{A}$ be the box-shaped matrix obtained by assigning to each integral point (i, j, k) of $\mathbf{B}$ the indeterminate x_{il} where l is the integer such that z_l is at the (j, k)-position in $\mathbf{C}$.

Lemma 3.5. *$\mathcal{A}$ is a weak box-shaped matrix of indeterminates.*

Proof. Clearly, each x-section of $\mathcal{A}$ has its ideal of 2×2 minors as the defining ideal of a Veronese surface, so its ideal of 2×2 minors is a prime ideal. Also, each y-section and z-section of $\mathcal{A}$ is a matrix of indeterminates, whence whose ideal of 2×2 minors is also a prime ideal. Moreover, x_{111} surely satisfies property (c) of $\mathcal{A}$ being a weak box-shaped matrix of indeterminates. It remains to show that

$$< I_2(\mathcal{A}), x_{(d+1)3\binom{n+1}{2}} > = \cap_{l=1}^{3} I_l.$$

For convenience, we let $r_1 = d+1, r_2 = 3, r_3 = \binom{n+1}{2}$, and consider $\mathcal{A}$ as a box-shaped matrix of size $r_1 \times r_2 \times r_3$. We shall first prove that

$$I_2 \cap I_3 \ = < I_2(\mathcal{A}), \{x_{ir_2r_3} | i = 1, \ldots, r_1\} > .$$

The proof will follow the same lines as that of part (a) of Lemma 1.1.

It is clear that $< I_2(\mathcal{A}), \{x_{ir_2r_3} | i = 1, \ldots, r_1\} > \subseteq I_2 \cap I_3$, so it remains to show the other inclusion. Let $F \in I_2 \cap I_3$. Doing exactly as we did before, we end up with $F = F' + G' + G$, where $F', G' \in I_2(\mathcal{A})$, and

$$G = \sum_{ik} G_{ik} x_{ir_2k},$$

where G_{ik}'s are independent of the variables x_{ijr_3}. Again, we have $G \in I_3$, so we can write

$$G = H + \sum_{i,j} H_{ij} x_{ijr_3},$$

where $H \in I_2(\mathcal{A}_3)$. We may assume that the H_{ir_2}'s are independent of all the variables $\{x_{ijr_3} | j \neq r_2\}$. By the nature of the 2×2 minors of $\mathcal{A}$ and the symmetry (in construction) of Catalecticant matrices, it can be seen that if a 2×2 minor of $\mathcal{A}$ has one indeterminate belonging to $\{x_{ijr_3} | (i, j, r_3) \in \mathbf{B}\}$ then it must have at least two adjacent indeterminates belonging to $\{x_{ijr_3} | (i, j, r_3) \in \mathbf{B}\}$. Thus, by re-grouping and rewriting, we can always assume that H is also independent of the indeterminates $\{x_{ijr_3} | (i, j, r_3) \in \mathbf{B}\}$. Now, clearly,

$$G = H + \sum_{i} H_{ir_2} x_{ir_2r_3} \in < I_2(\mathcal{A}), \{x_{ir_2r_3} | i = 1, \ldots, r_1\} > .$$

We have shown that $I_2 \cap I_3 \; = < I_2(\mathcal{A}), \{x_{ir_2r_3} | i = 1, \ldots, r_1\} >$.

It now follows, using the same arguments as in the proof of part (b) of Lemma 1.1, that

$$< I_2(\mathcal{A}), \{x_{ir_2r_3} | i = 1, \ldots, r_1\} > \; \cap I_3 = < I_2(\mathcal{A}), x_{r_1r_2r_3} > .$$

The lemma is proved. □

We now prove the main result of this section.

Theorem 3.6. *Suppose $\mathbb{X}$ is a set of $\binom{d+1}{2}$ points of $\mathbb{P}^2$ which are in generic position. Then, for each $t \geq d+1$ ($t = d+n$, $n \geq 1$), the projective embedding Λ_t of the blowup of $\mathbb{P}^2$ along $\mathbb{X}$, given by the linear system of curves of degree t going through the points in $\mathbb{X}$, is defined by $\binom{n+1}{2}d$ linear forms and the 2×2 minors of a box-shaped matrix of linear forms.*

Proof. Let $S = \mathfrak{k}[x_{ij}]$ be the homogeneous coordinate ring of $\mathbb{P}^p$. Let $\mathcal{A}$ be the weak box-shaped matrix of indeterminates as above, and again, let $I_2(\mathcal{A})$ be the ideal of 2×2 minors of $\mathcal{A}$ in $\mathfrak{k}[\mathcal{A}]$. We also let $\mathbf{I}$ be the ideal generated by $I_2(\mathcal{A})$ and all the linear equations in (3.0.2). Let $\mathcal{V}$ be the subscheme of $\mathbb{P}^p$ defined by $\mathbf{I}$.

It is easy to see that $\mathbf{I}$ contains all the equations in (3.0.2), (**) and (***), so as sets, $\mathcal{V} \subseteq \mathbf{V}$ (where $\mathbf{V}$ is the subvariety of $\mathbb{P}^p$ defined by the equations in (3.0.2), (**) and (***)).

Suppose now that $P = [\overline{w_1} : \overline{w_2} : \overline{w_3}] \in \mathbb{P}^2 \backslash \mathbb{X}$ and $Q = [\overline{x_{ij}}] = \varphi(P)$. Let $\overline{z_1} = \overline{w_1}^n, \overline{z_2} = \overline{w_1}^{n-1}\overline{w_2}, \ldots, \overline{z_u} = \overline{w_3}^n$ then $\overline{x_{ij}} = \overline{z_i}F_j(P)$. Consider a 2×2 minor $a_{(K,L,M,N)}$ of $\mathcal{A}$ corresponding to the 4 points K, L, M and N in the box-shaped realization of $\mathcal{A}$. There are 3 possibilities for the tuple (K, L, M, N).

Case 1. $K = (i,j,k), L = (m,j,p), M = (m,n,p)$ and $N = (i,n,k)$ for some integers i,j,k,m,n and p (when the projections of K, L, M, N on the zx-plane collapse to a line).

Case 2. $K = (i, j, k), L = (m, j, k), M = (m, n, p)$ and $N = (i, n, p)$ for some integers i, j, k, m, n and p (when the projections of K, L, M, N on the yz-plane collapse to a line).

Case 3. $K = (i, j, k), L = (m, n, k), M = (m, n, p)$ and $N = (i, j, p)$ for some integers $i, j, k, m, n,$ and p (when the projections of K, L, M, N on the xy-plane collapse to a line).

By the construction of $\mathcal{A}$ and the fact that $[\overline{z_1} : \ldots : \overline{z_u}]$ is in the Veronese surface, i.e. it satisfies all the 2×2 minors of $\mathbf{C}$, it is easy to check that Q satisfies the minors $a_{(K,L,M,N)}$. This is true for any $Q \in \varphi(\mathbb{P}^2 \backslash \mathbb{X})$ and any 2×2 minor $a_{(K,L,M,N)}$ of $\mathcal{A}$, so $\varphi(\mathbb{P}^2 \backslash \mathbb{X}) \subseteq \mathcal{V}$, whence $\mathbf{V} \subseteq \mathcal{V}$.

We have shown that in all cases, $\mathbf{V} \subseteq \mathcal{V}$. Hence, as sets, $\mathcal{V} = \mathbf{V} = \overline{\Lambda_t}$.

Now, by Proposition 1.10, we know that $I_2(\mathcal{A})$ is a prime ideal. Consider the following sequence of surjective ring homomorphisms

$$\mathfrak{k}[x_{ij}] \xrightarrow{\phi} \mathfrak{k}[w^\alpha t_j] \xrightarrow{\psi} \mathfrak{k}[w^\alpha F_j],$$

defined in the obvious way; that is, both ϕ and ψ send $\mathfrak{k}$ to $\mathfrak{k}$, and ϕ sends x_{ij} to $w^\alpha t_j$ where w^α is labelled z_i, and ψ sends $w^\alpha t_j$ to $w^\alpha F_j$.

We note that in proving equalities (3.0.3) and (3.0.4), we actually proved more. First, the proof of (3.0.3) and the fact that $I_2(\mathcal{A})$ is a prime ideal imply that $I_2(\mathcal{A})$ is the kernel of ϕ. Second, the proof of (3.0.4) shows that if we consider the equations in (3.0.2) as polynomials over the $w^\alpha t_j$s, then those polynomials are zero exactly when $t_j = F_j$ (since $t_j = F_j$ at all but a finite set of points $\mathbb{X}$). This implies that $\mathfrak{k}[w^\alpha t_j]/\mathfrak{a} \simeq \mathfrak{k}[w^\alpha F_j]$, where $\mathfrak{a}$ is the ideal generated by the images of the equations in (3.0.2) through ϕ. Thus, $\mathfrak{a}$ is the kernel of ψ. Now, by Lemma 3.1, we conclude that $\mathbf{I}$ is the kernel of $\psi \circ \phi$. In other words, $\mathbf{I}$ is the defining ideal of $\overline{\Lambda_t}$ embedded in $\mathbb{P}^p$ (since the homogeneous coordinate ring of $\overline{\Lambda_t}$ embedded in $\mathbb{P}^p$ is exactly $\mathfrak{k}[w^\alpha F_j]$). The theorem is proved. $\square$

Remark: When $t = d + 1$, our box-shaped matrix $\mathcal{A}$ collapses to be a normal matrix of size $3 \times (d + 1)$, and the above result coincides with that obtained by Geramita and Gimigliano in [**G-G**].

CHAPTER 4

Rees algebras of codimension two perfect ideals

Suppose $\mathbb{X}$ is an arbitrary set s of points in $\mathbb{P}^2$ and $I_{\mathbb{X}} = \oplus_{t\geq\alpha} I_t \subseteq R = \mathfrak{k}[w_1, w_2, w_3]$ its defining ideal. In this chapter, we study the asymptotic behaviour of the Rees algebras $\mathcal{R}(I_t)$ of the ideals generated by I_t, as t gets large.

In Section 4.1, we consider the case when the set $\mathbb{X}$ is in generic position. We look at Rees algebra of the ideal generated by $I_{\alpha+1}$, the second least degree piece of the defining ideal $I_{\mathbb{X}}$ of $\mathbb{X}$. When $s = \binom{d+1}{2}$, we write down the defining equations for this Rees algebra without any additional condition. When the number of points in $\mathbb{X}$ is arbitrary, we write down the defining equations for this Rees algebra for a general choice of the points in $\mathbb{X}$.

As an application, in Section 4.2, we revisit the problem of finding generating sets for the defining ideals of certain projective embeddings of the blowup of $\mathbb{P}^2$ along $\mathbb{X}$. We couple our results in Section 4.1 with the study on diagonal subalgebras of [**STV**] and [**CHTV**] to demonstrate a method for deriving a set of defining equations for some projective embeddings of the blowup of $\mathbb{P}^2$ along $\mathbb{X}$, when the number of points in $\mathbb{X}$ are arbitrary and $\mathbb{X}$ is general.

Results in Section 4.1 are also the starting points for the study of Section 4.3. In this section, we consider the general situation, when $\mathbb{X}$ is an arbitrary set of points (not necessarily in generic position). We investigate the Cohen-Macaulayness and the degrees of the defining equations for the Rees algebra $\mathcal{R}(I_t)$ of the ideal generated by I_t, as t gets large. In this section, we also discuss the property of having a Cohen-Macaulay bi-graded coordinate ring for a subscheme of the product scheme $\mathbb{P}^n \times \mathbb{P}^m$.

Before proceeding, let us briefly recall the notion of a Rees algebras. Suppose $I \subseteq A$ is a proper ideal of a commutative ring A with identity. The Rees algebra of I (with respect to A) is defined to be the subring $A[It]$ of the ring $A[t]$, and denoted by $\mathcal{R}_A(I)$ (or simply $\mathcal{R}(I)$ when there is no confusion about which ring is being discussed). We mostly work with the case where A is a polynomial ring over $\mathfrak{k}$, say $A = \mathfrak{k}[x_0, \ldots, x_n]$. In this situation, I is finitely generated. Suppose $I = (G_0, \ldots, G_m)$. An important tool in studying the Rees algebra of I is the following homogeneous surjective homomorphism:

$$\omega : A[y_0, \ldots, y_m] \to \mathcal{R}_A(I)$$

defined by sending A to A (identity map) and sending y_j to $G_j t$ for all j. Let $\mathfrak{a}$ be the kernel of ω. Then, $\mathfrak{a}$ is called the *defining ideal* of $\mathcal{R}_A(I)$. A system of generators for the ideal $\mathfrak{a}$ is called a system of *defining equations* for $\mathcal{R}_A(I)$. If we consider the minimal free resolution of $\mathcal{R}_A(I)$ as a $A[y_0, \ldots, y_m]$-module, then the Betti numbers of this resolution are called the *Betti numbers of the Rees algebra* $\mathcal{R}_A(I)$.

4.1. Ideal of a generic set of points

We start by considering the situation where our set of points is in generic position. In particular, let $\mathbb{X} = \{P_1, \ldots, P_s\}$ be a set of s distinct points in $\mathbb{P}^2$ which are in generic position. Let $I_{\mathbb{X}} = \oplus_{t \geq \alpha} I_t$ be the defining ideal of $\mathbb{X}$ in $R = \mathfrak{k}[w_1, w_2, w_3]$, and $\mathbb{P}^2(\mathbb{X})$ the blowup of $\mathbb{P}^2$ centered at $\mathbb{X}$. We now proceed by considering different cases depending on the number of points in $\mathbb{X}$.

4.1.1. Binomial coefficient number of points

Our argument in this case is very similar to the argument on the Room surfaces of [**G-G**]. We refer the reader to Theorem 1.2 of [**G-G**]. Suppose $\mathbb{X}$ is a set of $s = \binom{d+1}{2}$ points of $\mathbb{P}^2$ which are in generic position (for some integer d). Then, from [**G-M**], $\sigma(I_{\mathbb{X}}) = d$ and $I_{\mathbb{X}}$ is generated by I_d. By the Hilbert-Burch theorem (cf. [**Bu**], [**Ei-1**], [**C-G-O**]), these generators are the $d \times d$ minors of a $d \times (d+1)$ matrix, say $\mathbf{L}$, of linear forms :

$$\mathbf{L} = (L_{ij}),\ L_{ij} \in R_1 \text{ for } i = 1, 2, \ldots, d \text{ and } j = 1, 2, \ldots, d+1.$$

In this notation,

$$I_{\mathbb{X}} = (F_1, \ldots, F_{d+1}),\ F_i = (-1)^{i+1}\det(\mathbf{L} \setminus i^{\text{th}} \text{ column}).$$

We shall now establish the defining equations for the Rees algebra $\mathcal{R}(I_{d+1})$ of the ideal generated by I_{d+1}.

A system of generators of the vector space I_{d+1} is given by $3(d+1)$ forms $w_i F_j$ for $i = 1, 2, 3$ and $j = 1, \ldots, d+1$. Consider the rational map :

$$\varphi_{d+1} : \mathbb{P}^2 \dashrightarrow \mathbb{P}^N,\ N = 3(d+1) - 1,$$

given by $\varphi_{d+1}(P) = [w_i F_j]$ for any point $P \in \mathbb{P}^2 \backslash \mathbb{X}$. Let Γ_{d+1} and Λ_{d+1} be the graph and the image of φ_{d+1}, respectively, and let $\overline{\Gamma_{d+1}}$ and $\overline{\Lambda_{d+1}}$ be their closures in $\mathbb{P}^2 \times \mathbb{P}^N$ and $\mathbb{P}^N$, respectively. We use homogeneous coordinates $[x_{ij}]_{1 \leq j \leq d+1, 1 \leq i \leq 3}$ of $\mathbb{P}^N$ such that

$$\varphi_{d+1}([\overline{w_1} : \overline{w_2} : \overline{w_3}]) = [\overline{x_{ij}}],\ \text{where } \overline{x_{ij}} = \overline{w_i} F_j. \tag{4.1.1}$$

The vector space dimension of I_{d+1} is $2d+3$, so there must be d linear dependence relations between the $w_i F_j$'s. Those relations can be found by expanding the following zero determinants:

$$0 = \det \begin{pmatrix} L_{l1}\ L_{l2}\ \ldots\ L_{l,d+1} \\ \mathbf{L} \end{pmatrix} = \sum_{j=1}^{d+1} L_{lj} F_j,$$

for each $l = 1, 2, \ldots, d$. Now, let $L_{lj} = \sum_{i=1}^{3} \lambda_{lji} w_i$. Then, by grouping similar terms, we get a collection of dependence relations among the $w_i F_j$'s as follows:

$$\sum_{j=1}^{d+1} \sum_{i=1}^{3} \lambda_{lji} w_i F_j = 0,\ \forall\ l = 1, 2, \ldots, d.$$

This gives rise to a collection of equations, where the coordinates of the points in Γ_{d+1} satisfy:

$$\sum_{1\leq i\leq 3, 1\leq j\leq d+1} \lambda_{lji}x_{ij} \;=\; 0,\ \forall\ l=1,2,\ldots,d. \tag{4.1.2}$$

There are exactly d equations and, as it was proved in [**G-G**] and in Lemma 3.2, those equations are linearly independent, so they are indeed all the equations obtained from the linear dependence relations of the w_iF_js.

Consider the matrix

$$M=\begin{bmatrix} w_1 & x_{11} & x_{12} & \ldots & x_{1,d+1}\\ w_2 & x_{21} & x_{22} & \ldots & x_{2,d+1}\\ w_3 & x_{31} & x_{32} & \ldots & x_{3,d+1}\end{bmatrix}$$

From (4.1.1), it is easy to see that the points of Γ_{d+1} satisfy all the 2×2 minors of M. Denote the collection of these equations by (∇). Let

$$M'=\begin{bmatrix} x_{11} & x_{12} & \ldots & x_{1,d+1}\\ x_{21} & x_{22} & \ldots & x_{2,d+1}\\ x_{31} & x_{32} & \ldots & x_{3,d+1}\end{bmatrix},$$

and recall Proposition 1.1 of [**G-G**].

Proposition 4.1. *For each $Q=[\overline{x_{ij}}]\in\mathbb{P}^N$ satisfying the equations in* (4.1.2) *and the 2×2 minors of M', there exists a unique $P'=[\overline{w_1}:\overline{w_2}:\overline{w_3}]\in\mathbb{P}^2$ such that the coordinates of P' and Q satisfy*

$$(\dagger)\begin{cases} \overline{x_{2j}w_1}-\overline{x_{1j}w_2} &= 0\\ \overline{x_{3j}w_2}-\overline{x_{2j}w_3} &= 0 \quad \textit{for } j=1,2,\ldots,d+1.\\ \overline{x_{1j}w_3}-\overline{x_{3j}w_1} &= 0\end{cases}$$

Similar to what was done in [**G-G**], we have the following result.

Theorem 4.2. *Suppose $\mathbb{X}$ is a set of $s=\binom{d+1}{2}$ points in $\mathbb{P}^2$ which are in generic position, and $\overline{\Gamma_{d+1}}$ is defined as on page 52. Then, equations in* (4.1.2) *and (∇) (see before Proposition 4.1) are the defining equations of $\overline{\Gamma_{d+1}}$ in $\mathbb{P}^2\times\mathbb{P}^N$.*

Proof. Let $\mathbf{V}$ be the algebraic set in $\mathbb{P}^2 \times \mathbb{P}^N$ defined by all the bi-homogeneous equations in (4.1.2) and (∇). We shall first prove that $\mathbf{V} = \overline{\Gamma_{d+1}}$ as sets.

Clearly, the coordinates of the points of Γ_{d+1} satisfy all the equations in (4.1.2) and (∇), so as sets, $\Gamma_{d+1} \subseteq \mathbf{V}$, hence $\overline{\Gamma_{d+1}} \subseteq \mathbf{V}$. To prove the reverse inclusion, let $(P, Q) \in \mathbf{V}$ (where $P \in \mathbb{P}^2$ and $Q \in \mathbb{P}^N$). The coordinates of Q satisfy equations in (4.1.2) and the 2×2 minors of M', so by Proposition 4.1, and following the same argument as that of [**G-G**] or of Theorem 3.3, there exists a unique $P' = [\overline{w_1} : \overline{w_2} : \overline{w_3}]$ such that the coordinates of P' and Q satisfy (†), and Q must have the form

(4.1.3) $$Q = [\overline{w_1}c_1 : \overline{w_2}c_1 : \overline{w_3}c_1 : \ldots : \overline{w_1}c_{d+1} : \overline{w_2}c_{d+1} : \overline{w_3}c_{d+1}],$$

for some $c_1, c_2, \ldots, c_{d+1} \in \mathfrak{k}$. The equations in (4.1.2) now become

$$\mathbf{L}(P') \begin{bmatrix} c_1 \\ \vdots \\ c_{d+1} \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Thus, if $P' \notin \mathbb{X}$ then

(4.1.4) $$\begin{bmatrix} c_1 \\ \vdots \\ c_{d+1} \end{bmatrix} = \rho \begin{bmatrix} F_1(P') \\ \vdots \\ F_{d+1}(P') \end{bmatrix},$$

for some $\rho \in \mathfrak{k}$; and otherwise, if $P' \in \mathbb{X}$, then Q lies on the exceptional line corresponding the the blowup at P. Thus, $Q \in \overline{\Lambda_{d+1}}$.

We also note that the equations in (†) and the 2×2 minors of M' are exactly the 2×2 minors of M. Thus, Proposition 4.1 shows that for each $Q \in \overline{\Lambda_{d+1}}$, there exists a unique $P' \in \mathbb{P}^2$ such that the coordinates of P' and Q satisfy the 2×2 minors of M. This implies $P = P'$. Since the divisor $|I_{d+1}|$ is very ample (Theorem 0.6), the projection map $\overline{\Gamma_{d+1}} \to \overline{\Lambda_{d+1}}$ defined by sending $(P'', Q) \in \overline{\Gamma_{d+1}}$ to $Q \in \overline{\Lambda_{d+1}}$ is one-to-one and onto (Proposition 2.6); and so, for each $Q \in \overline{\Lambda_{d+1}}$, there exists a unique $P'' \in \mathbb{P}^2$, such that $(P'', Q) \in \overline{\Gamma_{d+1}}$. Moreover, the coordinates of every point on Γ_{d+1} satisfy (†), so the coordinates of every point on $\overline{\Gamma_{d+1}}$ also satisfy (†). Thus, the coordinates of (P'', Q) satisfy (†), and so all the 2×2 minors of M. Therefore, $P = P' = P''$, i.e. $(P, Q) \in \overline{\Gamma_{d+1}}$. We have shown that $\mathbf{V} \subseteq \overline{\Gamma_{d+1}}$. Hence, $\mathbf{V} = \overline{\Gamma_{d+1}}$.

In conclusion, the equations in (4.1.2) and the 2×2 minors of M describe $\overline{\Gamma_{d+1}}$ as a set. Furthermore, M is a matrix of indeterminates, so it is a well known fact that the 2×2 minors of M form a prime ideal (cf. [**Sha**], [**H-E**]). Similar to the last part of the proof of Theorem 3.6, we consider the following sequence of surjective ring homomorphisms:

$$R[x_{ij}] \xrightarrow{\phi} R[w_i t_j] \xrightarrow{\psi} R[w_i F_j t],$$

where ϕ sends R to R, and sends x_{ij} to $w_i t_j$; and ψ sends R to R, and sends $w_i t_j$ to $w_i F_j t$. Then from the proofs of equations (4.1.3) and (4.1.4), we further deduce that the 2×2 minors of M form the kernel of ϕ, and the images of the equations in (4.1.2) through ϕ form the kernel of ψ. Therefore, the 2×2 minors of M and the equations in (4.1.2) form the kernel of $\psi \circ \phi$, which is a prime ideal. Hence, the equations in (4.1.2) and the 2×2 minors of M form the defining ideal for $\overline{\Gamma_{d+1}}$ in $\mathbb{P}^2 \times \mathbb{P}^N$. The theorem is proved. □

This gives rise to the following result.

Theorem 4.3. *Let $I = \oplus_{t \geq d} I_t$ be the defining ideal of $s = \binom{d+1}{2}$ points in $\mathbb{P}^2$ which are in generic position. Then, the defining equations for the Rees algebra $\mathcal{R}(I_{d+1})$ of the ideal generated by I_{d+1} are the 2×2 minors of a $3 \times (d+2)$ matrix of linear forms. Moreover, $\mathcal{R}(I_{d+1})$ is Cohen-Macaulay, and has the same Betti numbers as that of the ideal of 2×2 minors of a generic $3 \times (d+2)$ matrix.*

Proof. The first statement of the theorem follows from Theorem 4.2 since the Rees algebra $\mathcal{R}(I_{d+1})$ is the bi-graded coordinate ring of $\overline{\Gamma_{d+1}}$ (Proposition 2.4). For the second statement of the theorem, we observe that the defining ideal of $\overline{\Gamma_{d+1}}$ is the ideal of 2×2 minors of a matrix of linear forms of size $3 \times (d+2)$, so codim $\overline{\Gamma_{d+1}} \leq 2(d+1)$ (see [**H-E**]). Furthermore, $\overline{\Gamma_{d+1}}$ is a surface in the product space $\mathbb{P}^2 \times \mathbb{P}^{2d+2}$ (after cutting out by the linear forms in (4.1.2)), so its codimension in $\mathbb{P}^2 \times \mathbb{P}^{2d+2}$ is exactly $2(d+1)$. This implies that the defining ideal of $\overline{\Gamma_{d+1}}$ is perfect, and has the same Betti numbers as that of the ideal of 2×2 minors of a generic $3 \times (d+2)$ matrix ([**H-E**]). The result then follows. □

Remark: The resolution of the ideal of minors of a generic matrix was computed by many authors (cf. [**La**], [**P-W**]). One can apply their results to get the Betti numbers for $\mathcal{R}(I_{d+1})$.

We lastly observe that it is not hard to extend the whole discussion to the case of reduced codimension 2 perfect ideals with linear presentation in a polynomial ring. More precisely, one can follow the same argument to obtain the following result.

Theorem 4.4. *Suppose $I \subseteq R = \mathfrak{k}[w_1, \dots, w_n]$ is a reduced codimension 2 perfect ideal with linear presentation (i.e. its Hilbert-Burch matrix has linear entries). Let $I = \oplus_{t \geq d} I_t$ be its homogeneous decomposition. Then, the Rees algebra $\mathcal{R}(I_{d+1})$ of the ideal generated by I_{d+1} is Cohen-Macaulay, and its defining equations are the 2×2 minors of an $n \times (d+2)$ matrix of linear forms. Moreover, the Betti numbers of $\mathcal{R}(I_{d+1})$ are the same as those of the ideal of 2×2 minors of a generic $n \times (d+2)$ matrix.*

4.1.2. Arbitrary number of generic points

Our argument in this section inherits a great deal from that of [**Gi-Lo**]. We refer the readers to Theorem 4.2 and Proposition 4.4 of [**Gi-Lo**]. Suppose $\mathbb{X}$ is a set of $s = \binom{d+1}{2} + k$ points in generic position (with $0 < k < d+1$). It follows from [**G-M**] that $I_{\mathbb{X}} = \oplus_{t \geq d} I_t$ is generated in degrees d and $d+1$, and $\sigma(I_{\mathbb{X}}) = d+1$. The ideal generation conjecture is true in $\mathbb{P}^2$ (cf. [**G-G-R**] or [**G-M**]), so we may take $\mathbb{X}$ general enough to have this conjecture satisfied. If we add the hypothesis that no $d+1$ points of $\mathbb{X}$ lie on a line, then the divisor $|I_{d+1}|$ is very ample on $\mathbb{P}^2(\mathbb{X})$ (Theorem 0.6). That is, the rational map from $\mathbb{P}^2$ to $\mathbb{P}^N$ given by a system of generators of the vector space I_{d+1} gives an embedding of $\mathbb{P}^2(\mathbb{X})$ (see Chapter 2). We shall now establish the defining equations for the Rees algebra $\mathcal{R}(I_{d+1})$ under these conditions.

The ideal generation conjecture states that $I_{\mathbb{X}}$ is minimally generated by $d-k+1$ forms of degree d, and h forms of degree $d+1$, where h is either 0 or $2k-d$, depending on whether $d \geq 2k$ or not. Moreover, by the Hilbert-Burch theorem (cf. [**Bu**], [**Ei-1**], [**C-G-O**]), these generators can be seen as the $\rho+1$ maximal minors of a $\rho \times (\rho+1)$ matrix **L**, where

$$\rho = \begin{cases} k & \text{if } d \leq 2k \\ d-k & \text{if } d \geq 2k. \end{cases}$$

In the case when $d < 2k$, the matrix **L** is given by

$$\mathbf{L} = \begin{bmatrix} L_{1,1} & \dots & L_{1,2k-d} & Q_{1,1} & \dots & Q_{1,d-k+1} \\ \vdots & & \vdots & \vdots & & \vdots \\ L_{k,1} & \dots & L_{k,2k-d} & Q_{k,1} & \dots & Q_{k,d-k+1} \end{bmatrix}.$$

where the $L_{i,j}$'s are linear forms and the $Q_{i,j}$'s are forms of degree 2 (see also [**Gi-Lo**]). We denote by F_j the minor obtained by deleting column $2k-d+j$ for $j = 1, \dots, d-k+1$, and by G_l the minor obtained by deleting column l for $l = 1, \dots, 2k-d$. In this case $I_{\mathbb{X}} = < F_1, \dots, F_{d-k+1}, G_1, \dots, G_{2k-d} >$, where $F_j \in I_d$ and $G_l \in I_{d+1}$ for all j and l.

In the case when $d \geq 2k$, the matrix **L** is given by

$$\mathbf{L} = \begin{bmatrix} Q_{1,1} & Q_{1,2} & \dots & Q_{1,d-k+1} \\ \vdots & \vdots & \dots & \vdots \\ Q_{k,1} & Q_{k,2} & \dots & Q_{k,d-k+1} \\ L_{1,1} & L_{1,2} & \dots & L_{1,d-k+1} \\ \vdots & \vdots & \dots & \vdots \\ L_{d-2k,1} & L_{d-2k,2} & \dots & L_{d-2k,d-k+1} \end{bmatrix}.$$

where, again, the $L_{i,j}$'s are linear forms and the $Q_{i,j}$'s are forms of degree 2 (see also [**Gi-Lo**]). We denote by F_j the minor obtained by deleting column j for all $j = 1, \dots, d-k+1$. In this case, $I_{\mathbb{X}} = < F_1, \dots, F_{d-k+1} >$, where $F_j \in I_d$ for all j.

If $d < 2k$, a minimal system of generators for the vector space I_{d+1} is given by $\{w_iF_j, G_l | i = 1,2,3; j = 1, \dots, d-k+1; l = 1, \dots, 2k-d\}$. On the other hand, if $d \geq 2k$, a system of generators for the vector space I_{d+1} is given by $\{w_iF_j | i = 1,2,3; j = 1, \dots, d-k+1\}$, but this may not be minimal. In this case, the system w_iF_j's gives $3(d-k+1)$ generators, while the vector space dimension of I_{d+1} is $2d-k+1$, so there must be $d-2k$ linear dependence relations among those generators.

Those relations can be found by expanding the following zero determinants :

$$0 = \det \begin{bmatrix} \mathbf{L} \\ L_{l,1} \dots L_{l,d-k+1} \end{bmatrix} = \begin{vmatrix} Q_{1,1} & Q_{1,2} & \dots & Q_{1,d-k+1} \\ \vdots & \vdots & \dots & \vdots \\ Q_{k,1} & Q_{k,2} & \dots & Q_{k,d-k+1} \\ L_{1,1} & L_{1,2} & \dots & L_{1,d-k+1} \\ \vdots & \vdots & \dots & \vdots \\ L_{d-2k,1} & L_{d-2k,2} & \dots & L_{d-2k,d-k+1} \\ L_{l,1} & L_{l,2} & \dots & L_{l,d-k+1} \end{vmatrix}.$$

Write $L_{l,j} = \sum_{i=1}^{3} \lambda_{lji} w_i$ for all $l = 1, \dots, d-2k$, then

$$0 = \begin{vmatrix} \mathbf{L} \\ L_{l,1} \dots L_{l,d-k+1} \end{vmatrix} = \sum_{j=1}^{d-k+1} L_{l,j} F_j = \sum_{j=1}^{d-k+1} \sum_{i=1}^{3} \lambda_{lji} w_i F_j. \tag{4.1.5}$$

Again, it was proved in [**Gi-Lo**] that these $d-2k$ equations are linearly independent, so they are indeed all the dependence relations there are.

We consider the rational map on $\mathbb{P}^2$

$$\varphi_{d+1} : \mathbb{P}^2 \dashrightarrow \mathbb{P}^N,$$

defined by sending each point $P = [\overline{w_1} : \overline{w_2} : \overline{w_3}] \notin \mathbb{X}$ to $[\overline{w_i} F_j(P) : G_l(P)]$ if $d < 2k$, or to $[\overline{w_i} F_j(P)]$ if $d \geq 2k$. Here, $(N+1)$ is the appropriate number of generators chosen for I_{d+1}. Again, let Γ_{d+1} and Λ_{d+1} be the graph and the image of φ_{d+1}, respectively, and let $\overline{\Gamma_{d+1}}$ and $\overline{\Lambda_{d+1}}$ be their closures in $\mathbb{P}^2 \times \mathbb{P}^N$ and $\mathbb{P}^N$, respectively. If $d < 2k$, we use $[x_{ij} : y_l]$ to represent the homogeneous coordinates of $\mathbb{P}^N$ such that $\varphi_{d+1}([\overline{w_1} : \overline{w_2} : \overline{w_3}]) = [\overline{x_{ij}} : \overline{y_l}]$, where $\overline{x_{ij}} = \overline{w_i} F_j(\overline{w_1}, \overline{w_2}, \overline{w_3})$ and $\overline{y_l} = G_l(\overline{w_1}, \overline{w_2}, \overline{w_3})$. If $d \geq 2k$, we use $[x_{ij}]$ to represent the homogeneous coordinates of $\mathbb{P}^N$ such that $\varphi_{d+1}([\overline{w_1} : \overline{w_2} : \overline{w_3}]) = [\overline{x_{ij}}]$, where $\overline{x_{ij}} = \overline{w_i} F_j(\overline{w_1}, \overline{w_2}, \overline{w_3})$.

When $d \geq 2k$, the linear dependence relations of the $w_i F_j$'s in (4.1.5) give rise to a collection of linear equations which are satisfied on Γ_{d+1} as follows.

$$\sum_{j=1}^{d-k+1} \sum_{i=1}^{3} \lambda_{lji} x_{ij} = 0, \text{ for all } l = 1, \dots, d-2k. \tag{4.1.6}$$

Now, consider the matrix

$$X = \begin{bmatrix} w_1 & x_{11} & \dots & x_{1,d-k+1} \\ w_2 & x_{21} & \dots & x_{2,d-k+1} \\ w_3 & x_{31} & \dots & x_{3,d-k+1} \end{bmatrix}.$$

Clearly on Γ_{d+1}, the homogeneous coordinates of the points satisfy the 2×2 minors of X. To proceed, similar to what was done in [**Gi-Lo**], we separate the two cases.

Case 1: $d < 2k$. In this case, for each $u = 1, \dots, k$, expanding the following zero determinant:

$$\det \mathbf{L}_u = \begin{vmatrix} \mathbf{L} \\ L_{u,1} \dots L_{u,2k-d}\ Q_{u,1} \dots Q_{u,d-k+1} \end{vmatrix},$$

we obtain

$$0 = \sum_{l=1}^{2k-d} L_{u,l} G_l + \sum_{j=1}^{d-k+1} Q_{u,j} F_j.$$

Let $L_{u,l} = \sum_{i=1}^{3} \lambda_{uli} w_i$ and $Q_{u,j} = \sum_{i,h=1}^{3} \gamma_{uihj} w_i w_h$. Then

$$0 = \sum_{l=1}^{2k-d} (\sum_{i=1}^{3} \lambda_{uli} w_i) G_l + \sum_{j=1}^{d-k+1} (\sum_{i,h=1}^{3} \gamma_{uihj} w_i w_h) F_j.$$

Rewriting this as

$$0 = \sum_{i=1}^{3} (\sum_{l=1}^{2k-d} \lambda_{uli} G_l) w_i + \sum_{i=1}^{3} (\sum_{j=1}^{d-k+1} \sum_{h=1}^{3} \gamma_{uihj} w_h F_j) w_i. \tag{4.1.7}$$

Also, for each $v = 1, \dots, d-k+1$, we get

$$\begin{aligned} 0 &= F_v \Big(\sum_{l=1}^{2k-d} (\sum_{i=1}^{3} \lambda_{uli} w_i) G_l + \sum_{j=1}^{d-k+1} (\sum_{i,h=1}^{3} \gamma_{uihj} w_i w_h) F_j \Big) \\ &= \sum_{i=1}^{3} (\sum_{l=1}^{2k-d} \lambda_{uli} G_l) w_i F_v + \sum_{i=1}^{3} (\sum_{j=1}^{d-k+1} \sum_{h=1}^{3} \gamma_{uihj} w_h F_j) w_i F_v. \end{aligned} \tag{4.1.8}$$

The equations in (4.1.7) and (4.1.8) give a collection of bi-homogeneous equations that are satisfied by the coordinates of the points in Γ_{d+1}:

$$\sum_{i=1}^{3} (\sum_{l=1}^{2k-d} \lambda_{uli} y_l) w_i + \sum_{i=1}^{3} (\sum_{j=1}^{d-k+1} \sum_{h=1}^{3} \gamma_{uihj} x_{hj}) w_i = 0,$$

for all $u = 1, \ldots, k$, and

$$\sum_{i=1}^{3}(\sum_{l=1}^{2k-d} \lambda_{uli} y_l) x_{iv} + \sum_{i=1}^{3}(\sum_{j=1}^{d-k+1}\sum_{h=1}^{3} \gamma_{uihj} x_{hj}) x_{iv} = 0,$$

for all $u = 1, \ldots, k$ and $v = 1, \ldots, d-k+1$.

Now, let $B = (b_{ui})_{1 \leq u \leq k, 1 \leq i \leq 3}$ be the matrix given by

$$b_{ui} = \sum_{l=1}^{2k-d} \lambda_{uli} y_l + \sum_{j=1}^{d-k+1}\sum_{h=1}^{3} \gamma_{uihj} x_{hj},$$

then the collection of the equations above can be rewritten as:

$$\sum_{i=1}^{3} b_{ui} w_i = 0, \text{ for any } u = 1, \ldots, k,$$

and

$$\sum_{i=1}^{3} b_{ui} x_{iv} = 0, \text{ for any } u = 1, \ldots, k \text{ and } v = 1, \ldots, d-k+1.$$

These equations are exactly all the entries of $B.X$ where B and X are the matrices as defined.

It follows also from [**Gi-Lo**] that the coordinates of all the points of Γ_{d+1} satisfy the 3×3 minors of B. We let $\mathbf{J}$ be the ideal in $\mathfrak{k}[\underline{w}, \underline{x}, \underline{y}]$ defined by the 3×3 minors of B, the 2×2 minors of X and the entries of $B.X$.

Case 2: $d \geq 2k$. Similar to what was done in the previous case, we expand the zero determinants:

$$\det \begin{bmatrix} \mathbf{L} \\ Q_{u,1} \ldots Q_{u,d-k+1} \end{bmatrix},$$

for $u = 1, \ldots, k$. We also let $B = (b_{ui})_{1 \leq u \leq k, 1 \leq i \leq 3}$ be the matrix given by

$$b_{ui} = \sum_{j=1}^{d-k+1}\sum_{h=1}^{3} \gamma_{uihj} x_{hj},$$

where $Q_{u,j} = \sum_{i,h=1}^{3} \gamma_{uihj} w_i w_h$. Again, let $\mathbf{J}$ be the ideal defined by the 3×3 minors of B, the 2×2 minors of X and the entries of $B.X$. Similar to the previous case, we can prove that the coordinates of the points on Γ_{d+1} satisfy the equations in $\mathbf{J}$.

We now establish a collection of equations which are similar to those in (†) as follows:

$$(\dagger\dagger)\begin{cases} x_{2j}w_1 - x_{1j}w_2 &= 0 \\ x_{3j}w_2 - x_{2j}w_3 &= 0 \quad \text{for } j = 1, 2, \ldots, d-k+1. \\ x_{1j}w_3 - x_{3j}w_1 &= 0 \end{cases}$$

From the proof of Proposition 4.4 of [**Gi-Lo**], one can deduce the following proposition:

Proposition 4.5. *If $Q \in \mathbb{P}^N$ has homogeneous coordinates which satisfy: the 3×3 minors of B, the 2×2 minors of*

$$Y = \begin{bmatrix} x_{11} & \ldots & x_{1,d-k+1} \\ x_{21} & \ldots & x_{2,d-k+1} \\ x_{31} & \ldots & x_{3,d-k+1} \end{bmatrix},$$

and the entries of $B.Y$, then there exists a unique point $P' \in \mathbb{P}^2$ such that the homogeneous coordinates of P' and Q satisfy the equations in (††).

We obtain our first result when the number of points in $\mathbb{X}$ is arbitrary.

Theorem 4.6. *Let $\mathbf{V}$ be the subvariety of $\mathbb{P}^2 \times \mathbb{P}^N$ defined by $\mathbf{J}$ if $d < 2k$, or defined by $\mathbf{J}$ and the equations in (4.1.6) if $d \geq 2k$ (pages 58, 60). Then $\mathbf{V} = \overline{\Gamma_{d+1}}$ as sets.*

Proof. The proof goes in the same manner of that of Theorem 4.2. First, since all the points on Γ_{d+1} satisfy the defining equations of $\mathbf{V}$, we have $\Gamma_{d+1} \subseteq \mathbf{V}$, whence $\overline{\Gamma_{d+1}} \subseteq \mathbf{V}$.

To prove the reverse inclusion, suppose $(P, Q) \in \mathbf{V}$, where $P \in \mathbb{P}^2$ and $Q \in \mathbb{P}^N$. By Proposition 4.5 and following the same argument as that of [**Gi-Lo**], there exists a unique $P' = [\overline{w_1} : \overline{w_2} : \overline{w_3}]$ such that the coordinates of P' and Q satisfy the equations in (††), and Q must have the form

$$Q = [\overline{w_1}c_1 : \overline{w_2}c_1 : \overline{w_3}c_1 : \ldots : \overline{w_1}c_{d-k+1} : \overline{w_2}c_{d-k+1} : \overline{w_3}c_{d-k+1}],$$

for some $c_1, \ldots, c_{d-k+1} \in \mathfrak{k}$, if $d \geq 2k$, or

$$Q = [\overline{w_1}c_1 : \overline{w_2}c_1 : \overline{w_3}c_1 : \ldots : \overline{w_1}c_{d-k+1} : \overline{w_2}c_{d-k+1} : \overline{w_3}c_{d-k+1} : \overline{y_1} : \ldots : \overline{y_{2k-d}}],$$

for some $c_1, \ldots, c_{d-k+1}, \overline{y_1}, \ldots, \overline{y_{2k-d}} \in \mathfrak{k}$, if $d < 2k$.

Now, substituting the coordinates of Q into the entries of the product matrix $B.X$ and the equations in (4.1.6), we get

$$\mathbf{L}(P') \begin{bmatrix} c_1 \\ \vdots \\ c_{d-k+1} \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix},$$

if $d \geq 2k$, or

$$\mathbf{L}(P') \begin{bmatrix} c_1 \\ \vdots \\ c_{d-k+1} \\ y_1 \\ \vdots \\ y_{2k-d} \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

if $d < 2k$. Thus, if $P' \notin \mathbb{X}$, then

$$\begin{bmatrix} c_1 \\ \vdots \\ c_{d-k+1} \end{bmatrix} = \rho \begin{bmatrix} F_1(P') \\ \vdots \\ F_{d-k+1}(P') \end{bmatrix},$$

if $d \geq 2k$, or

$$\begin{bmatrix} c_1 \\ \vdots \\ c_{d-k+1} \\ y_1 \\ \vdots \\ y_{2k-d} \end{bmatrix} = \rho \begin{bmatrix} F_1(P') \\ \vdots \\ F_{d-k+1}(P') \\ G_1(P') \\ \vdots \\ G_{2k-d}(P') \end{bmatrix},$$

if $d < 2k$ (for some $\rho \in \mathfrak{k}$); and if $P' \in \mathbb{X}$, then Q belongs to the exceptional line corresponding the the blowup at P'. Therefore, $Q \in \overline{\Lambda_{d+1}}$.

We note further that the equations in (††) are among the equations of $\mathbf{J}$, so Proposition 4.5 shows that for each $Q \in \overline{\Lambda_{d+1}}$, there exists a unique P' such that the coordinates of P' and Q satisfy the equations of $\mathbf{J}$ (when $d < 2k$), or of $\mathbf{J}$ and (4.1.6) (when $d \geq 2k$). Thus, $P = P'$. Now, with our assumption that there are no $d+1$ points of $\mathbb{X}$ lying on a line, the divisor $|I_{d+1}|$ is very ample (Theorem 0.6), so the projection that sends $(P'', Q) \in \mathbb{P}^2 \times \mathbb{P}^N$ to $Q \in \mathbb{P}^N$ is one-to-one and onto from $\overline{\Gamma_{d+1}}$

to $\overline{\Lambda_{d+1}}$ (Proposition 2.6). Thus, for each $Q \in \overline{\Lambda_{d+1}}$ there exists a unique $P'' \in \mathbb{P}^2$ such that $(P'', Q) \in \overline{\Gamma_{d+1}}$. Moreover, since $\overline{\Gamma_{d+1}} \subseteq \mathbf{V}$, the coordinates of P'' and Q satisfy the equations of $\mathbf{J}$ (when $d < 2k$), or of $\mathbf{J}$ and (4.1.6) (when $d \geq 2k$). Hence, $P = P' = P''$. In other words, $(P, Q) \in \overline{\Gamma_{d+1}}$. We just proved that $\mathbf{V} \subseteq \overline{\Gamma_{d+1}}$.

Hence, $\mathbf{V} = \Gamma_{d+1}$ as sets. □

This gives rise to the following result.

Theorem 4.7. *Suppose $\mathbb{X}$ is a set of $s = \binom{d+1}{2} + k$ points in $\mathbb{P}^2$ which are in generic position ($1 \leq k \leq d$). For a general choice of the points in $\mathbb{X}$, the defining equations of the Rees algebra $\mathcal{R}(I_{d+1})$ of the ideal generated by I_{d+1} are the equations in $\mathbf{J}$ if $d < 2k$, or the equations in $\mathbf{J}$ together with the equations in (4.1.6) if $d \geq 2k$ (see pages 58, 60). Moreover, $\mathcal{R}(I_{d+1})$ is Cohen-Macaulay.*

Proof. We first show that for a general choice of $\mathbb{X}$, the ideal $\mathbf{J}$ is prime and perfect. Similar to what was done in [**Gi-Lo**], we consider a new polynomial ring $R' = \mathfrak{k}[\underline{w}, \underline{x}, \underline{y}, \underline{z}]$ (in the case where $d \geq 2k$, $R' = \mathfrak{k}[\underline{w}, \underline{x}, \underline{z}]$), where $\underline{z} = \{z_{ui}\}_{1 \leq u \leq k, 1 \leq i \leq 3}$, and let $B' = (z_{ui})$. Then we can view $\mathbf{J}$ as the quotient ideal of $\mathbf{J}'$ in the ring $R'/(H_{ui})$, where $\mathbf{J}'$ is the ideal in R' defined by the 3×3 minors of B', the 2×2 minors of X and the entries of $B'.X$, and

$$H_{ui} = z_{ui} - b_{ui}, \text{ for all } u = 1, \ldots, k \text{ and } i = 1, 2, 3.$$

Since, $\mathbf{J}$ and $\mathbf{J}'$ are both bi-homogeneous, they are in particular also homogenous, so they define subvarieties of certain projective spaces. Let $\mathbf{W}$ be the subvariety of $\mathbb{P}^{N+3}$ defined by $\mathbf{J}$, and let $\mathbf{W}'$ be the subvariety of $\mathbb{P}^{N+3+3k}$ defined by $\mathbf{J}'$. Then $\mathbf{W}$ is obtained from $\mathbf{W}'$ by cutting $\mathbf{W}'$ with $3k$ hyperplanes H_{ui}. In other words, $\mathbf{W}$ is the intersection of $\mathbf{W}'$ and a $3k$-codimensional linear subspace of $\mathbb{P}^{N+3+3k}$. By Huneke's theorem ([**Hu**, Theorem 60]), we know that $\mathbf{W}'$ is an integral Cohen-Macaulay variety.

Let Ω' be the Grassmannian which parameterizes the linear subspaces of codimension $3k$ of $\mathbb{P}^{N+3+3k}$. It follows from Bertini's theorem (cf. [**Hart**], [**Jou**]) that the subset

$\mathcal{U}' \subseteq \Omega'$, such that for any $U' \in \mathcal{U}'$, $U' \cap \mathbf{W}'$ is again an integral Cohen-Macaulay variety, is non-empty and open. We also let Θ' be the Grassmannian which parameterizes the linear subspaces of codimension $3k$ of $\mathbb{P}^{N+3+3k}$ that lie inside the variety defined by the equations $w_1 = w_2 = w_3 = 0$. In [**Gi-Lo**, Theorem 4.2], the authors actually showed that for a general choice of $\mathbb{X}$, the $3k$-codimensional linear subspace of $\mathbb{P}^{N+3+3k}$ given by the hyperplanes $\{H_{ui} | u = 1, \ldots, k; i = 1, 2, 3\}$ is general in Θ'. Moreover, consider the element $T' \in \Theta' \subseteq \Omega'$ given by $3k$ linear equations $z_{ui} = 0$, then $T' \cap \mathbf{W}'$ is the subvariety of $\mathbb{P}^{N+3}$ given by the 2×2 minors of X, so it is an integral Cohen-Macaulay variety. In other words, $T' \in \mathcal{U}' \cap \Theta'$. Since Θ' is a closed subset of Ω', we deduce that $\mathcal{U}' \cap \Theta'$ is a non-empty open subset of Θ'. All these facts, put together, imply that for a general choice of $\mathbb{X}$, the $3k$-codimensional linear subspace of $\mathbb{P}^{N+3+3k}$ given by the hyperplanes $\{H_{ui} | u = 1, \ldots, k; i = 1, 2, 3\}$ is in $\mathcal{U}' \cap \Theta'$. In other words, for a general choice of $\mathbb{X}$, $\mathbf{W}$ is an integral Cohen-Macaulay subvariety of $\mathbb{P}^{N+3}$, which, in turn, implies that $\mathbf{J}$ is a perfect prime ideal.

For $d < 2k$, this and Theorem 4.6 clearly imply that $\mathbf{J}$ is the defining ideal of $\mathcal{R}(I_{d+1})$.

For $d \geq 2k$, let l be the least integer such that $3l \geq d - 2k$. Let Ω be the Grassmannian which parameterizes the linear subspaces of codimension $3l$ of $\mathbb{P}^{N+3}$, and let Θ be the Grassmannian which parameterizes the linear subspaces of codimension $3l$ of $\mathbb{P}^{N+3}$ that lie inside the variety defined by the equations $w_1 = w_2 = w_3 = 0$. Again, it follows from Bertini's theorem that the subset $\mathcal{U} \subseteq \Omega$, such that for any $U \in \mathcal{U}$, $U \cap \mathbf{W}$ is an integral Cohen-Macaulay variety, is non-empty and open (for a general choice of $\mathbb{X}$, $\mathbf{W}$ is an integral Cohen-Macaulay variety). One can follow a similar argument as above, and consider the element T in Θ given by $3l$ linear equations $\{x_{ij} = 0 | i = 1, 2, 3; j = d-k-l+2, \ldots, d-k+1\}$, to show that $\mathcal{U} \cap \Theta$ is a non-empty open subset of Θ. Moreover, from [**Gi-Lo**], it is easy to see that for a general choice of $\mathbb{X}$, the $3l$-codimensional linear subspace of $\mathbb{P}^{N+3}$ given by the equations in (4.1.6) and $3l - (d - 2k)$ other general hyperplanes is general in Θ. Thus, for a general choice of $\mathbb{X}$, the $3l$-codimensional subspace of $\mathbb{P}^{N+3}$ given by the equations in (4.1.6) and

$3l-(d-2k)$ other general hyperplanes is in $\mathcal{U}\cap\Theta$, i.e. this $3l$-codimensional subspace of $\mathbb{P}^{N+3}$ is general enough to intersect $\mathbf{W}$ at an integral Cohen-Macaulay variety. In particular, the hyperplanes given by the equations in (4.1.6) are general enough for a general choice of the points in $\mathbb{X}$. This and the fact in the previous paragraph show that for a general choice of $\mathbb{X}$, the hyperplanes H_{ui} and the hyperplanes defined by the equations in (4.1.6) are general enough to intersect $\mathbf{W}'$ at an integral Cohen-Macaulay variety. Hence, for a general choice of $\mathbb{X}$, $\mathbf{J}$ together with the equations in (4.1.6) form a perfect prime ideal, i.e. $\mathbf{J}$ and the equations in (4.1.6) form the defining ideal of $\mathcal{R}(I_{d+1})$.

The Cohen-Macaulayness of $\mathcal{R}(I_{d+1})$ follows from the perfection of its defining ideal. The theorem is proved. □

We finally observe that the same argument can be extended to a class of codimension 2 perfect ideals with Hilbert-Burch matrix in the form of $\mathbf{L}$ (i.e. constituted by rows and columns of linear forms or quadratics). The generality of the points in $\mathbb{X}$ transforms to the genericity of the Hilbert-Burch matrix of the ideal. One can follow the same argument to obtain the following result.

Theorem 4.8. *Suppose $I \subseteq R = \mathfrak{k}[w_1, \ldots, w_n]$ is a generic codimension 2 perfect ideal, whose Hilbert-Burch matrix looks like that of* $\mathbf{L}$. *Suppose also that $I = \oplus_{t\geq d} I_t$ is its homogeneous decomposition. Then, the defining equations of the Rees algebra $\mathcal{R}(I_{d+1})$ of the ideal generated by I_{d+1} are the $n \times n$ minors of a $k \times n$ matrix B of linear forms, the 2×2 minors of an $n \times (d-k+2)$ matrix X of linear forms, and the entries of $B.X$. Moreover, $\mathcal{R}(I_{d+1})$ is Cohen-Macaulay, and its defining ideal has the generic grade.*

4.2. Projective embeddings of blowup surfaces: Revisited

In this section, we revisit the problem addressed in Chapter 3, but in a more general situation. We consider the case when our set has an arbitrary number of points.

Suppose $\mathbb{X} = \{P_1, \ldots, P_s\} \subseteq \mathbb{P}^2$ is a set of $s = \binom{d+1}{2} + k$ distinct points which are in generic position ($1 \le k \le d$). The reader is referred back to Section 0.3 for a more detailed introduction to the problem. We shall now only briefly recall the setup and the notation. It follows from [**G-M**] that the defining ideal $I_{\mathbb{X}} \subseteq R = \mathfrak{k}[w_1, w_2, w_3]$ of $\mathbb{X}$ is generated in degrees d and $d+1$, and $\sigma(I_{\mathbb{X}}) = d+1$. Suppose $I_{\mathbb{X}} = \oplus_{t \ge d} I_t$ is the homogeneous decomposition of $I_{\mathbb{X}}$, and $\mathbb{P}^2(\mathbb{X})$ is the blowup of $\mathbb{P}^2$ along $\mathbb{X}$. For each $t \ge d+1$, we use I_t to define a rational map $\varphi_t : \mathbb{P}^2 \dashrightarrow \mathbb{P}^{m_t}$, and let Γ_t and Λ_t be the graph and the image of this map. Let also $\overline{\Gamma_t}$ and $\overline{\Lambda_t}$ be the closures of Γ_t and Λ_t inside $\mathbb{P}^2 \times \mathbb{P}^{m_t}$ and $\mathbb{P}^{m_t}$, respectively. Under the additional condition that the points in $\mathbb{X}$ are general (then there are no $d+1$ points of $\mathbb{X}$ lying on a line), the divisor $|I_t|$ is very ample on $\overline{\Gamma_t}$ for all $t \ge d+1$ (Theorem 0.6), and it gives a projective embedding $\overline{\Lambda_t}$ of the blowup surface $\mathbb{P}^2(\mathbb{X})$ (see Chapter 2). In this section, with the assumption that the points in $\mathbb{X}$ are general, we study $\overline{\Lambda_t}$ for all $t \ge d+1$.

We shall employ results of [**STV**] and [**CHTV**] to demonstrate a method of writing down the defining equations for $\overline{\Lambda_t}$ for any value of $t \ge d+1$. This method makes use of our results on the Rees algebras of the ideal generated by I_{d+1} in Theorems 4.3 and 4.7. We start by going through the notion of diagonal subalgebras introduced in [**STV**] and [**CHTV**].

Definition. Let A be a $\mathbb{Z}^2$-graded $\mathfrak{k}$-algebra (we say A is a *bi-graded* algebra). For any tuple $\Delta = (c, e)$ of integers, the *diagonal subalgebra* of A along Δ is defined as the $\mathbb{Z}$-graded algebra $A_\Delta = \oplus_{s \in \mathbb{Z}} A_{(cs,es)}$.

It was shown in [**STV**] that if A is a quotient algebra, $A = \mathfrak{k}[\underline{x}, \underline{t}]/J$, where $\underline{x}$ and $\underline{t}$ are two sets of variables, $\underline{x} = \{x_1, \ldots, x_r\}$ and $\underline{t} = \{t_1, \ldots, t_q\}$, and J is a bihomogeneous ideal of $\mathfrak{k}[\underline{x},\underline{t}]$, then $A_\Delta = \mathfrak{k}[\underline{x},\underline{t}]_\Delta / J_\Delta$. The generators of J_Δ as an ideal in $\mathfrak{k}[\underline{x},\underline{t}]_\Delta$ were given in [**STV**, Lemma 2.1]. We recall that lemma now (but note that the result in [**STV**] was stated only for $\Delta = (1,1)$, but it is true for any Δ).

Lemma 4.9 ([**STV**], Lemma 2.1)**.** *Suppose A and J are as above, and that J is generated by bihomogeneous polynomials $F_1, \ldots, F_l$ with $\deg F_i = (a_i, b_i)$. Let s_i be the least integer such that $cs_i \geq a_i$ and $es_i \geq b_i$. Then J_Δ is generated by the elements of the form $F_i M$ where M is a monomial of degree $(cs_i - a_i, es_i - b_i)$, $i = 1, \ldots, l$.*

Moreover, the $\mathfrak{k}$-algebra $\mathfrak{k}[\underline{x},\underline{t}]_\Delta$ is exactly the ordinary Segre product $\mathfrak{k}[\underline{x}]^{(c)} \underline{\otimes} \mathfrak{k}[\underline{t}]^{(e)}$ of the c^{th} Veronese subring of $\mathfrak{k}[\underline{x}]$ and the e^{th} Veronese subring of $\mathfrak{k}[\underline{t}]$. Denote by $\mathbf{S}$ the set $\{(\alpha, \beta) \in \mathbb{N}^r \times \mathbb{N}^q : |\alpha| = c, |\beta| = e\}$. Then we may represent $\mathfrak{k}[\underline{x},\underline{t}]_\Delta$ as a factor ring of the polynomial ring

$$\mathfrak{k}[\underline{y}] = \mathfrak{k}[y_{(\alpha,\beta)} : (\alpha, \beta) \in \mathbf{S}] \to \mathfrak{k}[\underline{x},\underline{t}]_\Delta,$$

by sending $y_{(\alpha,\beta)}$ to $\underline{x}^\alpha \underline{t}^\beta$. The kernel of $\mathfrak{k}[\underline{y}] \to \mathfrak{k}[\underline{x},\underline{t}]_\Delta$ is just the defining ideal of the Segre embedding $\mathbb{P}^{r-1} \times \mathbb{P}^{q-1} \hookrightarrow \mathbb{P}^{rq-1}$, which is the ideal of 2×2 minors of a generic $r \times q$ matrix $(y_{(\alpha,\beta)})_{(\alpha,\beta)\in\mathbf{S}}$ (see Corollary 1.6.1). Thus, we have the presentation of A_Δ as a factor ring of a polynomial ring:

$$\mathfrak{k}[\underline{y}] \to \mathfrak{k}[\underline{x},\underline{t}]_\Delta \to A_\Delta,$$

with the kernel given by the 2×2 minors of a generic $r \times q$ matrix and the generators of J_Δ (given in Lemma 4.9). Hence, using this method, one can write down the defining equations for A_Δ, for any Δ, knowing the defining equations for A (i.e. the generators of J).

We now demonstrate how this method can be used in our problem. It can be observed that the Rees algebra $\mathcal{R}(I_{d+1})$ is a subring of $R[t]$, so $\mathcal{R}(I_{d+1})$ inherits the bigradation of $R[t]$ in which $\deg(w_i) = (1,0)$ for all $i = 1,2,3$ and $\deg t = (0,1)$. Under this gradation of $\mathcal{R}(I_{d+1})$, it is easy to see that $k[I_t]$ is isomorphic to the diagonal subalgebra $\mathcal{R}(I_{d+1})_\Delta$, where $\Delta = (t,1)$ for any $t \geq d+1$. Furthermore, Theorem 4.7 gives the defining equations for the Rees algebra $\mathcal{R}(I_{d+1})$ for a general choice of the points in $\mathbb{X}$. Thus, our method above enables us to write down the defining equations for the embedding $\overline{\Lambda_t}$ for any $t \geq d+1$. We illustrate the idea by looking at an example. Most of the calculations in the following example are obtained by the help of the CoCoA package ([**CoCoA**]).

Example: Let us look at 8 general points of $\mathbb{P}^2$ ($s = 8, d = 3$ and $k = 2$). Suppose $\mathbb{X} = \{P_1, \dots, P_8\} \subseteq \mathbb{P}^2$, where $P_1 = [1:0:0], P_2 = [-1:2:-2], P_3 = [0:-1:1], P_4 = [1:-2:-2], P_5 = [-1:-1:0], P_6 = [-2:1:2], P_7 = [-2:-2:-2]$ and $P_8 = [-2:1:-2]$. The resolution of the defining ideal $I_{\mathbb{X}}$ of $\mathbb{X}$ is

$$0 \to R(-5)^2 \to R(-3)^2 \oplus R(-4) \to I_{\mathbb{X}} \to 0.$$

This resolution is the generic resolution for a set of 8 points in $\mathbb{P}^2$ (see [**G-M**]). The generators of $I_{\mathbb{X}}$ are the maximal minors of the following matrix:

$$\begin{bmatrix} 4w_1 - 4w_2 - 14w_3 & 2w_2w_3 + 10w_3^2 & -2w_1w_3 + 4w_2w_3 + w_3^2 \\ -12w_1 + 2w_2 + 17w_3 & -2w_2^2 + 6w_1w_3 - 10w_3^2 & 2w_1w_2 + 2w_2^2 + 2w_1w_3 - 5w_2w_3 - 5/2w_3^2 \end{bmatrix}$$

This matrix is in fact the Hilbert-Burch matrix of $I_{\mathbb{X}}$, and is given by CoCoA when taking the resolution of $R/I_{\mathbb{X}}$.

Suppose we want to find the defining equations for the embedding of $\mathbb{P}^2(\mathbb{X})$ given by the linear system of plane curves of degree 5 through these 8 points. Let $I_{\mathbb{X}} = \oplus_{t \geq 3} I_t$ be the homogeneous decomposition of $I_{\mathbb{X}}$, and denote by $\{F_1, F_2, G_1 \mid F_1, F_2 \in I_3, G_1 \in I_4\}$ the minimal system of generators for $I_{\mathbb{X}}$ obtained from the above Hilbert-Burch matrix. We notice that the Rees algebra $\mathcal{R}(I_4)$ is a quotient ring of the polynomial ring $\mathfrak{k}[\underline{w}, \underline{x}, \underline{y}]$, where $\underline{w} = \{w_1, w_2, w_3\}$, $\underline{x} = \{x_{ij}\}_{1 \leq i \leq 3, 1 \leq j \leq 2}$, and $\underline{y} = \{y_1\}$. The matrix B is:

$$\begin{bmatrix} -4x_{32} + 8y_1 & 2x_{31} + 4x_{32} - 8y_1 & 10x_{31} + x_{32} - 14y_1 \\ 2x_{22} + 6x_{31} + 2x_{32} - 12y_1 & -2x_{21} + 2x_{22} - 5x_{32} + 2y_1 & -10x_{31} - 5/2x_{32} + 17y_1 \end{bmatrix},$$

and the matrix X is given by:

$$X = \begin{bmatrix} w_1 & x_{11} & x_{12} \\ w_2 & x_{21} & x_{22} \\ w_3 & x_{31} & x_{32} \end{bmatrix}.$$

Thus, the defining equations for $\mathcal{R}(I_4)$ (obtained as in Theorem 4.7) are:

$2w_2x_{31} + 10w_3x_{31} - 4w_1x_{32} + 4w_2x_{32} + w_3x_{32} + 8w_1y_1 - 8w_2y_1 - 14w_3y_1$,
$2x_{21}x_{31} + 10x_{31}^2 - 4x_{11}x_{32} + 4x_{21}x_{32} + x_{31}x_{32} + 8x_{11}y_1 - 8x_{21}y_1 - 14x_{31}y_1$,
$2x_{22}x_{31} - 4x_{12}x_{32} + 4x_{22}x_{32} + 10x_{31}x_{32} + x_{32}^2 + 8x_{12}y_1 - 8x_{22}y_1 - 14x_{32}y_1$,
$- 2w_2x_{21} + 2w_1x_{22} + 2w_2x_{22} + 6w_1x_{31} - 10w_3x_{31} + 2w_1x_{32} - 5w_2x_{32} - 5/2w_3x_{32} - 12w_1y_1 + 2w_2y_1 + 17w_3y_1$,
$-2x_{21}^2 + 2x_{11}x_{22} + 2x_{21}x_{22} + 6x_{11}x_{31} - 10x_{31}^2 + 2x_{11}x_{32} - 5x_{21}x_{32} - 5/2x_{31}x_{32} - 12x_{11}y_1 + 2x_{21}y_1 + 17x_{31}y_1$,
$2x_{12}x_{22} - 2x_{21}x_{22} + 2x_{22}^2 + 6x_{12}x_{31} + 2x_{12}x_{32} - 5x_{22}x_{32} - 10x_{31}x_{32} - 5/2x_{32}^2 - 12x_{12}y_1 + 2x_{22}y_1 + 17x_{32}y_1$,
$- w_2x_{11} + w_1x_{21}$,
$- w_2x_{12} + w_1x_{22}$,
$- x_{12}x_{21} + x_{11}x_{22}$,
$- w_3x_{11} + w_1x_{31}$,
$- w_3x_{12} + w_1x_{32}$,
$- x_{12}x_{31} + x_{11}x_{32}$,
$- w_3x_{21} + w_2x_{31}$,
$- w_3x_{22} + w_2x_{32}$,
$- x_{22}x_{31} + x_{21}x_{32}$.

A system of generators for the $\mathfrak{k}$-vector space I_5 is given by $\{w^\alpha F_i, w_j G_1 \mid |\alpha| = 2, 1 \le i \le 2, 1 \le j \le 3\}$. This system has 21 generators, but the vector space dimension of I_5 is only 13. Thus, there are exactly 8 linear equations coming from the linear dependence relations among these generators. We first use these generators of I_5 to embed $\mathbb{P}^2(\mathbb{X})$ into $\mathbb{P}^{20}$. As we have shown above, the homogeneous coordinate ring of the embedded surface is the diagonal subalgebra $\mathcal{R}(I_4)_\Delta$ with $\Delta = (5,1)$. Here, the bi-gradation of $\mathcal{R}(I_4)$ is inherited from that of $\mathfrak{k}[\underline{w},\underline{x},\underline{y}]$, where $\deg w_i = (1,0)$ and $\deg x_{ij} = \deg y_1 = (4,1)$. We use $\{z_{ij} | i = 1,2,3; j = 1,\ldots,7\}$ to represent the coordinates of $\mathbb{P}^{20}$. Then the kernel of the map $\mathfrak{k}[\underline{z}] \to \mathfrak{k}[\underline{w},\underline{x},\underline{y}]_\Delta \to 0$ is the ideal of 2×2 minors of the following matrix:

$$Z = \begin{bmatrix} z_{11} & z_{12} & \cdots & z_{17} \\ z_{21} & z_{22} & \cdots & z_{27} \\ z_{31} & z_{32} & \cdots & z_{37} \end{bmatrix}.$$

These minors and the generators for the kernel of the presentation $\mathfrak{k}[\underline{w},\underline{x},\underline{y}]_\Delta \to \mathcal{R}(I_4)_\Delta \to 0$, which can be found from the equations of $\mathcal{R}(I_4)$ using Lemma 4.9, form the defining ideal for the embedding $\overline{\Lambda_5}$ sitting inside $\mathbb{P}^{20}$. We include a set of equations coming from that of $\mathcal{R}(I_4)$ in the Appendix (for those who are interested in seeing the actual equations). Note also that there are exactly 8 linear equations in the given set of equations (in the Appendix). Those equations are exactly the linear equations coming from the linear dependence relations among the generators of I_5 we have chosen. By factoring out these 8 linear equations, we obtain the defining ideal for the embedding $\overline{\Lambda_5}$ (sitting inside $\mathbb{P}^{12}$) of the blowup $\mathbb{P}^2(\mathbb{X})$ given by the linear system I_5.

Remark: One should note that the method we just illustrated has its disadvantages. First, it does not normally give a minimal set of generators. Secondly, certain defining equations of the Rees algebra may become redundant through the process of diagonalizing, leaving redundant high order equations. For instance, if we apply this method for a set of $\binom{d+1}{2}$ points in generic position using the result of Morey-Ulrich (Theorem 0.11), which gives the defining equations for the Rees algebra of the defining ideal of these points, then for each embedding we would end up with redundant cubics whereas the ideal of the embedding is known to be generated by quadratics. If instead of Theorem 0.11, we use Theorem 4.3, then we could eliminate those redundant cubics, but still end up with a non-minimal set of generators. The last and probably the most important disadvantage of this method is that it does not give any structure to the equations obtained (as opposed to being the ideal of minors of certain matrices or box-shaped matrices, or being the ideal of an Eisenbud-Buchsbaum variety, in the known cases), so it becomes incredibly hard if one wishes to study higher order syzygies among these equations.

4.3. Asymptotic behaviour of the Rees algebras

If instead of a generic set of points in $\mathbb{P}^2$, we start with an arbitrary set of points $\mathbb{X}$, then the Hilbert-Burch matrix of its defining ideal $I_{\mathbb{X}} = \oplus_{t \geq \alpha} I_t$ no longer possesses a nice structure as it had in Section 4.1. It then becomes difficult to decide whether the Rees algebra $\mathcal{R}(I_t)$ is Cohen-Macaulay or to find its defining equations for a specific value of t. It is, however, possible to answer questions on the Cohen-Macaulayness or the degrees of the generators of $\mathcal{R}(I_t)$ as t gets large. In this section, we address these questions.

We begin by discussing a few properties of subschemes of a product scheme $\mathbb{P}^n \times \mathbb{P}^m$. For details on the definitions of the product scheme $\mathbb{P}^n \times \mathbb{P}^m$, sheaves associated to bi-graded modules, and sheaf cohomology groups on $\mathbb{P}^n \times \mathbb{P}^m$, we refer the reader to [**STV**], [**Hyry**] and [**Vid**]. Let $S = \mathfrak{k}[x_0, \ldots, x_n, y_0, \ldots, y_m]$ be a polynomial ring over an algebraically closed field $\mathfrak{k}$ of characteristic 0. Then $\mathbb{P}^n \times \mathbb{P}^m$, by definition, is the bi-Proj of S with the natural bigradation on S with respect to $(x_0, \ldots, x_n)$ and $(y_0, \ldots, y_m)$. Let $\mathfrak{m} = (x_i y_j)_{i,j} \subseteq S$ be the bihomogeneous irrelevant ideal of S. We first recall the following known fact (cf. [**Hyry**], [**Vid**]).

Proposition 4.10. *Suppose M is a bi-graded S-module, and $\mathcal{M}$ is the sheaf on $\mathbb{P}^n \times \mathbb{P}^m$ associated to M. Then, we have an exact sequence*

$$0 \to H^0_{\mathfrak{m}}(M) \to M \to \oplus_{a,b} H^0(\mathcal{M}(a, b)) \to H^1_{\mathfrak{m}}(M) \to 0,$$

and isomorphisms

$$\oplus_{a,b} H^i(\mathcal{M}(a, b)) \simeq H^{i+1}_{\mathfrak{m}}(M) \ \forall\ i > 0.$$

It is of interest to study the Cohen-Macaulayness of the bi-graded coordinate ring of a variety in the product space $\mathbb{P}^n \times \mathbb{P}^m$. This is equivalent to the variety being *arithmetically Cohen-Macaulay* (a.CM). The definition of a.CM is given as follows.

Definition. Suppose $V \subseteq \mathbb{P}^n \times \mathbb{P}^m$ is a subscheme defined by the bihomogeneous ideal $I \subseteq S$. We say V is *a.CM* if the bi-graded coordinate ring of V (i.e. S/I) is a Cohen-Macaulay ring, or equivalently, if I is a perfect ideal in S.

On a product space $\mathbb{P}^n \times \mathbb{P}^m$ (for any n and m), let

$$\pi_1 : \mathbb{P}^n \times \mathbb{P}^m \longrightarrow \mathbb{P}^n \text{ and } \pi_2 : \mathbb{P}^n \times \mathbb{P}^m \longrightarrow \mathbb{P}^m$$

be the two projection maps. If V is a subscheme of $\mathbb{P}^n \times \mathbb{P}^m$, when working on V, by abuse of notation, we also use π_1 and π_2 for the projection maps restricted to V. A necessary and a sufficient condition for a subscheme of $\mathbb{P}^n \times \mathbb{P}^m$ to be a.CM is given in the following theorem.

Theorem 4.11. *Suppose $V \subseteq \mathbb{P}^n \times \mathbb{P}^m$ is a proper closed subscheme of dimension d of $\mathbb{P}^n \times \mathbb{P}^m$. Then,*

(1) *If V is a.CM, then $H^i(\mathcal{I}_V(a,b)) = 0$ for all $a, b \in \mathbb{Z}$ and $1 \leq i \leq d$, where $\mathcal{I}_V$ is the ideal sheaf of V in $\mathbb{P}^n \times \mathbb{P}^m$.*
(2) *Suppose $d \neq n, m$, and $H^i(\mathcal{I}_V(a,b)) = 0$ for all $a, b \in \mathbb{Z}$ and $1 \leq i \leq d$. If in addition, $H^{d+1}(\mathcal{I}_V(a,b)) = 0$ for all $a, b \geq 0$, and for every $j > 0$,*

$$R^j\pi_{1*}(\mathcal{O}_V(p,q)) = 0 \ \forall p \in \mathbb{Z}, q \geq 0,$$

and

$$R^j\pi_{2*}(\mathcal{O}_V(p,q)) = 0 \ \forall q \in \mathbb{Z}, p \geq 0,$$

then V is a.CM.

Proof. Similar criteria for varieties with negative a^*-invariants were given in [**Hyry**, Theorem 2.5]. We adopt his argument with a slight modification to prove our result. Let $\mathfrak{n} = (x_0, \dots, x_n, y_0, \dots, y_m) \subseteq S$ be the maximal homogeneous ideal of S. Let

$\mathfrak{n}_1 = (x_0, \ldots, x_n)$ and $\mathfrak{n}_2 = (y_0, \ldots, y_m)$ be the ideals in S generated by the two sets of variables with respect to the standard bi-gradation of S. Then $\mathfrak{n}_1 + \mathfrak{n}_2 = \mathfrak{n}$ and $\mathfrak{n}_1 \cap \mathfrak{n}_2 = \mathfrak{m}$. Let I_V and $S_V = S/I_V$ be the defining ideal and the coordinate ring of V, respectively. Then, the Krull dimension of S_V is $d+2$. It is also not hard to see that the Cohen-Macaulayness of S_V is equivalent to the condition that $H^i_{\mathfrak{n}}(S_V) = 0$ for all $i = 1, \ldots, d+1$, which is the same as the condition that $H^i_{\mathfrak{n}}(I_V) = 0$ for all $i = 1, \ldots, d+2$.

(1) Suppose that V is a.CM. Equivalently, $H^i_{\mathfrak{n}}(I_V) = 0$ for all $i = 1, \ldots, d+2$. Consider the following Mayer-Vietoris sequence of local cohomology:

$$\ldots \to H^i_{\mathfrak{n}}(I_V) \to H^i_{\mathfrak{n}_1}(I_V) \oplus H^i_{\mathfrak{n}_2}(I_V) \to H^i_{\mathfrak{m}}(I_V) \to H^{i+1}_{\mathfrak{n}}(I_V) \to \ldots$$

The condition $H^i_{\mathfrak{n}}(I_V) = 0$ for all $i = 1, \ldots, d+2$, implies that the homomorphism

$$H^i_{\mathfrak{n}_1}(I_V) \oplus H^i_{\mathfrak{n}_2}(I_V) \to H^i_{\mathfrak{m}}(I_V)$$

is an isomorphism for $1 \leq i \leq d+1$ and injective for $i = d+2$. Localizing $H^i_{\mathfrak{n}}(I_V)$ at the maximal ideals of $\oplus_{t \in \mathbb{Z}} S_{(0,t)}$ and $\oplus_{t \in \mathbb{Z}} S_{(t,0)}$, respectively, and making use of [**Hyry**, Lemma 1.1 and Lemma 2.3], we have:

$$H^i_{\mathfrak{n}_1}(I_V) = H^i_{\mathfrak{n}_2}(I_V) = 0 \ \forall i = 1, \ldots, d+1.$$

This implies $H^i_{\mathfrak{m}}(I_V) = 0$ for all $i = 1, \ldots, d+1$. Together with Proposition 4.10, it then follows that $H^i(\mathcal{I}_V(a,b)) = 0$ for all $a, b \in \mathbb{Z}$ and $i = 1, \ldots, d$.

(2) Suppose now that $H^i(\mathcal{I}_V(a,b)) = 0$ for all $a, b \in \mathbb{Z}$ and $i = 1, \ldots, d$, and $H^{d+1}(\mathcal{I}_V(a,b)) = 0$ for all $a, b \geq 0$. We observe the following. For $i = 1$, this is to say that the homomorphism $S_{(a,b)} \to \Gamma(V, \mathcal{O}_V(a,b))$ is surjective. In other words, the homomorphism

$$S_{V(a,b)} \to \Gamma(V, \mathcal{O}_V(a,b))$$

is an isomorphism. Furthermore, the vanishing of $H^i(\mathcal{I}_V(a,b))$ for all $a, b \in \mathbb{Z}$, and all $i = 1, \ldots, d$, is the same as having $H^i_{\mathfrak{m}}(I_V) = 0$ for all $i = 2, \ldots, d+1$. This is equivalent to having $H^i_{\mathfrak{m}}(S_V) = 0$ for $i = 1, \ldots, d$. Since $\mathfrak{m} \not\subseteq I_V$, $H^0_{\mathfrak{m}}(S_V)$ is clearly also 0. Lastly, the vanishing of $H^{d+1}(\mathcal{I}_V(a,b))$ for all $a, b \geq 0$ implies that $H^d(\mathcal{O}_V(a,b)) = 0$ for all $a, b \geq 0$, i.e. $[H^{d+1}_{\mathfrak{m}}(S_V)]_{(a,b)} = 0$ for all $a, b \geq 0$.

Suppose, in addition, for every $j > 0$, $R^j\pi_{1*}(\mathcal{O}_V(p,q)) = 0\ \forall p \in \mathbb{Z}, q \geq 0$ and $R^j\pi_{2*}(\mathcal{O}_V(p,q)) = 0\ \forall q \in \mathbb{Z}, p \geq 0$. We need to show that V is a.CM.

Let $T = \oplus_{t\in\mathbb{Z}} S_{V(t,0)}$ and $W = \text{Proj } T$, then $\pi_1 : V \to W$ is the canonical projection. For all $p \in \mathbb{Z}$ and $q \geq 0$, the Leray spectral sequence

$$E_2^{i,j} = H^i(W, R^j\pi_{1*}(\mathcal{O}_V(p,q))) \Rightarrow H^{i+j}(V, \mathcal{O}_V(p,q))$$

degenerates. Thus, the edge homomorphisms $H^i(W, \pi_{1*}(\mathcal{O}_V(p,q))) \to H^i(V, \mathcal{O}_V(p,q))$ are just isomorphisms for all $p \in \mathbb{Z}, q \geq 0$ and $i \geq 0$.

Let $T_{<q>}$ $(q \geq 0)$ be the T-module given by $T_{<q>} = \oplus_{t\in\mathbb{Z}} S_{V(t,q)}$, and let $\mathcal{T}_{<q>}$ be the sheaf associated to $T_{<q>}$ on W. It is easy to see that for $p \gg 0$,

$$\Gamma(W, \mathcal{T}_{<q>}(p)) = S_{V(p,q)} = \Gamma(V, \mathcal{O}_V(p,q)) = \Gamma(W, \pi_{1*}(\mathcal{O}_V(p,q))).$$

Also, $\pi_{1*}(\mathcal{O}_V(p,q)) \otimes \mathcal{O}_W(p') = \pi_{1*}(\mathcal{O}_V(p+p',q))$. Thus, the canonical homomorphism $\mathcal{T}_{<q>}(p) \to \pi_{1*}(\mathcal{O}_V(p,q))$ is an isomorphism for all $p \in \mathbb{Z}$ and $q \geq 0$. It now follows that the homomorphisms

$$H^i(W, \mathcal{T}_{<q>}(p)) \to H^i(W, \pi_{1*}(\mathcal{O}_V(p,q))) \to H^i(V, \mathcal{O}_V(p,q)),$$

similar to what was mentioned in [**Hyry**, Theorem 1.4], are isomorphisms for all $p \in \mathbb{Z}$, $q \geq 0$ and $i \geq 0$. Applying the five lemma on the diagram of [**Hyry**, Theorem 1.4], it then implies that the homomorphism

$$[H^i_{\mathfrak{n}_1}(S_V)]_{(p,q)} \to [H^i_{\mathfrak{m}}(S_V)]_{(p,q)}$$

is an isomorphism for all $p \in \mathbb{Z}$, $q \geq 0$ and $i \geq 0$. By symmetry, the homomorphism

$$[H^i_{\mathfrak{n}_2}(S_V)]_{(p,q)} \to [H^i_{\mathfrak{m}}(S_V)]_{(p,q)}$$

is also an isomorphism for all $q \in \mathbb{Z}$, $p \geq 0$ and $i \geq 0$. Thus, for all $i = 1, \ldots, d$,

$$[H^i_{\mathfrak{n}_1}(S_V)]_{(p,q)} = 0\ \forall p \in \mathbb{Z}, q \geq 0 \ \text{ and } \ [H^i_{\mathfrak{n}_2}(S_V)]_{(p,q)} = 0\ \forall q \in \mathbb{Z}, p \geq 0,$$

and

$$[H^{d+1}_{\mathfrak{n}_1}(S_V)]_{(p,q)} = 0 = [H^{d+1}_{\mathfrak{n}_2}(S_V)]_{(p,q)} \ \forall p, q \geq 0.$$

Moreover, it is also easy to see that, for all $i > 0$,

$$[H^i_{\mathfrak{n}_1}(S_V)]_{(p,q)} = 0 \text{ if } q < 0 \text{ and } [H^i_{\mathfrak{n}_2}(S_V)]_{(p,q)} = 0 \text{ if } p < 0.$$

Therefore, in the following Mayer-Vietoris sequence of local cohomology

$$\ldots \to H^i_{\mathfrak{n}}(S_V) \to H^i_{\mathfrak{n}_1}(S_V) \oplus H^i_{\mathfrak{n}_2}(S_V) \to H^i_{\mathfrak{m}}(S_V) \to H^{i+1}_{\mathfrak{n}}(S_V) \to \ldots$$

the homomorphisms

$$H^i_{\mathfrak{n}_1}(S_V) \oplus H^i_{\mathfrak{n}_2}(S_V) \to H^i_{\mathfrak{m}}(S_V)$$

are isomorphisms for $i = 1, \ldots, d$, and injective for $i = d + 1$. We get $H^i_{\mathfrak{n}}(S_V) = 0$ for all $i = 1, \ldots, d + 1$. This is equivalent to S_V being Cohen-Macaulay, i.e. V being a.CM. The theorem is proved. □

Now, suppose $F \in S$ is a bihomogeneous polynomial. By abuse of notation, we denote by (F) both the ideal generated by F in S and the subscheme of $\mathbb{P}^n \times \mathbb{P}^m$ defined by the equation $F = 0$.

Proposition 4.12. *Suppose V is a closed irreducible subscheme of $\mathbb{P}^n \times \mathbb{P}^m$ of dimension at least 2, and L is a general linear form in the indeterminates $\{y_j | j = 0, \ldots, m\}$. Then V is a.CM if and only if $V \cap (L)$, considered as a subscheme of (L), is a.CM.*

Proof. Let I be the defining ideal of V. Then I is a bihomogeneous ideal in S, so in particular, I is a homogeneous ideal in S. Let W then be the subscheme of $\mathbb{P}^{n+m+1}$ defined by I. We observe that our discussion only involves the perfection of the defining ideal of V (and $V \cap (L)$) which is the same as the defining ideal of W (and $W \cap (L)$). Thus, the proposition would remain the same if instead of V (and $V \cap (L)$) we look at W (and $W \cap (L)$). The proposition now follows from [**Mig**, Theorem 1.3.2]. □

From here onwards, we focus our attention back to the study of the asymptotic behaviour of the Rees algebras $\mathcal{R}(I_t)$ for the defining ideal of an arbitrary set of points in $\mathbb{P}^2$. Again, suppose $\mathbb{X} = \{P_1, \ldots, P_s\}$ is a set of s distinct arbitrary points in $\mathbb{P}^2$. Let $I_{\mathbb{X}} = \wp_1 \cap \ldots \cap \wp_s \subseteq R = \mathfrak{k}[w_1, w_2, w_3]$ be the defining ideal of $\mathbb{X}$, and suppose $I_{\mathbb{X}} = \oplus_{t \geq \alpha} I_t$ is its homogeneous decomposition.

Theorem 4.13. *Suppose $\mathbb{X} = \{P_1, \ldots, P_s\}$ is an arbitrary set of s points in $\mathbb{P}^2$, and $I_{\mathbb{X}} = \oplus_{t \geq \alpha} I_t \subseteq R = \mathfrak{k}[w_1, w_2, w_3]$ is its defining ideal. Then, there exists an integer d_0 such that for all $t \geq d_0$, the Rees algebra $\mathcal{R}(I_t)$ of the ideal generated by I_t is Cohen-Macaulay, and its defining ideal is generated by quadratics.*

Proof. Let $\sigma = \sigma(I_{\mathbb{X}})$ be the least integer at which the difference function of the Hilbert function of $\mathbb{X}$ equals 0, and take $d_0 = \max\{4, \sigma + 1, s + 1\}$ (note that σ is bounded by s, so we in fact only need to take $d_0 = \max\{4, s + 1\}$). We shall prove that this value of d_0 satisfies the requirements of the theorem.

Suppose t is an arbitrary integer which is bigger than or equal to d_0. We add $\binom{t}{2} - s$ general smooth points to $\mathbb{X}$ to obtain a set of points $\tilde{\mathbb{X}}$ with generic Hilbert function up to degree $t - 2$, i.e.

$$H_{\tilde{\mathbb{X}}} : 1\ 3\ 6\ \ldots\ \binom{t}{2}\ \binom{t}{2}\ \ldots.$$

We start by showing that the Rees algebra $\mathcal{R}(I_t)$ is Cohen-Macaulay using induction on the number $l = \binom{t}{2} - s$ of points we add to $\mathbb{X}$ to get $\tilde{\mathbb{X}}$.

If $l = 0$, then $\mathbb{X}$ is a set of $\binom{t}{2}$ points in $\mathbb{P}^2$ with the generic Hilbert function. It follows from [**G-M**] that the Hilbert-Burch matrix of $I_{\mathbb{X}}$ has linear entries, $I_{\mathbb{X}}$ is generated in degree $t - 1$, and $\sigma(I_{\mathbb{X}}) = t - 1$. It now follows from Theorem 4.3 that the Rees algebra $\mathcal{R}(I_t)$ is Cohen-Macaulay. The assertion that the Rees algebra $\mathcal{R}(I_t)$ is Cohen-Macaulay is true for the base case.

Suppose now that our assertion is true for a set of points $\mathbb{X}' = \mathbb{X} \cup \{P_{s+1}\}$, and we need to prove the assertion for the set of points $\mathbb{X}$. Let $I_{\mathbb{X}} = \oplus_{t \geq \alpha} I_t$ and $I'_{\mathbb{X}} = \oplus_{t \geq \alpha'} I'_t$ be the defining ideals of $\mathbb{X}$ and $\mathbb{X}'$ respectively. Since a general point P_{s+1} imposes one independent condition at degree t, a system of generators for I_t and for I'_t may be given by $\{F_0, \ldots, F_{r-1}, F_r\}$ and $\{F_0, \ldots, F_{r-1}\}$, respectively. Consider the following rational maps

$$\varphi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^r \text{ and } \varphi' : \mathbb{P}^2 \dashrightarrow \mathbb{P}^{r-1}$$

given by $\varphi(P) = [F_0(P) : \ldots : F_r(P)]$ and $\varphi'(P) = [F_0(P) : \ldots : F_{r-1}(P)]$. Let V and V' be the closure of the graphs of these maps in $\mathbb{P}^2 \times \mathbb{P}^r$ and $\mathbb{P}^2 \times \mathbb{P}^{r-1}$, respectively. Clearly, the Rees algebras $\mathcal{R}(I_t)$ and $\mathcal{R}(I'_t)$ are the bi-graded coordinate rings of V and V', respectively. By induction hypothesis, we know that V' is a.CM. We need to show that V is also a.CM.

Let $[y_0 : \ldots : y_r]$ represent the homogeneous coordinates of $\mathbb{P}^r$. Let $S = \mathfrak{k}[w_1, w_2, w_3, y_0, \ldots, y_r]$ and S_V be the bi-graded coordinate rings of $\mathbb{P}^2 \times \mathbb{P}^r$ and V, respectively. Let $H = V \cap (y_r)$. By Proposition 4.12, we only need to show that H, considered as a subscheme of $\mathbb{P}^2 \times \mathbb{P}^{r-1}$, is a.CM.

Clearly, $\pi_1(V) = \pi_1(V') = \mathbb{P}^2$. Let $\overline{V} = \pi_2(V)$ and $\overline{V'} = \pi_2(V')$. By the construction of $\tilde{\mathbb{X}}$, we know that $\sigma(I_{\mathbb{X}}) \leq \sigma(I_{\tilde{\mathbb{X}}}) = t-1$ and $\sigma(I_{\mathbb{X}'}) \leq \sigma(I_{\tilde{\mathbb{X}}}) = t-1$. Therefore, the divisors $|I_t|$ and $|I'_t|$ are very ample on V and V', respectively (Theorem 0.6). Thus, π_2 is one-to-one and onto from V to $\overline{V}$ and from V' to $\overline{V'}$ (Proposition 2.6). It is easy to see that since the coordinate y_r of $\mathbb{P}^r$ is chosen generally, H meets each exceptional curve of V no more than once. Thus, $\pi_1^{-1}(P)$, restricted to H, is at most one point, for every $P \in \mathbb{P}^2$. Since π_2 is one-to-one on V, it is also one-to-one on H, whence $\pi_2^{-1}(Q)$, restricted to H, is also at most one point, for every $Q \in \mathbb{P}^{r-1}$. Therefore, by [**Hart**, Corollary III.11.2], for every $j > 0$, one gets

$$R^j \pi_{1*}(\mathcal{O}_H(p,q)) = 0 \text{ and } R^j \pi_{2*}(\mathcal{O}_H(p,q)) = 0 \; \forall p, q \in \mathbb{Z}.$$

By Theorem 4.11, to show that H is a.CM, it is now enough to show $H^1(\mathcal{I}_H(a,b)) = 0$ for all $a, b \in \mathbb{Z}$, and $H^2(\mathcal{I}_H(a,b)) = 0$ for all $a, b \geq 0$, where $\mathcal{I}_H$ is the ideal sheaf of H in $\mathbb{P}^2 \times \mathbb{P}^{r-1}$.

It is easy to see that since V' is the projection of V on (y_r) centered at the point $P_{s+1} \times \varphi(P_{s+1})$, $H \subseteq V'$. Consider the following exact sequence

$$0 \to \mathcal{I}_{V'} \to \mathcal{I}_H \to \mathcal{I}_{H,V'} \to 0,$$

where $\mathcal{I}_{H,V'}$ is the ideal sheaf of H considered as a subscheme of V'. Since V' is a.CM, taking the cohomology groups, we get

$$H^1(\mathcal{I}_H(a,b)) = H^1(\mathcal{I}_{H,V'}(a,b)) \text{ and } H^2(\mathcal{I}_H(a,b)) \hookrightarrow H^2(\mathcal{I}_{H,V'}(a,b)).$$

Thus, it suffices to show that

$$H^1(\mathcal{I}_{H,V'}(a,b)) = 0 \ \forall a,b \in \mathbb{Z} \text{ and } H^2(\mathcal{I}_{H,V'}(a,b)) = 0 \ \forall a,b \geq 0.$$

Let E_0 be the pull back to V' of the class of a general line in $\mathbb{P}^2$, and let $E_1, \ldots, E_{s+1}$ be the classes of the exceptional divisors corresponding to the blowup at the points $P_1, \ldots, P_{s+1}$, respectively. Let $\overline{E_0}, \ldots, \overline{E_{s+1}}$ be the images of $E_0, \ldots, E_{s+1}$ through π_2, respectively, then they generate the Picard group of $\overline{V'}$. $\overline{V} \cap (y_r)$ is a hyperplane section of $\overline{V}$ and since the coordinates in $\mathbb{P}^r$ are chosen generally, we may assume that $\overline{V} \cap (y_r)$ belongs to the divisor class $|t\overline{E_0} - \overline{E_1} - \ldots - \overline{E_s}|$. Thus, $H \in |tE_0 - E_1 - \ldots - E_s|$. We have

$$\begin{aligned}
\mathcal{I}_{H,V'}(a,b) &= \mathcal{O}_{V'}(-H)(a,b) \\
&= \mathcal{O}_{V'}(-H) \otimes \pi_1^*(\mathcal{O}_{\mathbb{P}^2}(a)) \otimes \pi_2^*(\mathcal{O}_{\overline{V'}}(b)) \\
&= \mathcal{O}_{V'}(-H) \otimes \pi_1^*(\mathcal{O}_{\mathbb{P}^2}(a)) \otimes \pi_2^*(\mathcal{O}_{\overline{V'}}(bt\overline{E_0} - b\overline{E_1} - \ldots - b\overline{E_s} - b\overline{E_{s+1}})) \\
&= \mathcal{O}_{V'}(-H) \otimes \mathcal{O}_{V'}(aE_0) \otimes \mathcal{O}_{V'}(btE_0 - bE_1 - \ldots - bE_s - bE_{s+1}) \\
&= \mathcal{O}_{V'}((a+(b-1)t)E_0 - (b-1)E_1 - \ldots - (b-1)E_s - bE_{s+1})
\end{aligned}$$

Let $D_{a,b} = (a+(b-1)t)E_0 - (b-1)E_1 - \ldots - (b-1)E_s - bE_{s+1}$ on V'. We first prove $H^2(\mathcal{O}_{V'}(D_{a,b})) = 0$ for all $a, b \geq 0$. We shall use double induction on b and a.

For $b = 0$, we first have $D_{0,0} = -tE_0 + E_1 + \ldots + E_s$. The canonical divisor on V' is $K_{V'} = -3E_0 + E_1 + \ldots + E_{s+1}$. Let $H' = tE_0 - E_1 - \ldots - E_{s+1}$. From Theorem 0.6, H' is very ample on V'. We also have, $K_{V'}.D_{0,0} = 3t - s > -3t + s + 1 = K_{V'}.H'$. Thus,

$H^2(\mathcal{O}_{V'}(D_{0,0})) = 0$ (cf. [**Hart**, Lemma V.1.7]). Suppose now that $H^2(\mathcal{O}_{V'}(D_{a,0})) = 0$ is true for $a \geq 0$, we shall show that $H^2(\mathcal{O}_{V'}(D_{a+1,0})) = 0$. Indeed, consider the exact sequence

$$0 \to \mathcal{O}_{V'}(D_{a,0}) \to \mathcal{O}_{V'}(D_{a+1,0}) \to \mathcal{O}_{E_0}(D_{a+1,0}) \to 0.$$

Since $\deg \mathcal{O}_{E_0}(D_{a+1,0}) = a + 1 > 0$, and since E_0 is a rational curve, we have $H^1(\mathcal{O}_{E_0}(D_{a+1,0})) = 0$, and $H^2(\mathcal{O}_{E_0}(D_{a+1,0})) = 0$. Thus,

$$H^2(\mathcal{O}_{V'}(D_{a+1,0})) = H^2(\mathcal{O}_{V'}(D_{a,0})) = 0.$$

For $b = 1$, we first have $D_{0,1} = -E_{s+1}$. We also have $K_{V'}.D_{0,1} = 1 > -3t + s + 1 = K_{V'}.H'$. Thus, $H^2(\mathcal{O}_{V'}(D_{0,1})) = 0$ (cf. [**Hart**, Lemma V.1.7]). Suppose now that $H^2(\mathcal{O}_{V'}(D_{a,1})) = 0$ is true for $a \geq 0$, we shall show that $H^2(\mathcal{O}_{V'}(D_{a+1,1})) = 0$. Indeed, as before, consider the exact sequence

$$0 \to \mathcal{O}_{V'}(D_{a,1}) \to \mathcal{O}_{V'}(D_{a+1,1}) \to \mathcal{O}_{E_0}(D_{a+1,1}) \to 0.$$

Since $\deg \mathcal{O}_{E_0}(D_{a+1,1}) = a + 1 > 0$, and since E_0 is a rational curve, we have $H^1(\mathcal{O}_{E_0}(D_{a+1,1})) = H^2(\mathcal{O}_{E_0}(D_{a+1,1})) = 0$. Thus,

$$H^2(\mathcal{O}_{V'}(D_{a+1,1})) = H^2(\mathcal{O}_{V'}(D_{a,1})) = 0.$$

Now, suppose that $H^2(\mathcal{O}_{V'}(D_{a,b})) = 0$ is true for any $a \geq 0$ and some $b \geq 1$, we shall show that $H^2(\mathcal{O}_{V'}(D_{a,b+1})) = 0$. Indeed, consider the exact sequence

$$0 \to \mathcal{O}_{V'}(D_{a,b}) \to \mathcal{O}_{V'}(D_{a,b+1}) \to \mathcal{O}_{H'}(D_{a,b+1}) \to 0.$$

Since $\deg \mathcal{O}_{H'}(D_{a,b+1}) = (a + bt)t - b(s + 1) - 1 > 2\binom{t-1}{2} + 1 = 2g(H') + 1$, we have $H^1(\mathcal{O}_{H'}(D_{a,b+1})) = H^2(\mathcal{O}_{H'}(D_{a,b+1})) = 0$. Thus,

$$H^2(\mathcal{O}_{V'}(D_{a,b+1})) = H^2(\mathcal{O}_{V'}(D_{a,b})) = 0.$$

Hence, $H^2(\mathcal{O}_{V'}(D_{a,b})) = 0$ for all $a, b \geq 0$.

It remains now to prove that $H^1(\mathcal{O}_{V'}(D_{a,b})) = 0$ for all $a, b \in \mathbb{Z}$. Similar to what we did above, we can rewrite

$$\mathcal{O}_{V'}(D_{a,b}) = \mathcal{O}_{V'}(-E_{s+1})(a, b-1) = \mathcal{I}_{E_{s+1},V'}(a, b-1),$$

where $\mathcal{I}_{E_{s+1},V'}$ is the ideal sheaf of E_{s+1} considered as a subscheme of V'. It was proved in [**Gi-2**, Proposition 2.1] that

$$H^1(\mathcal{O}_{\overline{V'}}((n-1)t\overline{E_0} - (n-1)\overline{E_1} - \ldots - (n-1)\overline{E_s} - n\overline{E_{s+1}}) = 0 \ \forall n \in \mathbb{Z}$$

That is, $H^1(\mathcal{I}_{\overline{E_{s+1}},\overline{V'}}(n-1)) = 0$ for all $n \in \mathbb{Z}$, where $\mathcal{I}_{\overline{E_{s+1}},\overline{V'}}$ is the ideal sheaf of $\overline{E_{s+1}}$ considered as a subscheme of $\overline{V'}$. Therefore, $\overline{E_{s+1}}$, considered as a subscheme of $\overline{V'}$, is projectively CM (see [**Gi-2**] for definition of projectively CM). By considering the exact sequence

$$0 \to \mathcal{I}_{\overline{V'}} \to \mathcal{I}_{\overline{E_{s+1}}} \to \mathcal{I}_{\overline{E_{s+1}},\overline{V'}} \to 0,$$

where $\mathcal{I}_{\overline{V'}}$ and $\mathcal{I}_{\overline{E_{s+1}}}$ are the ideal sheaves of $\overline{V'}$ and $\overline{E_{s+1}}$ in $\mathbb{P}^{r-1}$, and from the fact that $\overline{V'}$ is projectively CM ([**Gi-2**]), we deduce that in $\mathbb{P}^{r-1}$, the homogeneous coordinate ring of $\overline{E_{s+1}}$ is CM. It is also clear that $E_{s+1} = P_{s+1} \times \overline{E_{s+1}}$. Thus, the coordinate ring of E_{s+1} in $\mathbb{P}^2 \times \mathbb{P}^{r-1}$ is CM, i.e. E_{s+1} is a.CM in $\mathbb{P}^2 \times \mathbb{P}^{r-1}$. Hence, by Theorem 4.11, $H^1(\mathcal{I}_{E_{s+1}}(a, b-1)) = 0$ for all $a, b \in \mathbb{Z}$, where $\mathcal{I}_{E_{s+1}}$ is the ideal sheaf of E_{s+1} in $\mathbb{P}^2 \times \mathbb{P}^{r-1}$. By considering the exact sequence

$$0 \to \mathcal{I}_{V'} \to \mathcal{I}_{E_{s+1}} \to \mathcal{I}_{E_{s+1},V'} \to 0,$$

and from the hypothesis that V' is a.CM, we conclude that $H^1(\mathcal{I}_{E_{s+1},V'}(a, b-1)) = 0$ for all $a, b \in \mathbb{Z}$. Hence,

$$H^1(\mathcal{O}_{V'}(D_{a,b})) = 0 \text{ for all } a, b \in \mathbb{Z}.$$

The assertion is proved, i.e. $\mathcal{R}(I_t)$ is Cohen-Macaulay for all $t \geq d_0$.

We now proceed to prove that the defining ideal of $\mathcal{R}(I_t)$ is generated by quadratics. Again, we use induction on l, the number of general smooth points being added to $\mathbb{X}$ to get $\tilde{\mathbb{X}}$. If $l = 0$, then it follows from Theorem 4.3 that the defining ideal of $\mathcal{R}(I_t)$ is generated by quadratics. Thus, the base case of the assertion that the defining ideal of the Rees algebra $\mathcal{R}(I_t)$ is generated by quadratics is proved.

Suppose now that this assertion is true for a set of points $\mathbb{X}' = \mathbb{X} \cup \{P_{s+1}\}$, and we need to show it for the set of points $\mathbb{X}$. Let V, V' and H be defined as before. By induction hypothesis, V' is generated by quadratics. We need to show that V is also generated by quadratics. Since V is a.CM, this amounts to showing that H is generated by quadratics. Moreover, since $H \subseteq V'$ and V' is generated by quadratics, we only need to show that $I_{H,V'}$ (the ideal of H in the coordinate ring of V') is generated in its quadratic degree. It suffices to show that the following multiplication maps are onto:

$$H^0(\mathcal{I}_{H,V'}(2,0)) \otimes H^0(\mathcal{O}_{V'}(1,0)) \twoheadrightarrow H^0(\mathcal{I}_{H,V'}(3,0)) \tag{4.3.1}$$

$$H^0(\mathcal{I}_{H,V'}(0,2)) \otimes H^0(\mathcal{O}_{V'}(0,1)) \twoheadrightarrow H^0(\mathcal{I}_{H,V'}(0,3)) \tag{4.3.2}$$

$$H^0(\mathcal{I}_{H,V'}(1,1)) \otimes H^0(\mathcal{O}_{V'}(1,0)) \twoheadrightarrow H^0(\mathcal{I}_{H,V'}(2,1)) \tag{4.3.3}$$

$$H^0(\mathcal{I}_{H,V'}(0,2)) \otimes H^0(\mathcal{O}_{V'}(1,0)) \twoheadrightarrow H^0(\mathcal{I}_{H,V'}(1,2)), \tag{4.3.4}$$

where $\mathcal{I}_{H,V'}$ is the ideal sheaf of H in V'.

To prove (4.3.1), for each integer a, as it was done above, we rewrite

$$\mathcal{I}_{H,V'}(a,0) = \mathcal{O}_{V'}(-H)(a,0) = \mathcal{O}_{V'}((a-t)E_0 + E_1 + \ldots + E_s).$$

Again, the canonical divisor on V' is $K_{V'} = -3E_0 + E_1 + \ldots + E_s + E_{s+1}$. Let

$$D_a = (a-t)E_0 + E_1 + \ldots + E_s \;\text{ and }\; H' = tE_0 - E_1 - \ldots - E_s - E_{s+1}.$$

As before, from Theorem 0.6, H' is every ample on V'. For $a = 2, 3$, we also have

$$(K_{V'} - D_a).H' = (t-a-3)t + 1 > -3t + s + 1 = K_{V'}.H'.$$

Therefore, it follows from [**Hart**, Lemma V.1.7] that, for $a = 2$ and 3,

$$H^0(\mathcal{O}_{V'}(D_a)) = H^2(\mathcal{O}_{V'}(K_{V'} - D_a)) = 0.$$

Hence, (4.3.1) follows vacuously.

To prove (4.3.2), for each integer b, as before, we rewrite

$$\begin{aligned}\mathcal{I}_{H,V'}(0,b) &= \mathcal{O}_{V'}((b-1)tE_0-(b-1)E_1-\ldots-(b-1)E_s-bE_{s+1})\\ &= \mathcal{I}_{E_{s+1},V'}(0,b-1).\end{aligned}$$

Thus, (4.3.2) is equivalent to

$$H^0(\mathcal{I}_{E_{s+1},V'}(0,1))\otimes H^0(\mathcal{O}_{V'}(0,1)) \twoheadrightarrow H^0(\mathcal{I}_{E_{s+1},V'}(0,2)).$$

This is indeed true, since E_{s+1} is a line in V', and hence, is obviously generated in degree 1.

Similarly, it can be shown that (4.3.3) and (4.3.4) are equivalent to

$$H^0(\mathcal{I}_{E_{s+1},V'}(1,0))\otimes H^0(\mathcal{O}_{V'}(1,0)) \twoheadrightarrow H^0(\mathcal{I}_{E_{s+1},V'}(2,0)),$$

and

$$H^0(\mathcal{I}_{E_{s+1},V'}(0,1))\otimes H^0(\mathcal{O}_{V'}(1,0)) \twoheadrightarrow H^0(\mathcal{I}_{E_{s+1},V'}(1,1)),$$

respectively. Those are again true since E_{s+1} is a line on V', and so, generated in degree 1. The theorem is proved. □

Remark: Most of our arguments do not use the fact that $\mathbb{X}$ is a set of points. Thus, if we couple our arguments together with Theorem 0.11, the Cohen-Macaulayness of Theorem 4.13 is actually true for any subscheme of $\mathbb{P}^2$ whose defining ideal satisfies condition G_3 (see page 11). An example of such a subscheme is any locally complete intersection subscheme of $\mathbb{P}^2$.

Inspired by the notion of having property N_p, introduced by Green (see [**G-L-1**], [**G-L-2**]), together with many pieces supportive evidences, we conjecture the following.

Conjecture: Suppose $\mathbb{X}$ is an arbitrary set of points in $\mathbb{P}^2$, and $I_{\mathbb{X}} = \oplus_{t\geq\alpha} I_t \subseteq R = \mathfrak{k}[w_1,w_2,w_3]$ is its defining ideal. Then, for any non-negative integer p, there exists an integer d_p such that for any $t \geq d_p$, the Rees algebra $\mathcal{R}(I_t)$ is Cohen-Macaulay, defined by quadratic equations, and the first p steps in its minimal free resolution are linear, i.e. the first p matrices are of linear forms.

Theorem 4.13 proves this conjecture for $p = 0$. We conclude this chapter and the thesis by raising the following question.

Question: For which classes of homogeneous ideals $I = \oplus_{t \geq \alpha} I_t$ of a polynomial ring can one study the asymptotic behaviour of the Rees algebras $\mathcal{R}(I_t)$ as t gets large, and obtain a similar result to that of Theorem 4.13?

Appendix

This appendix is in support to the Example in Section 4.2. The following are the equations obtained from that of $\mathcal{R}(I_4)$ using Lemma 4.9 (these equations are found with the help of the CoCoA package).

$2z_{13}z_{15} + 10z_{15}^2 - 4z_{11}z_{16} + 4z_{13}z_{16} + z_{15}z_{16} + 8z_{11}z_{17} - 8z_{13}z_{17} - 14z_{15}z_{17}$

$2z_{13}z_{25} + 10z_{15}z_{25} - 4z_{11}z_{26} + 4z_{13}z_{26} + z_{15}z_{26} + 8z_{11}z_{27} - 8z_{13}z_{27} - 14z_{15}z_{27}$

$2z_{13}z_{35} + 10z_{15}z_{35} - 4z_{11}z_{36} + 4z_{13}z_{36} + z_{15}z_{36} + 8z_{11}z_{37} - 8z_{13}z_{37} - 14z_{15}z_{37}$

$2z_{23}z_{25} + 10z_{25}^2 - 4z_{21}z_{26} + 4z_{23}z_{26} + z_{25}z_{26} + 8z_{21}z_{27} - 8z_{23}z_{27} - 14z_{25}z_{27}$

$2z_{23}z_{35} + 10z_{25}z_{35} - 4z_{21}z_{36} + 4z_{23}z_{36} + z_{25}z_{36} + 8z_{21}z_{37} - 8z_{23}z_{37} - 14z_{25}z_{37}$

$2z_{33}z_{35} + 10z_{35}^2 - 4z_{31}z_{36} + 4z_{33}z_{36} + z_{35}z_{36} + 8z_{31}z_{37} - 8z_{33}z_{37} - 14z_{35}z_{37}$

$2z_{14}z_{15} - 4z_{12}z_{16} + 4z_{14}z_{16} + 10z_{15}z_{16} + z_{16}^2 + 8z_{12}z_{17} - 8z_{14}z_{17} - 14z_{16}z_{17}$

$2z_{14}z_{25} - 4z_{12}z_{26} + 4z_{14}z_{26} + 10z_{15}z_{26} + z_{16}z_{26} + 8z_{12}z_{27} - 8z_{14}z_{27} - 14z_{16}z_{27}$

$2z_{14}z_{35} - 4z_{12}z_{36} + 4z_{14}z_{36} + 10z_{15}z_{36} + z_{16}z_{36} + 8z_{12}z_{37} - 8z_{14}z_{37} - 14z_{16}z_{37}$

$2z_{24}z_{25} - 4z_{22}z_{26} + 4z_{24}z_{26} + 10z_{25}z_{26} + z_{26}^2 + 8z_{22}z_{27} - 8z_{24}z_{27} - 14z_{26}z_{27}$

$2z_{24}z_{35} - 4z_{22}z_{36} + 4z_{24}z_{36} + 10z_{25}z_{36} + z_{26}z_{36} + 8z_{22}z_{37} - 8z_{24}z_{37} - 14z_{26}z_{37}$

$2z_{34}z_{35} - 4z_{32}z_{36} + 4z_{34}z_{36} + 10z_{35}z_{36} + z_{36}^2 + 8z_{32}z_{37} - 8z_{34}z_{37} - 14z_{36}z_{37}$

$-2z_{13}^2 + 2z_{11}z_{14} + 2z_{13}z_{14} + 6z_{11}z_{15} - 10z_{15}^2 + 2z_{11}z_{16} - 5z_{13}z_{16} - 5/2z_{15}z_{16} - 12z_{11}z_{17} + 2z_{13}z_{17} + 17z_{15}z_{17}$

$-2z_{13}z_{23} + 2z_{11}z_{24} + 2z_{13}z_{24} + 6z_{11}z_{25} - 10z_{15}z_{25} + 2z_{11}z_{26} - 5z_{13}z_{26} - 5/2z_{15}z_{26} - 12z_{11}z_{27} + 2z_{13}z_{27} + 17z_{15}z_{27}$

$-2z_{13}z_{33} + 2z_{11}z_{34} + 2z_{13}z_{34} + 6z_{11}z_{35} - 10z_{15}z_{35} + 2z_{11}z_{36} - 5z_{13}z_{36} - 5/2z_{15}z_{36} - 12z_{11}z_{37} + 2z_{13}z_{37} + 17z_{15}z_{37}$

$-2z_{23}^2 + 2z_{21}z_{24} + 2z_{23}z_{24} + 6z_{21}z_{25} - 10z_{25}^2 + 2z_{21}z_{26} - 5z_{23}z_{26} - 5/2z_{25}z_{26} - 12z_{21}z_{27} +$

$2z_{23}z_{27} + 17z_{25}z_{27}$

$-\ 2z_{23}z_{33} + 2z_{21}z_{34} + 2z_{23}z_{34} + 6z_{21}z_{35} - 10z_{25}z_{35} + 2z_{21}z_{36} - 5z_{23}z_{36} - 5/2z_{25}z_{36} - 12z_{21}z_{37} + 2z_{23}z_{37} + 17z_{25}z_{37}$

$-2z_{33}^2 + 2z_{31}z_{34} + 2z_{33}z_{34} + 6z_{31}z_{35} - 10z_{35}^2 + 2z_{31}z_{36} - 5z_{33}z_{36} - 5/2z_{35}z_{36} - 12z_{31}z_{37} + 2z_{33}z_{37} + 17z_{35}z_{37}$

$2z_{12}z_{14} - 2z_{13}z_{14} + 2z_{14}^2 + 6z_{12}z_{15} + 2z_{12}z_{16} - 5z_{14}z_{16} - 10z_{15}z_{16} - 5/2z_{16}^2 - 12z_{12}z_{17} + 2z_{14}z_{17} + 17z_{16}z_{17}$

$2z_{12}z_{24} - 2z_{13}z_{24} + 2z_{14}z_{24} + 6z_{12}z_{25} + 2z_{12}z_{26} - 5z_{14}z_{26} - 10z_{15}z_{26} - 5/2z_{16}z_{26} - 12z_{12}z_{27} + 2z_{14}z_{27} + 17z_{16}z_{27}$

$2z_{12}z_{34} - 2z_{13}z_{34} + 2z_{14}z_{34} + 6z_{12}z_{35} + 2z_{12}z_{36} - 5z_{14}z_{36} - 10z_{15}z_{36} - 5/2z_{16}z_{36} - 12z_{12}z_{37} + 2z_{14}z_{37} + 17z_{16}z_{37}$

$2z_{22}z_{24} - 2z_{23}z_{24} + 2z_{24}^2 + 6z_{22}z_{25} + 2z_{22}z_{26} - 5z_{24}z_{26} - 10z_{25}z_{26} - 5/2z_{26}^2 - 12z_{22}z_{27} + 2z_{24}z_{27} + 17z_{26}z_{27}$

$2z_{22}z_{34} - 2z_{23}z_{34} + 2z_{24}z_{34} + 6z_{22}z_{35} + 2z_{22}z_{36} - 5z_{24}z_{36} - 10z_{25}z_{36} - 5/2z_{26}z_{36} - 12z_{22}z_{37} + 2z_{24}z_{37} + 17z_{26}z_{37}$

$2z_{32}z_{34} - 2z_{33}z_{34} + 2z_{34}^2 + 6z_{32}z_{35} + 2z_{32}z_{36} - 5z_{34}z_{36} - 10z_{35}z_{36} - 5/2z_{36}^2 - 12z_{32}z_{37} + 2z_{34}z_{37} + 17z_{36}z_{37}$

$-\ z_{12}z_{13} + z_{11}z_{14},\ -z_{12}z_{23} + z_{11}z_{24},\ -z_{12}z_{33} + z_{11}z_{34}$

$-\ z_{22}z_{23} + z_{21}z_{24},\ -z_{22}z_{33} + z_{21}z_{34},\ -z_{32}z_{33} + z_{31}z_{34}$

$-\ z_{12}z_{15} + z_{11}z_{16},\ -z_{12}z_{25} + z_{11}z_{26},\ -z_{12}z_{35} + z_{11}z_{36}$

$-\ z_{22}z_{25} + z_{21}z_{26},\ -z_{22}z_{35} + z_{21}z_{36},\ -z_{32}z_{35} + z_{31}z_{36}$

$-\ z_{14}z_{15} + z_{13}z_{16},\ -z_{14}z_{25} + z_{13}z_{26},\ -z_{14}z_{35} + z_{13}z_{36}$

$-\ z_{24}z_{25} + z_{23}z_{26},\ -z_{24}z_{35} + z_{23}z_{36},\ -z_{34}z_{35} + z_{33}z_{36}$

$-\ 4z_{16} + 8z_{17} + 2z_{25} + 4z_{26} - 8z_{27} + 10z_{35} + z_{36} - 14z_{37}$

$2z_{14} + 6z_{15} + 2z_{16} - 12z_{17} - 2z_{23} + 2z_{24} - 5z_{26} + 2z_{27} - 10z_{35} - 5/2z_{36} + 17z_{37}$

$z_{13} - z_{21},\ z_{14} - z_{22},\ z_{15} - z_{31}$

$z_{16} - z_{32},\ z_{25} - z_{33},\ z_{26} - z_{34}.$

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