

THE REGULARITY OF POINTS IN MULTI-PROJECTIVE SPACES

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ABSTRACT. Let $I = \wp_1^{m_1} \cap \dots \cap \wp_s^{m_s}$ be the defining ideal of a scheme of fat points in $\mathbb{P}^{n_1} \times \dots \times \mathbb{P}^{n_k}$ with support in generic position. When all the m_i 's are 1, we explicitly calculate the Castelnuovo-Mumford regularity of I . In general, if at least one $m_i \geq 2$, we give an upper bound for the regularity of I , which extends a result of Catalisano, Trung and Valla.

INTRODUCTION

In this paper, we study the Castelnuovo-Mumford regularity of defining ideals of sets of points (reduced and non-reduced) in a multi-projective space $\mathbb{P}^{n_1} \times \dots \times \mathbb{P}^{n_k}$.

If $I \subseteq \mathbf{k}[x_0, \dots, x_n]$ is the defining ideal of a projective variety $X \subseteq \mathbb{P}^n$, then the Castelnuovo-Mumford regularity of I , denoted by $\text{reg}(I)$, is a very important invariant associated to X . It has been the objective of many authors to estimate $\text{reg}(I)$ since not only does it bound the degrees of a minimal set of defining equations for X , it also gives a uniform bound on the degrees of syzygies of I . The most fundamental situation is when X is a set of points. Examples of work on $\text{reg}(I)$ in this case can be seen in [5, 7, 8, 15]. Recently, many authors (cf. [4, 9, 10, 11, 16]) have been interested in extending our understanding of points in \mathbb{P}^n to sets of points in $\mathbb{P}^{n_1} \times \dots \times \mathbb{P}^{n_k}$. We continue this trend by studying $\text{reg}(I)$ when I defines a scheme of fat points in $\mathbb{P}^{n_1} \times \dots \times \mathbb{P}^{n_k}$.

In the context of \mathbb{N}^2 -graded rings, Aramova, Crona and De Negri [1] have introduced a finer notion of regularity that places bounds on each coordinate of the degree of a multi-graded syzygy. Extending the definition of regularity to multi-graded rings is also considered recently in [12, 13]. The usual notion of regularity could be treated as a bound on the total degree of the multi-graded syzygies. Our results, thus, naturally provide corresponding results for the new notion of regularity in multi-graded rings.

The \mathbb{N}^k -graded ring $R = \mathbf{k}[x_{1,0}, \dots, x_{1,n_1}, \dots, x_{k,0}, \dots, x_{k,n_k}]$ where $\deg x_{i,j} = e_i$, the i^{th} basis vector of \mathbb{N}^k , is the associated coordinate ring of $\mathbb{P}^{n_1} \times \dots \times \mathbb{P}^{n_k}$. Let $\mathbb{X} = \{P_1, \dots, P_s\}$ be a set of distinct points in $\mathbb{P}^{n_1} \times \dots \times \mathbb{P}^{n_k}$. The defining ideal of P_i is $\wp_i = (L_{1,1}, \dots, L_{1,n_1}, \dots, L_{k,1}, \dots, L_{k,n_k})$ with $\deg L_{i,j} = e_i$. If m_1, \dots, m_s are positive integers, then we want to study regularity of ideals of the form $I_Z = \wp_1^{m_1} \cap \dots \cap \wp_s^{m_s}$. Such an ideal I_Z defines a scheme of fat points $Z = m_1 P_1 + \dots + m_s P_s$ in $\mathbb{P}^{n_1} \times \dots \times \mathbb{P}^{n_k}$. The ideal I_Z is both \mathbb{N}^k -homogeneous, and homogeneous in the normal sense. Thus, when we refer to $\text{reg}(I_Z)$, we shall mean its regularity as a homogeneous ideal in R , where R is viewed as an \mathbb{N}^1 -graded ring.

A set of s points $\mathbb{X} = \{P_1, \dots, P_s\} \subseteq \mathbb{P}^{n_1} \times \dots \times \mathbb{P}^{n_k}$ is said to be in generic position if it has maximal Hilbert function $H_{\mathbb{X}}(\underline{i}) = \min\{\dim_{\mathbf{k}} R_{\underline{i}}, s\}$ for all $\underline{i} \in \mathbb{N}^k$, where $R = \bigoplus_{\underline{i}} R_{\underline{i}}$ is the \mathbb{N}^k -homogeneous decomposition of R . It is shown in [17] that sets of s points in generic position form an open subset of the Hilbert scheme of all sets of s points in $\mathbb{P}^{n_1} \times \dots \times \mathbb{P}^{n_k}$. Our main

2000 *Mathematics Subject Classification.* 13D02, 13D40, 14Q99.

Key words and phrases. regularity, points, fat points, multi-projective space.

Version: August 20, 2004.

results consist of explicitly calculating $\text{reg}(I_Z)$ when Z is in generic position and reduced (i.e. there is no multiplicity at each point), and giving a bound on $\text{reg}(I_Z)$ in general.

In the special case that each $m_i = 1$ and the set of points is in generic position, we show

$$\text{reg}(I_Z) = \max\{d_1 + 1, \dots, d_k + 1\}$$

where $d_i = \min\left\{d \in \mathbb{N} \mid \binom{d+n_i}{d} \geq s\right\}$ for each $i = 1, \dots, k$. To prove this we use the fact that I_Z is both \mathbb{N}^k -homogeneous and \mathbb{N}^1 -homogeneous to obtain information about $\text{reg}(I_Z)$. We also use the Bayer-Stillman criterion for detecting m -regularity [2].

We then show that if \mathbb{X} is generic position, and if $m_1 \geq m_2 \geq \dots \geq m_s$ with at least one $m_i \geq 2$, then

$$\text{reg}(I_Z) \leq \max\left\{m_1 + m_2 - 1, \left\lceil \frac{\sum_{i=1}^s m_i - 1}{n_1} \right\rceil, \dots, \left\lceil \frac{\sum_{i=1}^s m_i - 1}{n_k} \right\rceil\right\} + k.$$

Our strategy is to investigate the regularity index $\text{ri}(R/I_Z)$ of R/I_Z , considered as an \mathbb{N}^1 -graded ring, by extending the results of [5] for fat point schemes in \mathbb{P}^n to $\mathbb{P}^{n_1} \times \dots \times \mathbb{P}^{n_k}$, and then use the fact that $\text{reg}(I_Z) \leq \text{ri}(R/I_Z) + k$.

We have organized this papers as follows. In the first section we introduce the relevant information about regularity, the regularity index, and points in multi-projective spaces. In the second section we compute the regularity of a defining ideal of a set of points in generic position. In the last section we bound the regularity for a set of fat points with generic support.

Acknowledgments. The authors would like to thank A. Conca, for originally posing this question and some helpful discussions, and E. Guardo, for her comments on an earlier version of this paper. This work was begun when the second author visited the Università di Genova, and he would like to thank them for their hospitality. The second author also acknowledges the financial support of NSERC and INDAM while working on this project.

1. PRELIMINARIES

Throughout this paper \mathbf{k} denotes an algebraically closed field of characteristic zero. In this section, we recall the needed facts about the Castelnuovo-Mumford regularity, the regularity index, and points in multi-projective spaces. Let $S = \mathbf{k}[x_0, \dots, x_n]$ be a polynomial ring.

Definition 1.1. A graded S -module M is m -regular if there exists a free resolution

$$0 \longrightarrow \bigoplus_j S(-e_{r,j}) \longrightarrow \dots \longrightarrow \bigoplus_j S(-e_{1,j}) \longrightarrow \bigoplus_j S(-e_{0,j}) \longrightarrow M \longrightarrow 0$$

of M with $e_{i,j} - i \leq m$ for all i, j . The *Castelnuovo-Mumford regularity* (or simply, *regularity*) of M , denoted $\text{reg}(M)$, is the least integer m for which M is m -regular.

If $I \subseteq S$, then $\text{reg}(I) = \text{reg}(S/I) + 1$. The *saturation* \bar{I} of the ideal $I \subseteq S$ is the ideal

$$\bar{I} := \{F \in S \mid \text{for } i = 1, \dots, n, \text{ there exists an } r \text{ such that } x_i^r \cdot F \in I\}.$$

I is said to be *saturated* if $I = \bar{I}$. The regularity of a saturated ideal does not change if we add a non-zero divisor. In fact,

Lemma 1.2 ([2, Lemma 1.8]). *Let $I \subseteq S$ be a saturated ideal, and suppose h is a non-zero divisor of S/I . Then I is m -regular if and only if (I, h) is m -regular. Thus, $\text{reg}(I) = \text{reg}((I, h))$.*

The following theorem provides a means to determine if an ideal is m -regular.

Theorem 1.3 ([2, Theorem 1.10] Bayer-Stillman criterion for m -regularity). *Let $I \subseteq S$ be an ideal generated in degrees $\leq m$. The following conditions are equivalent:*

- (i) I is m -regular.
- (ii) There exists $h_1, \dots, h_j \in S_1$ for some $j \geq 0$ so that
 - (a) $((I, h_1, \dots, h_{i-1}) : h_i)_m = (I, h_1, \dots, h_{i-1})_m$ for $i = 1, \dots, j$, and
 - (b) $(I, h_1, \dots, h_j)_m = S_m$.

The *Hilbert function* $H_M : \mathbb{N} \rightarrow \mathbb{N}$ of a graded S -module M is defined by $H_M(t) := \dim_{\mathbf{k}} M_t$. It is well known (cf. [3, Theorem 4.1.3]) that there exists a unique polynomial $HP_M(t)$, called the *Hilbert polynomial* of M , such that $H_M(t) = HP_M(t)$ for $t \gg 0$.

Definition 1.4. The *regularity index* of an S -module M , denoted $\text{ri}(M)$, is defined to be

$$\text{ri}(M) := \min\{t \mid H_M(j) = HP_M(j) \text{ for all } j \geq t\}.$$

The regularity and regularity index of an S -module are then related as follows.

Lemma 1.5 ([14, Lemma 5.8]). *If M is a graded S -module, then*

$$\text{reg}(M) - \dim M + 1 \leq \text{ri}(M) \leq \text{reg}(M) - \text{depth } M + 1.$$

If $M = S/I$, then $\text{ri}(S/I) \leq \text{reg}(S/I) - \text{depth } S/I + 1 \leq \text{reg}(I)$. Hence, we have

Corollary 1.6. *If $I \subseteq S$, then for all $t \geq \text{reg}(I)$, $H_{S/I}(t) = HP_{S/I}(t)$.*

Our goal is to investigate $\text{reg}(I)$ when I defines either a reduced or non-reduced set of points in $\mathbb{P}^{n_1} \times \dots \times \mathbb{P}^{n_k}$ whose support is in generic position.

Let $R = \mathbf{k}[x_{1,0}, \dots, x_{1,n_1}, \dots, x_{k,0}, \dots, x_{k,n_k}]$, with $\deg x_{i,j} = e_i$ where e_i is the i^{th} basis vector of \mathbb{N}^k , be the \mathbb{N}^k -graded coordinate ring of $\mathbb{P}^{n_1} \times \dots \times \mathbb{P}^{n_k}$. Let $R_{e_i} = \mathbf{k}[x_{i,0}, \dots, x_{i,n_i}]$ be the graded coordinate ring of \mathbb{P}^{n_i} for $i = 1, \dots, k$. If $P \in \mathbb{P}^{n_1} \times \dots \times \mathbb{P}^{n_k}$ is a point, then the ideal $\wp \subseteq R$ associated to P is the prime ideal $\wp = (L_{1,1}, \dots, L_{1,n_1}, \dots, L_{k,1}, \dots, L_{k,n_k})$ with $\deg L_{i,j} = e_i$. Suppose $\mathbb{X} = \{P_1, \dots, P_s\}$ is a set of distinct points in $\mathbb{P}^{n_1} \times \dots \times \mathbb{P}^{n_k}$, and m_1, \dots, m_s are s positive integers. Let

$$I_Z = \wp_1^{m_1} \cap \wp_2^{m_2} \cap \dots \cap \wp_s^{m_s}$$

where \wp_i is the defining ideal of P_i , then I_Z defines a scheme of fat points $Z = m_1 P_1 + \dots + m_s P_s$ in $\mathbb{P}^{n_1} \times \dots \times \mathbb{P}^{n_k}$ with support \mathbb{X} . When $m_i = 1$ for all i , $Z \equiv \mathbb{X}$ is reduced, and we usually use $I_{\mathbb{X}}$ instead of I_Z .

Since $\text{ht}(\wp_i) = \sum_{j=1}^k n_j$ for each i , it follows that $\text{K-dim } R/I_Z = k$. Thus, by Lemma 1.5 we have

$$\text{reg}(I_Z) \leq \text{ri}(R/I_Z) + k.$$

Note that we have equality if $k = 1$ because then $\text{depth } R/I_Z = 1$.

We shall find it useful to consider R/I_Z as both an \mathbb{N}^k -graded ring and as an \mathbb{N}^1 -graded ring. We shall, therefore, use $\mathcal{H}_Z(\underline{t})$ to denote the multi-graded Hilbert function $\mathcal{H}_Z(\underline{t}) := \dim_{\mathbf{k}}(R/I_Z)_{\underline{t}}$ with $\underline{t} = (t_1, \dots, t_k) \in \mathbb{N}^k$, and $H_Z(t)$ to denote the \mathbb{N}^1 -graded Hilbert function $H_Z := H_{R/I_Z}$. Because $(R/I_Z)_t = \bigoplus_{t_1 + \dots + t_k = t} (R/I_Z)_{t_1, \dots, t_k}$, we have the identity:

$$H_Z(t) = \sum_{t_1 + \dots + t_k = t} \mathcal{H}_Z(t_1, \dots, t_k) \text{ for all } t \in \mathbb{N}.$$

Definition 1.7. A set of s points $\mathbb{X} = \{P_1, \dots, P_s\} \subseteq \mathbb{P}^{n_1} \times \dots \times \mathbb{P}^{n_k}$ is said to be in *generic position* if

$$\mathcal{H}_{\mathbb{X}}(\underline{t}) = \min \left\{ \dim_{\mathbb{K}} R_{\underline{t}} = \binom{t_1 + n_1}{n_1} \cdots \binom{t_k + n_k}{n_k}, s \right\} \text{ for all } \underline{t} \in \mathbb{N}^k.$$

Further results about points in $\mathbb{P}^{n_1} \times \dots \times \mathbb{P}^{n_k}$ can be found in [16, 17].

Remark 1.8. If $I \subseteq R$ is an \mathbb{N}^k -homogeneous ideal, then the \mathbb{N}^k -graded minimal free resolution of I is

$$0 \longrightarrow \mathcal{F}_r \longrightarrow \mathcal{F}_{r-1} \longrightarrow \dots \longrightarrow \mathcal{F}_0 \longrightarrow I \longrightarrow 0$$

where $\mathcal{F}_i = \bigoplus_j R(-d_{i,j,1}, -d_{i,j,2}, \dots, -d_{i,j,k})$. Since I is also homogeneous in the normal sense, the above resolution also gives a graded minimal free resolution of I :

$$0 \longrightarrow \mathcal{F}'_r \longrightarrow \mathcal{F}'_{r-1} \longrightarrow \dots \longrightarrow \mathcal{F}'_0 \longrightarrow I \longrightarrow 0$$

where $\mathcal{F}'_i = \bigoplus_j R(-d_{i,j,1} - d_{i,j,2} - \dots - d_{i,j,k})$ where we view R as \mathbb{N}^1 -graded. So if I is an \mathbb{N}^k -homogeneous ideal with $k \geq 2$, $\text{reg}(I)$ can be interpreted as a crude invariant that bounds the total degree of the multi-graded syzygies.

The following lemma, which generalizes [16, Lemma 3.3], enables us to find non-zero divisors of specific multi-degrees.

Lemma 1.9. *Suppose $\mathbb{X} = \{P_1, \dots, P_s\}$ is a set of distinct points in $\mathbb{P}^{n_1} \times \dots \times \mathbb{P}^{n_k}$, \wp_1, \dots, \wp_s are the defining ideals of P_1, \dots, P_s , respectively, and m_1, \dots, m_s are positive integers. Set $I_Z = \wp_1^{m_1} \cap \dots \cap \wp_s^{m_s}$, and fix an $i \in \{1, \dots, k\}$. Then there exists a form $L \in R_{e_i}$ such that \bar{L} is a non-zero divisor in R/I_Z .*

2. THE REGULARITY OF THE DEFINING IDEAL OF POINTS IN GENERIC POSITION

Let $\mathbb{X} \subseteq \mathbb{P}^{n_1} \times \dots \times \mathbb{P}^{n_k}$ be a set of s reduced points in generic position. In this section we calculate the Castelnuovo-Mumford regularity of the defining ideal of \mathbb{X} .

For each $i = 1, \dots, k$, set $d_i := \min \left\{ d \mid \binom{d+n_i}{d} \geq s \right\}$, and let $D := \max\{d_1 + 1, \dots, d_k + 1\}$. Note that if $n_i = \min\{n_1, \dots, n_k\}$, then $D = d_i + 1$. Beginning with a combinatorial lemma, we use this notation to describe some of the properties of points in generic position.

Lemma 2.1. *Let $n \geq 1$. Then, for all $a, b \geq 1$,*

$$\binom{a+b+n}{a+b} \leq \binom{a+n}{a} \binom{b+n}{b}.$$

Proof. Because

$$\binom{a+b+n}{a+b} = \frac{(a+b+n) \cdots (a+1+n)}{(a+b)(a+b-1) \cdots (a+1)} \binom{a+n}{a}$$

it is enough to show that the inequality

$$\frac{(a+b+n)(a+b-1+n) \cdots (a+1+n)}{(a+b) \cdots (a+1)} \leq \binom{b+n}{b}$$

is true. This is equivalent to showing that

$$\frac{(a+b+n)(a+b-1+n) \cdots (a+1+n)}{(b+n)(b-1+n) \cdots (1+n)} \leq \frac{(a+b)(a+b-1) \cdots (a+1)}{b(b-1) \cdots 2 \cdot 1}.$$

Rewriting the above expression, we see that we need to show that

$$\left[1 + \frac{a}{b+n}\right] \left[1 + \frac{a}{b-1+n}\right] \cdots \left[1 + \frac{a}{1+n}\right] \leq \left[1 + \frac{a}{b}\right] \left[1 + \frac{a}{b-1}\right] \cdots \left[1 + \frac{a}{1}\right].$$

But since $\left[1 + \frac{a}{b+n-j}\right] \leq \left[1 + \frac{a}{b-j}\right]$ for $j = 0, \dots, b-1$ we are finished. \square

Corollary 2.2. *Let $\mathbb{X} \subseteq \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ be s points in generic position. If $(t_1, \dots, t_k) \in \mathbb{N}^k$ is such that $t_1 + \cdots + t_k = D - 1$, then $\mathcal{H}_{\mathbb{X}}(t_1, \dots, t_k) = s$.*

Proof. Suppose that $n_i = \min\{n_1, \dots, n_k\}$, and hence, $D - 1 = d_i$. Lemma 2.1 then gives

$$\binom{t_1 + n_1}{t_1} \binom{t_2 + n_2}{t_2} \cdots \binom{t_k + n_k}{t_k} \geq \binom{t_1 + n_i}{t_1} \binom{t_2 + n_i}{t_2} \cdots \binom{t_k + n_i}{t_k} \geq \binom{d_i + n_i}{d_i}$$

Since $\binom{d_i + n_i}{d_i} \geq s$, we have $\mathcal{H}_{\mathbb{X}}(t_1, \dots, t_k) = s$. \square

Proposition 2.3. *Let $I_{\mathbb{X}}$ be the defining ideal of s points $\mathbb{X} \subseteq \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ in generic position.*

- (i) *As an \mathbb{N}^1 -graded ideal, $I_{\mathbb{X}}$ is generated by forms of degree $\leq D$.*
- (ii) *As an \mathbb{N}^1 -graded ring, $R/I_{\mathbb{X}}$ has Hilbert polynomial $HP_{R/I_{\mathbb{X}}}(t) = s \binom{t+k-1}{k-1}$.*
- (iii) *Fix an $i \in \{1, \dots, k\}$ and let L be the non-zero divisor of Lemma 1.9 of degree e_i . If $\underline{t} = (t_1, \dots, t_k) \in \mathbb{N}^k$ is such that $t_1 + \dots + t_k \geq D$ and $t_i > 0$, then $(I_{\mathbb{X}}, L)_{\underline{t}} = R_{\underline{t}}$.*

Proof. For (i) it suffices to show that for all $\underline{t} = (t_1, \dots, t_k) \in \mathbb{N}^k$ with $t_1 + \cdots + t_k \geq D + 1$, $(I_{\mathbb{X}})_{\underline{t}}$ contains no new minimal generators. If $\underline{t} \in \mathbb{N}^k$ is such a tuple, then there exists $l, m \in \{1, \dots, k\}$, not necessarily distinct, such that $\underline{t} - e_l - e_m \in \mathbb{N}^k$. By Corollary 2.2 it follows that $\mathcal{H}_{\mathbb{X}}(\underline{t} - e_l - e_m) = \mathcal{H}_{\mathbb{X}}(\underline{t} - e_l) = s$ since $t_1 + \cdots + t_k - 2 \geq D - 1$. Now apply the results of [17] to conclude that $(I_{\mathbb{X}})_{\underline{t}}$ contains no minimal generators.

Since \mathbb{X} is in generic position, for $t \gg 0$ we have

$$H_{\mathbb{X}}(t) = \sum_{t_1 + \cdots + t_k = t} \mathcal{H}_{\mathbb{X}}(t_1, \dots, t_k) = \sum_{t_1 + \cdots + t_k = t} s = s \binom{t+k-1}{k-1}.$$

Since $HP_{R/I_{\mathbb{X}}}$ is the unique polynomial that agrees with $H_{\mathbb{X}}$ for $t \gg 0$, (ii) now follows.

To prove (iii) we only consider the case $i = 1$. Since \bar{L} is a non-zero divisor, the exact sequence

$$0 \longrightarrow (R/I_{\mathbb{X}})(-e_1) \xrightarrow{\times \bar{L}} R/I_{\mathbb{X}} \longrightarrow R/(I_{\mathbb{X}}, L) \longrightarrow 0$$

implies that

$$\mathcal{H}_{R/(I_{\mathbb{X}}, L)}(t_1, \dots, t_k) = \mathcal{H}_{\mathbb{X}}(t_1, \dots, t_k) - \mathcal{H}_{\mathbb{X}}(t_1 - 1, t_2, \dots, t_k) \quad \text{for all } \underline{t} \in \mathbb{N}^k$$

where $\mathcal{H}_{\mathbb{X}}(t_1 - 1, t_2, \dots, t_k) = 0$ if $t_1 - 1 < 0$. Now suppose that $t_1 + \dots + t_k \geq D$ with $t_1 > 0$. Since $(t_1 - 1) + t_2 + \cdots + t_k \geq D - 1$, by Corollary 2.2 we have $\mathcal{H}_{\mathbb{X}}(t_1, \dots, t_k) = \mathcal{H}_{\mathbb{X}}(t_1 - 1, t_2, \dots, t_k) = s$. Thus $\mathcal{H}_{R/(I_{\mathbb{X}}, L)}(t_1, \dots, t_k) = 0$, or equivalently, $(I_{\mathbb{X}}, L)_{t_1, \dots, t_k} = R_{t_1, \dots, t_k}$. \square

Theorem 2.4. *Let $I_{\mathbb{X}}$ be the defining ideal of s points $\mathbb{X} \subseteq \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ in generic position. Then*

$$\text{reg}(I_{\mathbb{X}}) = \max\{d_1 + 1, \dots, d_k + 1\}$$

where $d_i := \min \left\{ d \mid \binom{d+n_i}{d} \geq s \right\}$ for $i = 1, \dots, k$.

Proof. Without loss of generality, we assume that $n_1 \geq n_2 \geq \dots \geq n_k \geq 1$. It thus suffices to show that $\text{reg}(I_{\mathbb{X}}) = d_k + 1 = \max\{d_1 + 1, \dots, d_k + 1\}$.

We first show that $\text{reg}(I_{\mathbb{X}}) > d_k$. By Lemma 1.9 there is a non-zero divisor \bar{L} of $R/I_{\mathbb{X}}$ with $\deg L = e_k$. As an \mathbb{N} -homogeneous element of R , $\deg L = 1$. Since $I_{\mathbb{X}}$ is saturated, by Lemma 1.2 it is enough to show $\text{reg}(I_{\mathbb{X}}, L) > d_k$.

From the short exact sequence

$$0 \longrightarrow (R/I_{\mathbb{X}})(-1) \xrightarrow{\times \bar{L}} R/I_{\mathbb{X}} \longrightarrow R/(I_{\mathbb{X}}, L) \longrightarrow 0.$$

of \mathbb{N}^1 -graded rings, and from Proposition 2.3 (ii) we deduce that

$$HP_{R/(I_{\mathbb{X}}, L)}(t) = HP_{R/I_{\mathbb{X}}}(t) - HP_{R/I_{\mathbb{X}}}(t-1) = s \binom{t + (k-2)}{k-2}.$$

If we can show that $HP_{R/(I_{\mathbb{X}}, L)}(d_k) \neq HP_{R/I_{\mathbb{X}}}(d_k)$, then by Corollary 1.6, we can conclude that $\text{reg}(I_{\mathbb{X}}, L) > d_k$. So, write $HP_{R/(I_{\mathbb{X}}, L)}(d_k) = A + B$ where

$$A := \sum_{t_1 + \dots + t_{k-1} = d_k} \mathcal{H}_{R/(I_{\mathbb{X}}, L)}(t_1, \dots, t_{k-1}, 0) \text{ and } B := \sum_{t_1 + \dots + t_k = d_k, t_k > 0} \mathcal{H}_{R/(I_{\mathbb{X}}, L)}(t_1, t_2, \dots, t_k).$$

From the short exact sequence

$$0 \longrightarrow (R/I_{\mathbb{X}})(-e_k) \xrightarrow{\times \bar{L}} R/I_{\mathbb{X}} \longrightarrow R/(I_{\mathbb{X}}, L) \longrightarrow 0$$

of \mathbb{N}^k -graded rings, we have

$$\mathcal{H}_{R/(I_{\mathbb{X}}, L)}(t_1, \dots, t_k) = \mathcal{H}_{R/I_{\mathbb{X}}}(t_1, \dots, t_k) - \mathcal{H}_{R/I_{\mathbb{X}}}(t_1, \dots, t_{k-1}, t_k - 1)$$

where $\mathcal{H}_{R/I_{\mathbb{X}}}(t_1, \dots, t_{k-1}, t_k - 1) = 0$ if $t_k = 0$. Thus

$$A = \sum_{t_1 + \dots + t_{k-1} = d_k} \mathcal{H}_{R/I_{\mathbb{X}}}(t_1, \dots, t_{k-1}, 0).$$

Since $t_1 + \dots + t_{k-1} = d_k$, by Corollary 2.2 we have $\mathcal{H}_{R/I_{\mathbb{X}}}(t_1, \dots, t_{k-1}, 0) = s$. Hence,

$$A = \sum_{t_1 + \dots + t_{k-1} = d_k} s = s \binom{d_1 + k - 2}{k - 2} = HP_{R/(I_{\mathbb{X}}, L)}(d_k).$$

On the other hand, because $d_k = \min \left\{ d \mid \binom{d+n_k}{d} \geq s \right\}$

$$\begin{aligned} B &\geq \mathcal{H}_{R/(I_{\mathbb{X}}, L)}(0, \dots, 0, d_k) = \mathcal{H}_{\mathbb{X}}(0, \dots, 0, d_k) - \mathcal{H}_{\mathbb{X}}(0, \dots, 0, d_k - 1) \\ &= s - \binom{d_k - 1 + n_k}{d_k - 1} > 0. \end{aligned}$$

Thus, $HP_{R/(I_{\mathbb{X}}, L)}(d_k) = HP_{R/I_{\mathbb{X}}}(d_k) + B > HP_{R/I_{\mathbb{X}}}(d_k)$, as desired.

We now show that $\text{reg}(I_{\mathbb{X}}) \leq d_k + 1$ by demonstrating that $I_{\mathbb{X}}$ is $(d_k + 1)$ -regular. By Proposition 2.3 (i), as an \mathbb{N}^1 -graded ideal $I_{\mathbb{X}}$ is generated by elements of degree $\leq d_k + 1$. For each $i \in \{1, \dots, k\}$, by Lemma 1.9 there exists a non-zero divisor $\bar{L}_i \in R/I_{\mathbb{X}}$ with $\deg L_i = e_i$. After a change of variables in the $x_{1,j}$'s, a change of variables in the $x_{2,j}$'s, etc., we can assume that $L_i = x_{i,0}$ for $i = 1, \dots, k$.

By the Bayer-Stillman criterion (Theorem 1.3), to show that $I_{\mathbb{X}}$ is $(d_k + 1)$ -regular, it is enough to prove:

- (a) $((I_{\mathbb{X}}, x_{1,0}, \dots, x_{j-1,0}) : x_{j,0})_{d_k+1} = (I_{\mathbb{X}}, x_{1,0}, \dots, x_{j-1,0})_{d_k+1}$ for $j = 1, \dots, k$,
 (b) $(I_{\mathbb{X}}, x_{1,0}, \dots, x_{k,0})_{d_k+1} = R_{d_k+1}$.

Proof of (a). We need to only show the non-trivial inclusion $[(I_{\mathbb{X}}, x_{1,0}, \dots, x_{j-1,0}) : x_{j,0}]_{d_k+1} \subseteq (I_{\mathbb{X}}, x_{1,0}, \dots, x_{j-1,0})_{d_k+1}$ for each j . If $j = 1$, then the statement holds because $x_{1,0}$ is a non-zero divisor.

So, suppose $j > 1$. Set $J := [(I_{\mathbb{X}}, x_{1,0}, \dots, x_{j-1,0}) : x_{j,0}]$. Because J is also \mathbb{N}^k -homogeneous, if $F \in J_{d_k+1}$, then can assume that $\deg F = \underline{t} = (t_1, \dots, t_k)$ with $t_1 + \dots + t_k = d_k + 1$. There are now two cases to consider.

In the first case, one of $t_1, \dots, t_{j-1} > 0$. Suppose $t_l > 0$ with $1 \leq l \leq (j-1)$. Then by Proposition 2.3 (iii) we have $F \in R_{\underline{t}} \subseteq (I_{\mathbb{X}}, x_{l,0})_{\underline{t}} \subseteq (I_{\mathbb{X}}, x_{1,0}, \dots, x_{j-1,0})_{\underline{t}}$. Since $(I_{\mathbb{X}}, x_{1,0}, \dots, x_{j-1,0})_{\underline{t}} \subseteq (I_{\mathbb{X}}, x_{1,0}, \dots, x_{j-1,0})_{d_k+1}$ (as vector spaces), we are finished.

In the second case, $t_1 = t_2 = \dots = t_{j-1} = 0$. Then $Fx_{j,0} \in (I_{\mathbb{X}}, x_{1,0}, \dots, x_{j-1,0})_{0, \dots, 0, t_j+1, \dots, t_k}$. But since

$$(I_{\mathbb{X}}, x_{1,0}, \dots, x_{j-1,0})_{0, \dots, 0, t_j+1, \dots, t_k} = (I_{\mathbb{X}})_{0, \dots, 0, t_j+1, \dots, t_k},$$

we have $Fx_{j,0} \in (I_{\mathbb{X}})_{0, \dots, 0, t_j+1, \dots, t_k}$. But because $x_{j,0}$ is a non-zero divisor of $R/I_{\mathbb{X}}$,

$$F \in (I_{\mathbb{X}})_{0, \dots, 0, t_j, \dots, t_k} \subseteq (I_{\mathbb{X}}, x_{1,0}, \dots, x_{j-1,0})_{0, \dots, 0, t_j, \dots, t_k} \subseteq (I_{\mathbb{X}}, x_{1,0}, \dots, x_{j-1,0})_{d_k+1}.$$

Proof of (b). Since $R_{d_k+1} = \bigoplus_{t_1+\dots+t_k=d_k+1} R_{t_1, \dots, t_k}$ and because $(I_{\mathbb{X}}, x_{1,0}, \dots, x_{k,0})$ is also \mathbb{N}^k -homogeneous, it is enough to show that $R_{\underline{t}} \subseteq (I_{\mathbb{X}}, x_{1,0}, \dots, x_{k,0})_{\underline{t}}$ for all $\underline{t} = (t_1, \dots, t_k) \in \mathbb{N}^k$ with $t_1 + \dots + t_k = d_k + 1$. But for any $\underline{t} \in \mathbb{N}^k$ with $t_1 + \dots + t_k = d_k + 1$, there exists at least one $t_l > 0$. Thus, by Proposition 2.3 (iii) we have $R_{\underline{t}} \subseteq (I_{\mathbb{X}}, x_{l,0})_{\underline{t}} \subseteq (I_{\mathbb{X}}, x_{1,0}, \dots, x_{k,0})_{\underline{t}}$, thus completing the proof of (b).

Since we have just shown $d_k < \text{reg}(I_{\mathbb{X}}) \leq d_k + 1$, the desired conclusion now follows. \square

Remark 2.5. If \mathbb{X} is a set of s points in generic position in \mathbb{P}^n , we recover the well known result that $\text{reg}(I_{\mathbb{X}}) = d + 1$ where $d = \min\{l \mid \binom{l+n}{n} \geq s\}$.

3. BOUNDING THE REGULARITY OF FAT POINTS IN $\mathbb{P}^{n_1} \times \dots \times \mathbb{P}^{n_k}$

Let $\mathbb{X} = \{P_1, \dots, P_s\} \subseteq \mathbb{P}^{n_1} \times \dots \times \mathbb{P}^{n_k}$ and $m_1 \geq \dots \geq m_s \in \mathbb{N}^+$. Suppose \wp_i is the defining ideal of P_i for $i = 1, \dots, s$. Let $I = I_Z = \wp_1^{m_1} \cap \dots \cap \wp_s^{m_s}$. In this section, we give an upper bound for $\text{reg}(I)$ when \mathbb{X} is in generic position. If we consider R/I as an \mathbb{N}^1 -graded ring, then by Lemma 1.5

$$\text{reg}(I) = \text{reg}(R/I) + 1 \leq \text{ri}(R/I) + \dim R/I = \text{ri}(R/I) + k.$$

To bound $\text{reg}(I)$, it is therefore enough to bound $\text{ri}(R/I)$. For convenience, we assume that $n_1 \geq \dots \geq n_k$. In the sequel, we shall also abuse notation by writing L for the form $L \in \mathbf{k}[x_{j,0}, \dots, x_{j,n_j}]$, the hyperplane L in \mathbb{P}^{n_j} defined by L , and the subvariety of $\mathbb{P}^{n_1} \times \dots \times \mathbb{P}^{n_k}$ defined by L .

Lemma 3.1. *If \wp is the defining ideal of point $P \in \mathbb{P}^{n_1} \times \dots \times \mathbb{P}^{n_k}$, then*

$$\text{ri}(R/\wp^a) = a - k \text{ for all } a \geq 1.$$

Proof. Since \wp defines a complete intersection of height $\sum_{i=1}^k n_i$, Lemma 1.5 gives $\text{ri}(R/\wp^a) = \text{reg}(R/\wp^a) - k + 1$. The conclusion follows since $\text{reg}(\wp^a) = a \text{reg}(\wp) = a$ by [6, Theorem 3.1]. \square

Lemma 3.2. *Suppose P_1, \dots, P_r, P are points in generic position in $\mathbb{P}^{n_1} \times \dots \times \mathbb{P}^{n_k}$, and let \wp_i be the defining ideal of P_i and let \wp be the defining ideal of P . Let m_1, \dots, m_r and a be positive integers, $J = \wp_1^{m_1} \cap \dots \cap \wp_r^{m_r}$, and $I = J \cap \wp^a$. Then*

$$\text{ri}(R/I) \leq \max \{a - k, \text{ri}(R/J), \text{ri}(R/(J + \wp^a))\}.$$

Furthermore, $R/(J + \wp^a)$ is artinian.

Proof. The short exact sequence of \mathbb{N}^1 -graded rings

$$0 \longrightarrow R/I \longrightarrow R/J \oplus R/\wp^a \longrightarrow R/(J + \wp^a) \longrightarrow 0$$

yields $H_{R/I}(t) = H_{R/J}(t) + H_{R/\wp^a}(t) - H_{R/(J + \wp^a)}(t)$. Combining this with Lemma 3.1 gives

$$\text{ri}(R/I) \leq \max \{a - k, \text{ri}(R/J), \text{ri}(R/(J + \wp^a))\}.$$

To show that $R/(J + \wp^a)$ is artinian, we need to show that there exists b such that for all $\underline{t} = (t_1, \dots, t_k) \in \mathbb{N}^k$, if there is $t_j \geq b$, then $(R/(J + \wp^a))_{\underline{t}} = 0$. So, it suffices to show that there exists such a b so that for all $\underline{t} = (t_1, \dots, t_k)$ with $t_j \geq b$ for some j , then all monomials of R of degree \underline{t} are in $(J + \wp^a)$. Suppose M is a monomial in R of degree \underline{t} . Then $M = N_1 N_2 \dots N_k$ where N_l are monomials in $\{x_{l,0}, \dots, x_{l,n_l}\}$ and of degree t_l . It is enough to show $N_j \in (J + \wp^a)$.

Let Q_1, \dots, Q_r, Q be the projections of P_1, \dots, P_r, P in \mathbb{P}^{n_j} . Since the points are in generic position, the projections are distinct. Let $\mathcal{Q}_1, \dots, \mathcal{Q}_r$ and \mathcal{Q} be the defining ideals of Q_1, \dots, Q_r, Q in $A = \mathbf{k}[x_{j,0}, \dots, x_{j,n_j}]$. Then it is easy to see that $A/(\mathcal{Q}_1^{m_1} \cap \dots \cap \mathcal{Q}_r^{m_r} + \mathcal{Q}^a)$ is artinian. As well, $\mathcal{Q}_1^{m_1} \cap \dots \cap \mathcal{Q}_r^{m_r} \subseteq J$ and $\mathcal{Q}^a \subseteq \wp^a$, and thus $\mathcal{Q}_1^{m_1} \cap \dots \cap \mathcal{Q}_r^{m_r} + \mathcal{Q}^a \subseteq (J + \wp^a)$, and this is what needs to be shown. \square

From Lemma 3.2, to estimate $\text{ri}(R/I)$ we need to estimate $\text{ri}(R/(J + \wp^a))$, or equivalently, the least integer t such that $(R/(J + \wp^a))_t = 0$, when this ring is considered as \mathbb{N}^1 -graded.

Lemma 3.3. *With the same hypotheses as in Lemma 3.2, and considering the \mathbb{N}^1 -gradation, we have*

- (i) $H_{R/(J + \wp^a)}(t) = \sum_{i=0}^{a-1} \dim_{\mathbf{k}} [(J + \wp^i)/(J + \wp^{i+1})]_t$ for all $t \geq 0$.
- (ii) If $P = [1 : 0 : \dots : 0] \times \dots \times [1 : 0 : \dots : 0]$ then $[(J + \wp^i)/(J + \wp^{i+1})]_t = 0$ if and only if either $i > t$, or $i < t$ and $GM \in (J + \wp^{i+1})$ for every monomial M of degree i in $\{x_{1,1}, \dots, x_{1,n_1}, \dots, x_{k,1}, \dots, x_{k,n_k}\}$, and every monomial G of degree $t - i$ in $\{x_{1,0}, x_{2,0}, \dots, x_{k,0}\}$.

Proof. The first assertion follows from the short exact sequences:

$$0 \longrightarrow (J + \wp^i)/(J + \wp^{i+1}) \longrightarrow R/(J + \wp^{i+1}) \longrightarrow R/(J + \wp^i) \longrightarrow 0$$

where $i = 0, \dots, a - 1$.

To prove (ii), if $i > t$, then $(J + \wp^i)_t = (J + \wp^{i+1})_t = J_t$. So suppose $i < t$. We see that $\wp = (x_{1,1}, \dots, x_{1,n_1}, \dots, x_{k,1}, \dots, x_{k,n_k})$. Thus $((J + \wp^i)/(J + \wp^{i+1}))_t = 0$ if and only if $(\wp^i)_t \subseteq (J + \wp^{i+1})_t$ if and only if $FM \in (J + \wp^{i+1})$ for every monomial M of degree i in $\{x_{1,1}, \dots, x_{1,n_1}, \dots, x_{k,1}, \dots, x_{k,n_k}\}$ and every form $F \in R_{t-i}$. But because $(J + \wp^{i+1})$ is \mathbb{N}^k -homogeneous, we can take F to be \mathbb{N}^k -homogeneous, and so $F = G + H$ where G is a monomial of degree $t - i$ in $x_{1,0}, \dots, x_{k,0}$ and $H \in \wp$. Since $HM \in \wp^{i+1}$, we have $((J + \wp^i)/(J + \wp^{i+1}))_t = 0$ if and only if $GM \in (J + \wp^{i+1})_t$, as desired. \square

Lemma 3.4. *Let P_1, \dots, P_r, P be points in generic position in $\mathbb{P}^{n_1} \times \dots \times \mathbb{P}^{n_k}$ with $n_1 \geq \dots \geq n_k$, and let $m_1 \geq \dots \geq m_r$ be positive integers. Set $J = \wp_1^{m_1} \cap \dots \cap \wp_r^{m_r}$. Suppose $\underline{a} = (a_1, \dots, a_k) \in \mathbb{N}^k$ is such that $n_k \left(\sum_{i=1}^k a_i \right) \geq \sum_{i=1}^r m_i$ and $\sum_{i=1}^k a_i \geq m_1$. Then we can find a_j hyperplanes $L_{j,1}, \dots, L_{j,a_j}$ in \mathbb{P}^{n_j} , that is, $L_{j,l} \in \mathbf{k}[x_{j,0}, \dots, x_{j,n_j}]$ for all $l = 1, \dots, a_j$, such that*

$$L = \prod_{j=1}^k \left(\prod_{l=1}^{a_j} L_{j,l} \right) \in J$$

and L avoids P .

Proof. If $r \leq n_j$ for all j , then for each j we can find a linear form $L_j \in \mathbf{k}[x_{j,0}, \dots, x_{j,n_j}]$ that passes through P_1, \dots, P_r and avoids P . If we take $L_{j,l} = L_j$ for all j , we have

$$L = \prod_{j=1}^k L_j^{a_j} \in \wp_1^{|\underline{a}|} \cap \dots \cap \wp_r^{|\underline{a}|} \subseteq \wp_1^{m_1} \cap \dots \cap \wp_r^{m_r} = J,$$

where $|\underline{a}| = \sum_{i=1}^k a_i$, since $|\underline{a}| \geq m_1 \geq \dots \geq m_r$. Moreover, L avoids P .

Suppose now that $n_k \leq n_{k-1} \leq \dots \leq n_{l+1} < r \leq n_l \leq \dots \leq n_1$. We shall use induction on $\sum_{i=1}^r m_i$. Note that if $\sum_{i=1}^r m_i \leq n_k$ then the conclusion follows since in this case $r \leq n_k \leq n_j$ for all j . If $a_k = a_{k-1} = \dots = a_{l+1} = 0$, then the conclusion follows as in the case $r \leq n_j$ for all j . Suppose there is $p \in \{l+1, \dots, k\}$ such that $a_p \neq 0$. Choose a hyperplane L_1 in \mathbb{P}^{n_p} ($L_1 \in \mathbf{k}[x_{p,0}, \dots, x_{p,n_p}]$) that avoids P and passes through P_1, \dots, P_{n_p} . Since $n_k \left(\sum_{i=1}^k a_i \right) \geq \sum_{i=1}^r m_i$, we have

$$\begin{aligned} n_k \left(\sum_{i=1}^k a_i \right) - n_k &\geq \sum_{i=1}^r m_i - n_k \geq \sum_{i=1}^r m_i - n_p \\ &= (m_1 - 1) + \dots + (m_{n_p} - 1) + m_{n_p+1} + \dots + m_r. \end{aligned}$$

If we set $(b_1, \dots, b_{p-1}, b_p, b_{p+1}, \dots, b_k) = (a_1, \dots, a_{p-1}, a_p - 1, a_{p+1}, \dots, a_k)$, then we have

$$n_k \left(\sum_{i=1}^k b_i \right) = n_k \left(\sum_{i=1}^k a_i \right) - n_k \geq (m_1 - 1) + \dots + (m_{n_p} - 1) + m_{n_p+1} + \dots + m_r.$$

By induction there exists $L_{j,1}, \dots, L_{j,b_j}$ in \mathbb{P}^{n_j} for all j that avoids P such that

$$L = \prod_{j=1}^k \left(\prod_{l=1}^{b_j} L_{j,l} \right) \in \wp_1^{m_1-1} \cap \dots \cap \wp_{n_p}^{m_{n_p}-1} \cap \wp_{n_p+1}^{m_{n_p+1}} \cap \dots \cap \wp_r^{m_r}.$$

If we take $L \cdot L_1$ we have the conclusion since $L_1 \in \wp_1 \cap \dots \cap \wp_{n_p}$ (the a_p hyperplanes in \mathbb{P}^{n_p} are $L_{p,1}, \dots, L_{p,b_p}$ and L_1). \square

Proposition 3.5. *Let P_1, \dots, P_r, P be points in generic position in $\mathbb{P}^{n_1} \times \dots \times \mathbb{P}^{n_k}$ with $n_1 \geq \dots \geq n_k$. Suppose $m_1 \geq \dots \geq m_r \geq a$ are positive integers. Set $J = \wp_1^{m_1} \cap \dots \cap \wp_r^{m_r}$. Let t be the least integer such that $n_k t \geq \sum_{i=1}^r m_i + a - 1$. Then*

$$\text{ri}(R/(J + \wp^a)) \leq \max\{m_1 + a - 1, t\}.$$

Proof. Without loss of generality take $P = [1 : 0 : \dots : 0] \times \dots \times [1 : 0 : \dots : 0]$. Then $\wp = (x_{1,1}, \dots, x_{1,n_1}, \dots, x_{k,1}, \dots, x_{k,n_k})$. If $r \leq n_j$ for all j , then we can find a hyperplane L_j in \mathbb{P}^{n_j} , i.e., $L_j \in \mathbf{k}[x_{j,0}, \dots, x_{j,n_j}]$, containing P_1, \dots, P_r and avoids P for each j . Then $L_j \in \wp_1 \cap \dots \cap \wp_r$ for all j .

Suppose $G = x_{1,0}^{a_1} \cdots x_{k,0}^{a_k}$ is a monomial of degree m_1 in $\{x_{1,0}, \dots, x_{k,0}\}$. Then $L := L_1^{a_1} \cdots L_k^{a_k} \in \wp_1^{m_1} \cap \cdots \cap \wp_r^{m_r} \subseteq \wp_1^{m_1} \cap \cdots \cap \wp_r^{m_r} = J$. We can rewrite $L_j = x_{j,0} + H_j$ where $H_j \in (x_{j,1}, \dots, x_{j,n_j}) \subseteq \wp$. Then $L \in J$ implies $G \in J + \wp$. Thus, for any monomial M of degree i in \wp^i for some $0 \leq i \leq a-1$, $GM \in J + \wp^{i+1}$. Since $a-1 \geq i$, this implies that for any monomial \bar{G} of degree $m_1 + a - 1 - i$ in $\{x_{1,0}, \dots, x_{k,0}\}$, and any monomial M of degree i in \wp^i , $\bar{G}M \in (J + \wp^{i+1})$ because \bar{G} is divisible by a monomial of degree m_1 . By Lemma 3.3, this implies that $\text{ri}(R/(J + \wp^a)) \leq m_1 + a - 1$.

Suppose now that $r > n_k$. Since $n_1 \geq \dots \geq n_k$, by a change of coordinates we may assume that

$$\begin{aligned} P_1 &= [0 : 1 : 0 : \cdots : 0] \times [0 : 1 : 0 : \cdots : 0] \times \cdots \times [0 : 1 : 0 : \cdots : 0] \\ &\vdots \\ P_{n_k} &= \underbrace{[0 : \cdots : 0 : 1 : 0 : \cdots : 0]}_{n_k} \times \underbrace{[0 : \cdots : 0 : 1 : 0 : \cdots : 0]}_{n_k} \times \cdots \times [0 : \cdots : 0 : 1] \end{aligned}$$

So for $0 \leq j \leq n_k$, $\wp_j = (\{x_{l,q} \mid l = 1, \dots, k, q \neq j\})$.

Let $h = \max\{m_1 + a - 1, t\}$ and $0 \leq i \leq a - 1$. Suppose now that $G = x_{1,0}^{a_1} \cdots x_{k,0}^{a_k}$ is a monomial of degree $h - i$ in $\{x_{1,0}, \dots, x_{k,0}\}$, and $M = \prod_{l=1}^k \prod_{q \neq 0} x_{l,q}^{c_{l,q}}$ is a monomial of degree i in \wp^i . Because of Lemma 3.3 we need to show that $GM \in (J + \wp^{i+1})$.

It can be seen that

$$M \in \wp_1^{i - \sum_{l=1}^k c_{l,1}} \cap \wp_2^{i - \sum_{l=1}^k c_{l,2}} \cap \cdots \cap \wp_{n_k}^{i - \sum_{l=1}^k c_{l,n_k}}.$$

We also have, since $i \leq a - 1$,

$$\sum_{i=1}^k a_i = h - i \geq m_1 \geq \max \left\{ m_1 - i + \sum_{i=1}^k c_{l,1}, \dots, m_{n_k} - i + \sum_{i=1}^k c_{l,n_k} \right\}$$

and

$$\begin{aligned} n_k \left(\sum_{j=1}^k a_j \right) &= n_k(h - i) = n_k h - i n_k \\ &\geq \sum_{j=1}^r m_j + a - 1 - i n_k \geq \sum_{j=1}^r m_j + i - i n_k \\ &\geq \sum_{j=1}^r m_j + \sum_{l=1}^k \sum_{q=1}^{n_k} c_{l,q} - i n_k \\ &= (m_1 - i + \sum_{l=1}^k c_{l,1}) + \cdots + (m_{n_k} - i + \sum_{l=1}^k c_{l,n_k}) + m_{n_k+1} + \cdots + m_r. \end{aligned}$$

Using Lemma 3.4, there exists $L_{j,1}, \dots, L_{j,a_j} \in \mathbf{k}[x_{j,0}, \dots, x_{j,n_j}]$ for each $1 \leq j \leq k$ such that

$$L = \prod_{j=1}^k \left(\prod_{q=1}^{a_j} L_{j,q} \right) \in \wp_1^{m_1 - i + \sum c_{l,1}} \cap \cdots \cap \wp_{n_k}^{m_{n_k} - i + \sum c_{l,n_k}} \cap \wp_{n_k+1}^{m_{n_k+1}} \cap \cdots \cap \wp_r^{m_r},$$

and L avoids P . This implies that $LM \in J$.

Since $L_{j,q}$ avoids P we can write $L_{j,q} = x_{j,0} + H_{j,q}$ where $H_{j,q} \in (x_{j,1}, \dots, x_{j,n_j}) \subseteq \wp$. Then $L = x_{1,0}^{a_1} \cdots x_{k,0}^{a_k} + N$ where $N \in \wp$. Thus, since $LM \in J$, then $GM \in (J + \wp^{i+1})$ which is what we need to prove. \square

Theorem 3.6. *Suppose P_1, \dots, P_s are points in generic position in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ ($s \geq 2$ and $n_1 \geq \dots \geq n_k$), and $m_1 \geq m_2 \geq \dots \geq m_s$ are positive integers. Set $I = \wp_1^{m_1} \cap \cdots \cap \wp_s^{m_s}$. Then*

$$\text{ri}(R/I) \leq \max \left\{ m_1 + m_2 - 1, \left\lceil \frac{\sum_{i=1}^s m_i - 1}{n_k} \right\rceil \right\}$$

where $\lceil q \rceil$ denotes the smallest integer t such that $t \geq q$.

Proof. Note that $n_1 \geq \dots \geq n_k$, so

$$\left\lceil \frac{\sum_{i=1}^s m_i - 1}{n_k} \right\rceil = \max \left\{ \left\lceil \frac{\sum_{i=1}^s m_i - 1}{n_j} \right\rceil \right\}_{j=1}^k$$

Also, $\min\{t \mid n_k t \geq q\} = \left\lceil \frac{q}{n_k} \right\rceil$. So, if we take $q = \sum_{i=1}^r m_i + m_{r+1} - 1$, and use Proposition 3.5 and Lemma 3.2, together with induction successively, we will have the conclusion. \square

We obtain an immediate corollary which gives a bound on the regularity of the defining ideal of a scheme of fat points in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$.

Corollary 3.7. *With the hypotheses as in Theorem 3.6 we have*

$$\text{reg}(I) \leq \max \left\{ m_1 + m_2 - 1, \left\lceil \frac{\sum_{i=1}^s m_i - 1}{n_k} \right\rceil \right\} + k.$$

Remark 3.8. When $k = 1$ we recover the result of [5] which was proved to be sharp. One may expect the bound in Corollary 3.7 to be sharp also.

REFERENCES

- [1] A. Aramova, K. Crona, E. De Negri, Bigeneric initial ideals, diagonal subalgebras and bigraded Hilbert functions. *JPPA* **150** (2000) 215–235.
- [2] D. Bayer, M. Stillman, A criterion for detecting m -regularity. *Invent. Math.* **87** (1987) 1–11.
- [3] W. Bruns, J. Herzog, *Cohen-Macaulay rings* (Revised Edition). Cambridge University Press, 1993.
- [4] M.V. Catalisano, A.V. Geramita, A. Gimigliano, Tensor rank, secant varieties to Segre varieties, and fat points in multiprojective spaces. In *The Curves Seminar at Queen's. Vol XIII*, Queen's Papers in Pure and Appl. Math. **119** (2000) 223–246.
- [5] M.V. Catalisano, N.V. Trung, G. Valla, A sharp bound for the regularity index of fat points in general position, *Proc. Amer. Math. Soc.* **118** (1993), 717–724.
- [6] A. Conca, J. Herzog, Castelnuovo-Mumford Regularity of Products of Ideals. (2001) Preprint. [math.AC/0210065](#)
- [7] G. Fatabbi, A. Lorenzini, On a sharp bound for the regularity index of any set of fat points. *JPA* **161** (2001) 91–111.
- [8] A. Gimigliano, Regularity of linear systems of plane curves, *J. Algebra* **124** (1989) 447–460.
- [9] S. Giuffrida, R. Maggioni, A. Ragusa, On the postulation of 0-dimensional subschemes on a smooth quadric. *Pacific J. Math.* **155** (1992), no. 2, 251–282.
- [10] E. Guardo, Fat point schemes on a smooth quadric. *JPA* **162** (2001) 183–208.
- [11] E. Guardo, A. Van Tuyl, Fat Points in $\mathbb{P}^1 \times \mathbb{P}^1$ and their Hilbert functions. To appear in *Can. J. Math.* (2002) [math.RA/0203071](#)
- [12] J. W. Hoffman, H. Wang, Castelnuovo-Mumford Regularity in Biprojective Spaces. (2003) Preprint. [math.AG/0212033](#)
- [13] D. Maclagan, G. G. Smith, Multigraded Castelnuovo-Mumford Regularity. (2003) Preprint. [math.AC/0305214](#)

- [14] J. Migliore, U. Nagel, Liaison and related topics: Notes from the Torino workshop/school. (2002) To appear in Rendiconti del Seminario Matematico-Università e Politecnico di Torino. **math.AG/0205161**
- [15] N.V. Trung, G. Valla, Upper bounds for the regularity index of fat points, J. Algebra **176** (1995) 182-209.
- [16] A. Van Tuyl, The border of the Hilbert function of a set of points in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$. JPAA **176** (2002) 223-247.
- [17] A. Van Tuyl, On the defining ideal of a set of points in multi-projective space. (2003) Preprint.

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