

SPLITTABLE IDEALS AND THE RESOLUTIONS OF MONOMIAL IDEALS

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ABSTRACT. We provide a new combinatorial approach to study the minimal free resolutions of edge ideals, that is, quadratic square-free monomial ideals. With this method we can recover most of the known results on resolutions of edge ideals with fuller generality, and at the same time, obtain new results. Past investigations on the resolutions of edge ideals usually reduced the problem to computing the dimensions of reduced homology or Koszul homology groups. Our approach circumvents the highly nontrivial problem of computing the dimensions of these groups and turns the problem into combinatorial questions about the associated simple graph. We also show that our technique extends successfully to the study of graded Betti numbers of arbitrary square-free monomial ideals viewed as facet ideals of simplicial complexes.

1. INTRODUCTION

Let $R = k[x_1, \dots, x_n]$ be a polynomial ring over an arbitrary field k . If I is a homogeneous ideal of R , then associated to I is a *minimal graded free resolution*

$$0 \rightarrow \bigoplus_j R(-j)^{\beta_{l,j}(I)} \rightarrow \bigoplus_j R(-j)^{\beta_{l-1,j}(I)} \rightarrow \dots \rightarrow \bigoplus_j R(-j)^{\beta_{0,j}(I)} \rightarrow I \rightarrow 0$$

where the maps are exact, $l \leq n$, and $R(-j)$ is the R -module shifted by j . The number $\beta_{i,j}(I)$, the ij th *graded Betti number* of I , is an invariant of I equal to the number of minimal generators of degree j in the i th syzygy module.

In this paper we shall study the graded Betti numbers of *monomial ideals*. The book of Miller and Sturmfels [20] contains a comprehensive introduction and list of references on this topic. We shall concentrate on ideals which are generated by square-free quadratic monomials so that we may exploit the natural bijection

$$\left\{ \begin{array}{l} \text{square-free quadratic monomial} \\ \text{ideals } I \subseteq R = k[x_1, \dots, x_n] \end{array} \right\} \leftrightarrow \{\text{simple graphs } G \text{ on } n \text{ vertices}\}.$$

By a simple graph we mean an undirected graph with no loops or multiple edges, but not necessarily connected. The bijection is defined by mapping the graph G with edge set E_G and vertices $V_G = \{x_1, \dots, x_n\}$ to the square-free monomial ideal

$$\mathcal{I}(G) = (\{x_i x_j \mid \{x_i, x_j\} \in E_G\}) \subseteq k[x_1, \dots, x_n]. \quad (1.1)$$

(The graph of n isolated vertices is mapped to $I = (0)$ which we shall also consider as a square-free quadratic monomial ideal.) Note that (1.1) implies that $\beta_{0,j}(\mathcal{I}(G)) =$

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$|E_G|$ if $j = 2$, and 0 if $j \neq 2$. Thus, we shall be interested in formulas for $\beta_{i,j}(\mathcal{I}(G))$ with $i \geq 1$. The ideal $\mathcal{I}(G)$ is commonly called the *edge ideal* of G .

Edge ideals, which were first introduced by Villarreal [24], are the focus of an ongoing program in algebraic combinatorics. Many authors have been interested in establishing a dictionary between the algebraic invariants of $\mathcal{I}(G)$ and the combinatorial data associated to the graph G . The references [3, 5, 9, 10, 11, 13, 14, 17, 18, 19, 21, 23, 24, 25, 26, 27] form a partial list of references on this topic.

Because the edge ideal is a square-free monomial ideal, Hochster's formula [16] and its variant, the Eagon-Reiner formula [2], provide us with tools to study the numbers $\beta_{i,j}(\mathcal{I}(G))$. More precisely, the ideal $\mathcal{I}(G)$ can be associated with a simplicial complex $\Delta(G)$ via the Stanley-Reisner correspondence. The numbers $\beta_{i,j}(\mathcal{I}(G))$ are then related to the dimensions of the reduced homology groups of subcomplexes of $\Delta(G)$, or as in the case of Eagon and Reiner's formula, the Alexander dual of $\Delta(G)$. An examination of the papers [3, 11, 17, 18, 19, 21, 27] reveals that these formulas provide the basis for most of the known results on the numbers $\beta_{i,j}(\mathcal{I}(G))$. The exception to this observation is [27] which uses Koszul homology.

In this paper we use the notion of a *splittable monomial ideal*, as first defined by Eliahou and Kervaire [4], to introduce a new technique to the study the numbers $\beta_{i,j}(\mathcal{I}(G))$. Our approach, which has the advantage that we can avoid the highly nontrivial problem of computing the dimensions of reduced homology or Koszul homology groups, allows us to recover many of the known results with fuller generality, and at the same time, provides new results. The use of splittable monomial ideals also provides a unified combinatorial perspective for most of the known results.

A monomial ideal I is *splittable* if there exists two monomial ideals J and K such that $I = J + K$, and furthermore, the generators of $J \cap K$ satisfy certain technical conditions (see Definition 2.1). Splittable ideals allow us to relate $\beta_{i,j}(I)$ to the graded Betti numbers of the "smaller" ideals J, K and $J \cap K$ (see Theorem 2.2). Given an edge ideal $\mathcal{I}(G)$, our goal is to find a splitting $\mathcal{I}(G) = J + K$ so that J, K , and $J \cap K$ are related to edge ideals of subgraphs of G , and therefore produce a recursive like formula for the graded Betti numbers of $\mathcal{I}(G)$.

Sections 3 and 4 of the paper are devoted to two natural candidates for a splitting of $\mathcal{I}(G)$. If $e = \{u, v\}$ is any edge of G , then it is clear that

$$\mathcal{I}(G) = (uv) + \mathcal{I}(G \setminus e) \tag{1.2}$$

where $G \setminus e$ is the subgraph of G with the edge e removed. Similarly, if v is any vertex of G , and if $N(v) = \{v_1, \dots, v_d\}$ denotes the distinct neighbors of v , then

$$\mathcal{I}(G) = (vv_1, vv_2, \dots, vv_d) + \mathcal{I}(G \setminus \{v\}) \tag{1.3}$$

where $G \setminus \{v\}$ is the subgraph of G with vertex v and edges incident to v removed. Observe that (vv_1, \dots, vv_d) is the edge ideal of the complete bipartite graph $\mathcal{K}_{1,d}$.

In general (1.2) and (1.3) will not be splittings of $\mathcal{I}(G)$. We therefore call e a *splitting edge* of G if (1.2) is a splitting, and similarly, we say v is a *splitting vertex* if (1.3) is a splitting. Theorems 3.4 and 4.2 then characterize which edges and vertices of G can have this property. An edge $e = \{u, v\}$ is a splitting edge if the set of neighbors of u (or v) is a subset of $N(v) \cup \{v\}$ (or $N(u) \cup \{u\}$). Splitting vertices are more ubiquitous; a vertex v is a splitting vertex provided v is not an isolated vertex or the vertex of degree d of $\mathcal{K}_{1,d}$.

We adopt the convention that for any ideal I , $\beta_{-1,j}(I) = 1$ if $j = 0$, and $\beta_{-1,j}(I) = 0$ otherwise. Our first main result is the following formulas for $\beta_{i,j}(\mathcal{I}(G))$:

Theorem 1.1. *Let G be a simple graph with edge ideal $\mathcal{I}(G)$.*

(i) (Theorem 3.6) *Suppose $e = \{u, v\}$ is a splitting edge of G . Set $H = G \setminus (N(u) \cup N(v))$. If $n = |N(u) \cup N(v)| - 2$, then for all $i \geq 1$*

$$\beta_{i,j}(\mathcal{I}(G)) = \beta_{i,j}(\mathcal{I}(G \setminus e)) + \sum_{l=0}^i \binom{n}{l} \beta_{i-l-1, j-2-l}(\mathcal{I}(H)).$$

(ii) (Theorem 4.6) *Let v be a splitting vertex of G with $N(v) = \{v_1, \dots, v_d\}$. Set*

$$G_i := G \setminus (N(v) \cup N(v_i)) \text{ for } i = 1, \dots, d,$$

and let $G_{(v)}$ be the subgraph of G consisting of all edges incident to v_1, \dots, v_d except those that are also incident to v . Then for all $i, j \geq 0$

$$\beta_{i,j}(\mathcal{I}(G)) = \beta_{i,j}(\mathcal{I}(\mathcal{K}_{1,d})) + \beta_{i,j}(\mathcal{I}(G \setminus \{v\})) + \beta_{i-1,j}(L)$$

where $L = v\mathcal{I}(G_{(v)}) + vv_1\mathcal{I}(G_1) + \dots + vv_d\mathcal{I}(G_d)$.

Our formula in Theorem 1.1 (i) unifies all known results about $\beta_{i,j}(\mathcal{I}(G))$ when G is a forest. Since a leaf is a splitting edge, we recover the recursive formula for the graded Betti numbers of forests (cf. Corollary 3.9) as first given by Jacques and Katzman [17, 18]. At the same time our recursive formula is more general since it applies to all leaves, and not only the special leaf required in the argument of [17, 18]. Theorem 1.1 (i) also allows us to give a combinatorial proof (cf. Corollary 3.11) of Zheng's formula [27] for the regularity of $\mathcal{I}(G)$ in terms of the number of disconnected edges in G when G is a forest. The above formula fails to be recursive in general because the subgraphs may not contain splitting edges, thus preventing us from reiterating the process. However, it is still general enough to provide new results on the projective dimension and regularity of edge ideals (cf. Corollary 3.7).

The formula in Theorem 1.1 (ii) is not recursive because it involves computing $\beta_{i,j}(L)$ where L is not an edge ideal. Yet, this formula proves to be very effective in studying the *linear strand* of the minimal free resolution of edge ideals. We can give new combinatorial proofs for many of the results on the linear strand of edge ideals; for example, we can recover (cf. Corollary 4.7) a result of Eisenbud, et al. [3] on the $N_{2,p}$ property of edge ideals, a notion closely tied to the N_p property introduced by Green [12]. Precisely, Theorem 1.1 (ii) enables us to give a new proof to the fact that $\mathcal{I}(G)$ has property $N_{2,p}$ for $p > 1$ if and only if every minimal cycle of G^c , the complementary graph of G , has length at least $p + 3$. Theorem 1.1 (ii) also allows us to recover the formulas for the Betti numbers in the linear strand first shown in [21] (cf. Corollary 4.9). At the same time, we can provide new results on the projective dimension and regularity of edge ideals (cf. Corollary 4.4).

In the final section we extend the scope of this paper by considering the graded Betti numbers of facet ideals. The facet ideal was introduced by Faridi [6, 7] to generalize an edge ideal. Since any square-free monomial ideal can be realized as the facet ideal of a simplicial complex, our method thus works for a large class of monomial ideals. Let Δ be a simplicial complex on the vertex set $V_\Delta = \{x_1, \dots, x_n\}$. The *facet ideal* $\mathcal{I}(\Delta)$ of Δ is defined to be

$$\mathcal{I}(\Delta) = \left(\left\{ \prod_{x \in F} x \mid F \text{ is a facet of } \Delta \right\} \right) \subseteq k[x_1, \dots, x_n].$$

If F is a facet of Δ , then we say that F is a *splitting facet* if $\mathcal{I}(\Delta) = (\prod_{x \in F} x) + \mathcal{I}(\Delta')$, where $\Delta' = \Delta \setminus F$ is the subcomplex of Δ with the facet F removed, is a splitting of $\mathcal{I}(\Delta)$. Our next main result is a higher dimension analogue of Theorem 1.1 (i) relating the graded Betti number of $\mathcal{I}(\Delta)$ to those of facet ideals of subcomplexes of Δ ; see Definition 5.1 for unexplained terminology.

Theorem 1.2 (Theorem 5.5). *Let F be a splitting facet of a simplicial complex Δ . Let $\Delta' = \Delta \setminus F$ and $\Omega = \Delta \setminus \text{conn}_\Delta(F)$. Then for all $i \geq 1$ and $j \geq 0$,*

$$\beta_{i,j}(\mathcal{I}(\Delta)) = \beta_{i,j}(\mathcal{I}(\Delta')) + \sum_{l_1=0}^i \sum_{l_2=0}^{j-|F|} \beta_{l_1-1,l_2}(\mathcal{I}(\overline{\text{conn}}_\Delta(F))) \beta_{i-l_1-1,j-|F|-l_2}(\mathcal{I}(\Omega)).$$

Similar to Theorem 1.1, our formula in Theorem 1.2 is recursive for simplicial forests (cf. Theorems 5.6 and 5.8). Consequently, there exists a large class of square-free monomial ideals that can be examined via a recursive formula. The recursive formula of Jacques and Katzman for forests [18] becomes a special case of this result. Moreover, formulas for the graded Betti numbers in the linear strand of facet ideals of simplicial forests are recovered, generalizing results of Zheng [27].

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2. PRELIMINARIES

For completeness we gather together the needed results and definitions on simple graphs, simplicial complexes, resolutions, and splittable ideals. Readers familiar with this material may wish to continue directly to the next section.

2.1. Graph terminology, simplicial complexes, edge and facet ideals. In this paper G will denote a finite simple graph (undirected, no loops or multiple edges, but not necessarily connected). We denote by V_G and E_G the set of vertices and edges, respectively, of G .

If $V_G = \{x_1, \dots, x_n\}$, then we associate to G a polynomial ring $R = k[x_1, \dots, x_n]$ (here, by abuse of notation, we use the x_i s to denote both the vertices in V_G and the variables in the polynomial ring). For simplicity we write $uv \in E_G$ instead of $\{u, v\} \in E_G$. Also, by abuse of notation, we use uv to denote both the edge uv and the monomial uv in the edge ideal. In particular, $\mathcal{I}(G) = (\{uv \mid uv \in E_G\}) \subseteq R$.

A vertex y is a *neighbor* of x if $xy \in E_G$. Set $N(x) := \{y \in V_G \mid xy \in E_G\}$, the set of all neighbors of x in G . The *degree* of a vertex $x \in V_G$, denoted by $\deg_G x$, is the number of edges incident to x . When there is no confusion, we shall omit G and write $\deg x$. Observe that $\deg x = |N(x)|$ since G is simple.

If $e \in E_G$, we shall write $G \setminus e$ for the subgraph of G with the edge e deleted. If $S = \{x_{i_1}, \dots, x_{i_s}\} \subseteq V_G$, we shall write $G \setminus S$ for the subgraph of G with the vertices of S (and their incident edges) deleted. We further write G_S to denote the *induced subgraph* of G on S (i.e., the subgraph of G whose vertex set is S and whose edges are edges of G connecting vertices in S). We say that $C = (x_1 x_2 \dots x_l x_1)$ is a *cycle* of G if $x_i x_{i+1} \in E_G$ for $i = 1, \dots, l$ (where $x_{l+1} = x_1$). The *complete graph* \mathcal{K}_n of size n is the graph whose vertex set V has n vertices and whose edges are $\{uv \mid u \neq v \in V\}$. A complete graph \mathcal{K}_n which is a subgraph of G is called a

n -clique of G . The *complete bipartite* graph $\mathcal{K}_{m,n}$ is the graph whose vertex set can be divided into two disjoint subsets A and B such that $|A| = m, |B| = n$, and the edges of the graph are $\{uv \mid u \in A, v \in B\}$.

A *simplicial complex* Δ over a vertex set $V_\Delta = \{x_1, \dots, x_n\}$ is a collection of subsets of V_Δ , with the property that $\{x_i\} \in \Delta$ for all i , and if $F \in \Delta$ then all subsets of F are also in Δ . Elements of Δ are called *faces*. The *dimension* of a face F , denoted by $\dim F$, is defined to be $|F| - 1$, where $|F|$ denotes the cardinality of F . The *dimension* of Δ , denoted by $\dim \Delta$, is defined to be the maximal dimension of a face in Δ . The maximal faces of Δ under inclusion are called *facets*. If all facets of Δ have the same dimension d , then Δ is said to be *pure d -dimensional*.

We usually denote the simplicial complex Δ with facets F_1, \dots, F_q by $\Delta = \langle F_1, \dots, F_q \rangle$; here, the set $\mathcal{F}(\Delta) = \{F_1, \dots, F_q\}$ is often referred to as the *facet set* of Δ . If F is a facet of Δ , say $F = F_q$, then we denote by $\Delta \setminus F$ the simplicial complex obtained by removing F from the facet set of Δ , i.e., $\Delta \setminus F = \langle F_1, \dots, F_{q-1} \rangle$. Throughout the paper, by a *subcomplex* of a simplicial complex Δ , we shall mean a simplicial complex whose facet set is a subset of the facet set of Δ . If Δ' is a subcomplex of Δ , then we denote by $\Delta \setminus \Delta'$ the simplicial complex obtained from Δ by removing from its facet set all facets of Δ' .

We say that two facets F and G of Δ are *connected* if there exists a *chain* of facets of Δ , $F = F_0, F_1, \dots, F_m = G$, such that $F_i \cap F_{i+1} \neq \emptyset$ for any $i = 0, \dots, m-1$. The simplicial complex Δ is said to be *connected* if any two of its facets are connected.

To a simplicial complex Δ over the vertex set $V_\Delta = \{x_1, \dots, x_n\}$ we associate an ideal $\mathcal{I}(\Delta)$ in the polynomial ring $R = k[x_1, \dots, x_n]$. We write F to denote both a facet of Δ and the monomial $\prod_{x \in F} x$. In particular, $\mathcal{I}(\Delta) = (\{F \mid F \in \mathcal{F}(\Delta)\}) \subseteq R$.

A facet F of Δ is a *leaf* if either F is the only facet of Δ , or there exists a facet G in Δ , $G \neq F$, such that $F \cap G \subseteq F' \cap G$ for every facet $F' \in \Delta$, $F' \neq F$. The simplicial complex Δ is called a *tree* if Δ is connected and every nonempty connected subcomplex of Δ (including Δ itself) has a leaf. We call Δ a *forest* if every connected component of Δ is a tree.

2.2. Resolutions, Betti numbers, and splittable ideals. Let $\mathcal{G}(I)$ denote the minimal set of generators of a monomial ideal I ; this set is uniquely determined (cf. Lemma 1.2 of [20]). The following definition and result play an essential role throughout the paper.

Definition 2.1 (see [4]). A monomial ideal I is *splittable* if I is the sum of two nonzero monomial ideals J and K , that is, $I = J + K$, such that

- (1) $\mathcal{G}(I)$ is the disjoint union of $\mathcal{G}(J)$ and $\mathcal{G}(K)$.
- (2) there is a *splitting function*

$$\begin{aligned} \mathcal{G}(J \cap K) &\rightarrow \mathcal{G}(J) \times \mathcal{G}(K) \\ w &\mapsto (\phi(w), \psi(w)) \end{aligned}$$

satisfying

- (a) for all $w \in \mathcal{G}(J \cap K)$, $w = \text{lcm}(\phi(w), \psi(w))$.
- (b) for every subset $S \subset \mathcal{G}(J \cap K)$, both $\text{lcm}(\phi(S))$ and $\text{lcm}(\psi(S))$ strictly divide $\text{lcm}(S)$.

If J and K satisfy the above properties, then we say $I = J + K$ is a *splitting* of I .

Theorem 2.2 (Eliahou-Kervaire [4] Fatabbi [8]). *Suppose I is a splittable monomial ideal with splitting $I = J + K$. Then for all $i, j \geq 0$,*

$$\beta_{i,j}(I) = \beta_{i,j}(J) + \beta_{i,j}(K) + \beta_{i-1,j}(J \cap K).$$

Recall that for an ideal I generated by elements of degree at least d , the Betti numbers $\beta_{i,i+d}(I)$ form the so-called *linear strand* of I (see [3, 15]). An ideal I generated by elements of degree d is said to have a *linear resolution* if the only nonzero graded Betti numbers are those in the linear strand. Of particular interest are also the following invariants which measure the “size” of the minimal graded free resolution of I . The *regularity* of I , denoted $\text{reg}(I)$, is defined by

$$\text{reg}(I) := \max\{j - i \mid \beta_{i,j}(I) \neq 0\}.$$

The *projective dimension* of I , denoted $\text{pd}(I)$, is defined to be

$$\text{pd}(I) := \max\{i \mid \beta_{i,j}(I) \neq 0\}.$$

When I is a splittable ideal, Theorem 2.2 implies the following result:

Theorem 2.3. *If I is a splittable monomial ideal with splitting $I = J + K$, then*

- (i) $\text{reg}(I) = \max\{\text{reg}(J), \text{reg}(K), \text{reg}(J \cap K) - 1\}$.
- (ii) $\text{pd}(I) = \max\{\text{pd}(J), \text{pd}(K), \text{pd}(J \cap K) + 1\}$.

The following results shall be required throughout the paper. The lemma is well known. See, for example, Lemma 2.1 and Corollary 2.2 of [18].

Lemma 2.4. *Let $R = k[x_1, \dots, x_n]$ and $S = k[y_1, \dots, y_m]$, and let $I \subseteq R$ and $J \subseteq S$ be homogeneous ideals. Then*

$$\beta_{i,j}(R/I \otimes S/J) = \sum_{l_1=0}^i \sum_{l_2=0}^j \beta_{l_1,l_2}(R/I) \beta_{i-l_1,j-l_2}(S/J).$$

Remark 2.5. If $I, J \subseteq R = k[x_1, \dots, x_n]$ are square-free monomial ideals such that none of the x_i s appearing in the minimal generators of I appear in the minimal generators of J , then $R/I \otimes R/J = R/(I + J)$. Lemma 2.4 thus implies

$$\beta_{i,j}(R/(I + J)) = \sum_{l_1=0}^i \sum_{l_2=0}^j \beta_{l_1,l_2}(R/I) \beta_{i-l_1,j-l_2}(R/J).$$

Remark 2.6. If G is a simple graph with two connected components, i.e., $G = G_1 \cup G_2$, with $V_G = V_{G_1} \cup V_{G_2}$ and $V_{G_1} \cap V_{G_2} = \emptyset$, then Remark 2.5 implies that to calculate $\beta_{i,j}(\mathcal{I}(G))$, it is enough to calculate the graded Betti numbers of the edge ideals $\mathcal{I}(G_1)$ and $\mathcal{I}(G_2)$. More generally, if G has $n \geq 2$ components, by repeated applying Remark 2.5, to calculate $\beta_{i,j}(\mathcal{I}(G))$ it suffices to calculate the Betti numbers of the edge ideals associated to each connected component of G .

Theorem 2.7. *Suppose that $G = \mathcal{K}_{1,d}$. Then for $i \geq 0$*

$$\beta_{i,j}(\mathcal{I}(G)) = \begin{cases} \binom{d}{i+1} & \text{if } j = i + 2 \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Since $G = \mathcal{K}_{1,d}$, it follows that $\mathcal{I}(G) = (vv_1, \dots, vv_d) \subseteq R = k[v, v_1, \dots, v_d]$. The conclusion now follows from the fact that v_1, \dots, v_d is a regular sequence on R , and that $\beta_{i,j}(\mathcal{I}(G)) = \beta_{i,j-1}((v_1, \dots, v_d))$. \square

3. SPLITTING EDGES

Let G be a simple graph with edge ideal $\mathcal{I}(G)$ and $e = uv \in E_G$. If we set $J = (uv)$ and $K = \mathcal{I}(G \setminus e)$, then $\mathcal{I}(G) = J + K$. In general this may not be a splitting of $\mathcal{I}(G)$. The goal of this section is to determine when J and K give a splitting of $\mathcal{I}(G)$, and furthermore, how this splitting can be used to ascertain information about the numbers $\beta_{i,j}(\mathcal{I}(G))$.

We begin by assigning a name to an edge for which there is a splitting.

Definition 3.1. An edge $e = uv$ is a *splitting edge* if $\mathcal{I}(G) = (uv) + \mathcal{I}(G \setminus e)$ is a splitting.

Lemma 3.2. Let $J = (uv)$ and $K = \mathcal{I}(G \setminus e)$ with $e = uv \in E_G$. If $N(u) \setminus \{v\} = \{u_1, \dots, u_n\}$, $N(v) \setminus \{u\} = \{v_1, \dots, v_m\}$, and $H = G \setminus (N(u) \cup N(v))$, then

$$J \cap K = uv((u_1, \dots, u_n, v_1, \dots, v_m) + \mathcal{I}(H)).$$

Proof. Because $J = (uv)$ and $K = \mathcal{I}(G \setminus e)$ are both monomial ideals,

$$J \cap K = (\{\text{lcm}(uv, m) \mid m \in \mathcal{G}(K)\}).$$

Each $m \in \mathcal{G}(K)$ corresponds to an edge of $G \setminus e$. There are three cases for this edge: (1) it is incident to either u or v , (2) it is not incident to u or v , but is incident to a neighbor of either u or v , or (3) it is not incident to any vertex in $N(u) \cup N(v)$.

If m is in cases (1) and (2), then $\text{lcm}(uv, m)$ is in $uv(u_1, \dots, u_n, v_1, \dots, v_m)$. If m is in case (3), $\text{lcm}(uv, m)$ belongs to $uv\mathcal{I}(H)$. The statement follows. \square

We in fact obtain the following description for $\mathcal{G}(J \cap K)$.

Corollary 3.3. Let $e = uv \in E_G$, $J = (uv)$ and $K = \mathcal{I}(G \setminus e)$. If $A = N(u) \setminus \{v\}$ and $B = N(v) \setminus \{u\}$, then

$$\begin{aligned} \mathcal{G}(J \cap K) = & \{uvu_i \mid u_i \in A \setminus B\} \cup \{uvv_i \mid v_i \in B \setminus A\} \cup \{uvwz_i \mid z_i \in A \cap B\} \cup \\ & \{uvm \mid m \in \mathcal{I}(H)\}. \end{aligned}$$

The above description of $\mathcal{G}(J \cap K)$ will enable us to identify splitting edges.

Theorem 3.4. An edge $e = uv$ is a splitting edge of G if and only if $N(u) \subseteq (N(v) \cup \{v\})$ or $N(v) \subseteq (N(u) \cup \{u\})$.

Proof. (\Leftarrow). Without loss of generality, we shall assume that $N(u) \subseteq (N(v) \cup \{v\})$. This condition and Corollary 3.3 then imply that

$$\mathcal{G}(J \cap K) = \{uvv_i \mid v_i \in N(v) \setminus \{u\}\} \cup \{uvm \mid m \in \mathcal{I}(H)\}.$$

To show that $e = uv$ is splitting edge, it suffices to verify that the function $\mathcal{G}(J \cap K) \rightarrow \mathcal{G}(J) \times \mathcal{G}(K)$ defined by

$$w \mapsto (\phi(w), \psi(w)) = \begin{cases} (uv, vv_i) & \text{if } w = uvv_i \\ (uv, m) & \text{if } w = uvm \end{cases}$$

satisfies conditions (a) and (b) of Definition 2.1. Indeed, condition (a) is immediate. So, suppose $S \subseteq \mathcal{G}(J \cap K)$. Our description of $\mathcal{G}(J \cap K)$ implies all elements of S are divisible by uv . Moreover, $\text{lcm}(S)$ must have degree at least three. Thus, $\text{lcm}(\phi(S)) = uv$ strictly divides $\text{lcm}(S)$. Furthermore, u does not divide $\text{lcm}(\psi(S))$ implying that $\text{lcm}(\psi(S))$ strictly divides $\text{lcm}(S)$. Condition (b) now follows.

(\Rightarrow) We prove the contrapositive. Suppose that $e = uv$ is an edge such that $N(u) \not\subseteq (N(v) \cup \{v\})$ and $N(v) \not\subseteq (N(u) \cup \{u\})$. Hence, there exists vertices x and y such that $ux, vy \in E_G$, but $uy, vx \notin E_G$.

We now show that no splitting function can exist. If there was a splitting function $\mathcal{G}(J \cap K) \rightarrow \mathcal{G}(J) \times \mathcal{G}(K)$ our splitting function must have the form

$$w \mapsto (\phi(w), \psi(w)) = (uv, \psi(w))$$

since $\mathcal{G}(J) = \{uv\}$. By Corollary 3.3 it follows that $uvx, uvy \in \mathcal{G}(J \cap K)$. Condition (a) of Definition 2.1, would imply $uvx = \text{lcm}(\phi(uvx), \psi(uvx)) = \text{lcm}(uv, \psi(uvx))$. Thus $\psi(uvx) = x, vx, ux$ or uvx . But since $\psi(uvx) \in \mathcal{G}(K)$ and $vx \notin E_G$, this forces $\psi(uvx) = ux$. By a similar argument, $\psi(uvy) = vy$.

The subset $S = \{uvx, uvy\} \subseteq \mathcal{G}(J \cap K)$ now fails to satisfy Definition 2.1 (b) since $\text{lcm}(S) = \text{lcm}(\psi(S)) = uvxy$. Thus $e = uv$ is not a splitting edge. \square

The following identity for the numbers of $\beta_{i-1,j}(J \cap K)$ can now be derived.

Lemma 3.5. *Let $e = uv$ be a splitting edge of G with $N(u) \subseteq (N(v) \cup \{v\})$. If $N(v) \setminus \{v\} = \{v_1, \dots, v_n\}$, $J = (uv)$, and $K = \mathcal{I}(G \setminus e)$, then for $i \geq 1$ and all $j \geq 0$*

$$\beta_{i-1,j}(J \cap K) = \sum_{l=0}^i \binom{n}{l} \beta_{i-l-1,j-2-l}(\mathcal{I}(H))$$

where $\mathcal{I}(H)$ is the edge ideal of $H = G \setminus \{u, v, v_1, \dots, v_n\}$.

Proof. When $e = uv$ is a splitting edge, the conclusion of Lemma 3.2 becomes

$$J \cap K = uv((v_1, \dots, v_n) + \mathcal{I}(H)).$$

where $H = G \setminus \{u, v, v_1, \dots, v_n\}$. Set $L = (v_1, \dots, v_n) + \mathcal{I}(H)$. Since no generator of L is divisible by either u or v , we have that uv is a nonzero divisor on R/L . As a consequence

$$\beta_{i-1,j}(uvL) = \beta_{i-1,j-2}(L) = \beta_{i,j-2}(R/L).$$

Observe that none of the generators of $\mathcal{I}(H)$ are divisible by v_i for $i = 1, \dots, n$. Now apply Remark 2.5 to compute $\beta_{i,j-2}(R/L)$ and use the fact that the graded Betti numbers of $R/(v_1, \dots, v_n)$ are given by the Koszul resolution. \square

We now state and prove the main theorem of this section.

Theorem 3.6. *Let $e = uv$ be a splitting edge of G , and set $H = G \setminus (N(u) \cup N(v))$. If $n = |N(u) \cup N(v)| - 2$, then for all $i \geq 1$ and all $j \geq 0$*

$$\beta_{i,j}(\mathcal{I}(G)) = \beta_{i,j}(\mathcal{I}(G \setminus e)) + \sum_{l=0}^i \binom{n}{l} \beta_{i-l-1,j-2-l}(\mathcal{I}(H)).$$

Proof. Because $e = uv$ is a splitting edge, we can assume without loss of generality that $N(u) \subseteq (N(v) \cup \{v\})$. So $N(u) \cup N(v) = \{u, v, v_1, \dots, v_n\}$ with $\{v_1, \dots, v_n\} = N(v) \setminus \{v\}$. The desired formula is a result of combining Theorem 2.2 with Lemma 3.5 and using the fact that $\beta_{i,j}((uv)) = 0$ if $i \geq 1$. \square

Corollary 3.7. *With the hypotheses and notation as in Theorem 3.6, we have*

- (i) $\text{reg}(\mathcal{I}(G)) = \max\{2, \text{reg}(\mathcal{I}(G \setminus e)), \text{reg}(\mathcal{I}(H)) + 1\}$.
- (ii) $\text{pd}(\mathcal{I}(G)) = \max\{\text{pd}(\mathcal{I}(G \setminus e)), \text{pd}(\mathcal{I}(H)) + n + 1\}$.

Proof. Set $L = J \cap K$ and $G' = G \setminus e$. By Corollary 2.3 we have

$$\operatorname{reg}(\mathcal{I}(G)) = \max\{\operatorname{reg}((uv)), \operatorname{reg}(\mathcal{I}(G')), \operatorname{reg}(L) - 1\}.$$

Since $\operatorname{reg}((uv)) = 2$, we only need to verify that $\operatorname{reg}(L) = \operatorname{reg}(\mathcal{I}(H)) + 2$. This is indeed true by Lemma 3.5. This proves (i). Similarly, Corollary 2.3 implies

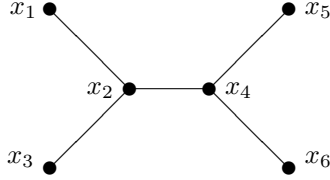
$$\operatorname{pd}(\mathcal{I}(G)) = \max\{\operatorname{pd}((uv)), \operatorname{pd}(\mathcal{I}(G')), \operatorname{pd}(L) + 1\}.$$

Now clearly $\operatorname{pd}((uv)) = 0$. By Lemma 3.5 we have

$$\operatorname{pd}(L) = \operatorname{pd}(R/((v_1, \dots, v_n) + \mathcal{I}(H))) - 1 = n + \operatorname{pd}(R/\mathcal{I}(H)) - 1.$$

Since $\operatorname{pd}(R/\mathcal{I}(H)) = \operatorname{pd}(\mathcal{I}(H)) + 1$, the assertion (ii) follows. \square

Example 3.8. The above corollary implies that removing a splitting edge e may decrease both the regularity and projective dimension, that is, $\operatorname{reg}(\mathcal{I}(G)) \geq \operatorname{reg}(\mathcal{I}(G \setminus e))$ and $\operatorname{pd}(\mathcal{I}(G)) \geq \operatorname{pd}(\mathcal{I}(G \setminus e))$. However, if e is not a splitting edge, then it may happen that $\operatorname{reg}(\mathcal{I}(G \setminus e))$, respectively $\operatorname{pd}(\mathcal{I}(G \setminus e))$, is larger than $\operatorname{reg}(\mathcal{I}(G))$, respectively $\operatorname{pd}(\mathcal{I}(G))$. For example, consider the graph G below:



The edge x_2x_4 is not a splitting edge. The resolution of $\mathcal{I}(G)$ is

$$0 \rightarrow R^2(-4) \rightarrow R^6(-3) \rightarrow R^5(-2) \rightarrow \mathcal{I}(G) \rightarrow 0$$

and the resolution of $\mathcal{I}(G \setminus e)$ is

$$0 \rightarrow R(-6) \rightarrow R^4(-5) \rightarrow R^2(-3) \oplus R^4(-4) \rightarrow R^4(-2) \rightarrow \mathcal{I}(G \setminus e) \rightarrow 0.$$

We have $\operatorname{pd}(\mathcal{I}(G \setminus e)) = 3 > 2 = \operatorname{pd}(\mathcal{I}(G))$ and $\operatorname{reg}(\mathcal{I}(G \setminus e)) = 3 > 2 = \operatorname{reg}(\mathcal{I}(G))$.

We end this section by using Theorem 3.6 to give new proofs for known results about the the graded Betti numbers of forests. We begin by recovering the recursive formula of [17, 18] found via a different means. In fact, our result is more general since it applies to any leaf of G , while [17, 18] required that a special leaf be removed.

Corollary 3.9. *Let $e = uv$ be any leaf of a forest G . If $\deg v = n$ and $N(v) = \{u, v_1, \dots, v_{n-1}\}$, then for $i \geq 1$ and $j \geq 0$*

$$\beta_{i,j}(\mathcal{I}(G)) = \beta_{i,j}(\mathcal{I}(T)) + \sum_{l=0}^i \binom{n-1}{l} \beta_{i-l-1, j-2-l}(\mathcal{I}(H))$$

where $T = G \setminus e = G \setminus \{u\}$ and $H = G \setminus \{u, v, v_1, \dots, v_{n-1}\}$.

Proof. The hypotheses imply that $\deg u = 1$. Since $N(u) \subseteq (N(v) \cup \{v\})$, uv is a splitting edge. Now apply Theorem 3.6. \square

Applying Corollary 3.7 allows us to rediscover Theorem 4.8 of [18].

Corollary 3.10. *With the notation as in the previous corollary,*

$$\operatorname{pd}(\mathcal{I}(G)) = \max\{\operatorname{pd}(\mathcal{I}(T)), \operatorname{pd}(\mathcal{I}(H)) + n\}.$$

We say two edges u_1v_1 and u_2v_2 of a simple graph G are *disconnected* if (a) $\{u_1, v_1\} \cap \{u_2, v_2\} = \emptyset$, and (b) $u_1u_2, u_1v_2, v_1u_2, v_1v_2$ are not edges of G . When G is a forest, Theorem 3.6 can be used to give a new proof of Zheng's result (Theorem 2.18 of [27]) relating $\text{reg}(\mathcal{I}(G))$ to the number of disconnected edges.

Corollary 3.11. *Let G be a forest with edge ideal $\mathcal{I}(G)$. Then $\text{reg}(\mathcal{I}(G)) = j + 1$ where j is the maximal number of pairwise disconnected edges in G .*

Proof. We use induction on $|E_G|$. The formula is clearly true for $|E_G| = 1$.

Suppose $|E_G| > 1$, and let $e = uv$ be any leaf of G with $\deg u = 1$. By Corollary 3.7 we have

$$\text{reg}(\mathcal{I}(G)) = \max\{2, \text{reg}(\mathcal{I}(T)), \text{reg}(\mathcal{I}(H)) + 1\}$$

where $T = G \setminus e = G \setminus \{u\}$ and $H = G \setminus (\{v\} \cup N(v))$. By induction $\text{reg}(\mathcal{I}(T)) = j_1 + 1$ where j_1 is the maximal number of pairwise disconnected edges of T , and $\text{reg}(\mathcal{I}(H)) = j_2 + 1$ where j_2 is the maximal number of pairwise disconnected edges of H . Since $\mathcal{I}(T)$ has at least one edge, $j_1 + 1 \geq 2$. So

$$\text{reg}(\mathcal{I}(G)) = \max\{j_1 + 1, j_2 + 2\}.$$

If we let j denote the maximal number of pairwise disconnected edges of G , then to complete the proof it suffices for us to show that $j = \max\{j_1, j_2 + 1\}$.

Let \mathcal{E}_1 be the set of the j_1 pairwise disconnected edges of T . The edges of \mathcal{E}_1 are also a set of pairwise disconnected edges of G . Thus $|\mathcal{E}_1| = j_1 \leq j$. If \mathcal{E}_2 is a set of j_2 pairwise disconnected edges of H , we claim that $\mathcal{E}_2 \cup \{uv\}$ is a set of pairwise disconnected edges of G . Indeed, uv does not share a vertex with any edge in H . The only edges that are adjacent to uv are vv_i with $v_i \in N(v) \setminus \{u\}$. No edge of \mathcal{E}_2 can share a vertex with these edges since none of the vertices of $N(v)$ belong to H . Thus $|\mathcal{E}_2 \cup \{uv\}| = j_2 + 1 \leq j$. Thus $j \geq \max\{j_1, j_2 + 1\}$.

Suppose that $j > \max\{j_1, j_2 + 1\}$. Let \mathcal{E} be a set of j pairwise disconnected edges of G . If $uv \notin \mathcal{E}$, then \mathcal{E} is also a set of pairwise disconnected edges of T , and so $j = |\mathcal{E}| \leq j_1$, a contradiction. If $uv \in \mathcal{E}$, then $\mathcal{E} \setminus \{uv\}$ is a set of pairwise disconnected edges of H . But this would imply that $j - 1 \leq j_2$, again a contradiction. Hence $j = \max\{j_1, j_2 + 1\}$. \square

4. SPLITTING VERTICES

Let G be a simple graph, and let v be a vertex of G with $N(v) = \{v_1, \dots, v_d\}$. This section complements the results of the previous section by determining when $\mathcal{I}(G) = J + K$ with $J = (vv_1, \dots, vv_d)$ and $K = \mathcal{I}(G \setminus \{v\})$ is a splitting of $\mathcal{I}(G)$.

If v is an isolated vertex of G , then $\beta_{i,j}(\mathcal{I}(G)) = \beta_{i,j}(\mathcal{I}(G \setminus \{v\}))$ for all $i, j \geq 0$. If $\deg v = d > 0$ and if $G \setminus \{v\}$ consists of isolated vertices, then $G = \mathcal{K}_{1,d}$, the complete bipartite graph of size $1, d$; in this case the graded Betti numbers of $\mathcal{I}(G)$ follow from Theorem 2.7. If $v \in V_G$ is in neither of these two cases, we give it the following name.

Definition 4.1. A vertex $v \in V_G$ is a *splitting vertex* if $\deg v = d > 0$ and $G \setminus \{v\}$ is not the graph of isolated vertices.

This name makes sense in light of the following theorem.

Theorem 4.2. *Let v be a splitting vertex of G with $N(v) = \{v_1, \dots, v_d\}$, and set $J = (vv_1, \dots, vv_d)$ and $K = \mathcal{I}(G \setminus \{v\})$. Then $\mathcal{I}(G) = J + K$ is a splitting of $\mathcal{I}(G)$.*

Proof. It is clear that $\mathcal{I}(G) = J + K$. As well, $\mathcal{G}(\mathcal{I}(G)) = \mathcal{G}(J) \cup \mathcal{G}(K)$ is a disjoint union because v divides all elements of $\mathcal{G}(J)$ but divides no element of $\mathcal{G}(K)$.

Now consider the ideal $J \cap K = (vv_1, \dots, vv_d) \cap \mathcal{I}(H)$ where $H = G \setminus \{v\}$. Then

$$J \cap K = (\{\text{lcm}(m_1, m_2) \mid m_1 \in \{vv_1, \dots, vv_d\}, m_2 \in \mathcal{G}(\mathcal{I}(H))\}).$$

Thus

$$\begin{aligned} \mathcal{G}(J \cap K) &= \{vv_iv_j \mid v_iv_j \in E_G\} \cup \{vv_iv_j \mid v_iv_j \in E_G\} \cup \\ &\quad \{vv_iv_j y_k \mid y_j y_k \in E_G \text{ but } v_iv_j, v_iv_k \notin E_G\} \end{aligned} \quad (4.1)$$

where y_i denotes a vertex in $V_G \setminus \{v, v_1, \dots, v_d\}$. Note that the three sets are disjoint.

We define a splitting function $\mathcal{G}(J \cap K) \rightarrow \mathcal{G}(J) \times \mathcal{G}(K)$ as follows. If $w \in \mathcal{G}(J \cap K)$, then define $\phi : \mathcal{G}(J \cap K) \rightarrow \mathcal{G}(J)$ and $\psi : \mathcal{G}(J \cap K) \rightarrow \mathcal{G}(K)$ by

$$\phi(w) = \begin{cases} vv_i & \text{if } w = vv_iv_j \text{ and } i < j \\ vv_i & \text{if } w = vv_iv_j \\ vv_i & \text{if } w = vv_iv_j y_k \end{cases} \quad \text{and } \psi(w) = \begin{cases} v_iv_j & \text{if } w = vv_iv_j \\ v_iv_j & \text{if } w = vv_iv_j \\ y_j y_k & \text{if } w = vv_iv_j y_k. \end{cases}$$

By construction, the map given by $w \mapsto (\phi(w), \psi(w))$ has the property that $w = \text{lcm}(\phi(w), \psi(w))$. It suffices to verify condition (b) of (2) in Definition 2.1.

So, suppose $S \subseteq \mathcal{G}(J \cap K)$. If S contains a monomial divisible by some variable $y \notin \{v, v_1, \dots, v_d\}$, then $\text{lcm}(\phi(S))$ strictly divides $\text{lcm}(S)$ since y does not divide $\text{lcm}(\phi(S))$. Otherwise, we must have $S \subseteq \{vv_iv_j \mid v_iv_j \in E_G\}$. In this case, let f be the maximal index such that v_f appears in a monomial of S . Then, by the definition of ϕ , v_f does not divide $\phi(w)$ for any $w \in S$. Thus, v_f does not divide $\text{lcm}(\phi(S))$. Therefore, $\text{lcm}(\phi(S))$ strictly divides $\text{lcm}(S)$. It is clear that $\text{lcm}(\psi(S))$ strictly divides $\text{lcm}(S)$ because v does not divide $\text{lcm}(\psi(S))$. The theorem is proved. \square

The following result is an immediate consequence of our description in (4.1).

Corollary 4.3. *With the notation as in the previous theorem, set*

$$\begin{aligned} G_i &:= G \setminus (N(v) \cup N(v_i)) \text{ for } i = 1, \dots, d, \text{ and} \\ G_{(v)} &:= G_{\{v_1, \dots, v_d\}} \cup \{e \in E_G \mid e \text{ incident to one of } v_1, \dots, v_d, \text{ but not } v\}. \end{aligned}$$

Then

$$J \cap K = v\mathcal{I}(G_{(v)}) + vv_1\mathcal{I}(G_1) + vv_2\mathcal{I}(G_2) + \dots + vv_d\mathcal{I}(G_d).$$

Theorem 4.2 gives us some partial results on how the projective dimension and regularity behave under removing any (splitting or non-splitting) vertex.

Corollary 4.4. *Let G be a simple graph, and let $v \in V_G$ be any vertex. Then*

- (i) $\text{reg}(\mathcal{I}(G)) \geq \max\{2, \text{reg}(\mathcal{I}(G \setminus \{v\}))\}$.
- (ii) $\text{pd}(\mathcal{I}(G)) \geq \max\{d - 1, \text{pd}(\mathcal{I}(G \setminus \{v\}))\}$ where $d = \deg v$.

Proof. If v is not a splitting vertex, then (i) and (ii) are immediate from the fact that either $\mathcal{I}(G) = \mathcal{I}(G \setminus \{v\})$, or $\mathcal{I}(G) = \mathcal{I}(\mathcal{K}_{1,d})$ and $\mathcal{I}(G \setminus \{v\}) = (0)$. If v is a splitting vertex, then $\mathcal{I}(G) = (vv_1, \dots, vv_d) + \mathcal{I}(G \setminus \{v\})$ is a splitting. Now use Corollary 2.3 and the fact that $\text{reg}(\mathcal{I}(\mathcal{K}_{1,d})) = 2$ and $\text{pd}(\mathcal{I}(\mathcal{K}_{1,d})) = d - 1$. \square

Remark 4.5. Jacques proved (Proposition 2.1.4 of [17]) statement (ii) when the vertex v is a terminal vertex, i.e., adjacent to at most one other vertex of G .

Applying Theorems 2.2 and 4.2 and Corollary 4.3 we obtain our next main result.

Theorem 4.6. *Let v be a splitting vertex of G with $N(v) = \{v_1, \dots, v_d\}$. Let $G_{(v)}$ and G_i ($i = 1, \dots, d$) be defined as in Corollary 4.3. Then*

$$\beta_{i,j}(\mathcal{I}(G)) = \beta_{i,j}(\mathcal{I}(\mathcal{K}_{1,d})) + \beta_{i,j}(\mathcal{I}(G \setminus \{v\})) + \beta_{i-1,j}(L)$$

where $L = v\mathcal{I}(G_{(v)}) + vv_1\mathcal{I}(G_1) + \dots + vv_d\mathcal{I}(G_d)$ and $\mathcal{K}_{1,d}$ is the complete bipartite graph of size $1, d$.

Theorem 4.6 allows us to give a new combinatorial proof for an interesting result due to Eisenbud, et al. [3]. We say that a cycle $C = (x_1x_2 \dots x_qx_1)$ of G has a *chord* if there exists some $j \not\equiv i+1 \pmod{q}$ such that x_ix_j is an edge of C . We call a cycle C a *minimal cycle* if C has length at least 4 and contains no chord. An ideal I is said to satisfy *property $N_{2,p}$* for some $p \geq 1$ if I is generated by quadratics and its minimal free resolution is linear up to the p th step, i.e., $\beta_{i,j}(I) = 0$ for all $0 \leq i < p$ and $j > i+2$.

Corollary 4.7 (see Theorem 2.1 of [3]). *Let G be a simple graph with edge ideal $\mathcal{I}(G)$. Then $\mathcal{I}(G)$ satisfies property $N_{2,p}$ with $p > 1$ if and only if every minimal cycle in G^c has length $\geq p+3$.*

Proof. We use induction on $n = |V_G|$. Our assertion is vacuously true for $n \leq 3$. Suppose $n \geq 4$. We may assume that G has no isolated vertices. Since the edge ideal of the complete bipartite graph $\mathcal{K}_{1,n-1}$ has a linear resolution by Theorem 2.7, our statement is also vacuously true in this case.

Suppose G is not the complete bipartite graph $\mathcal{K}_{1,n-1}$. Clearly G now has a splitting vertex, say v . Set $N(v) = \{v_1, \dots, v_d\}$, and let $G_i = G \setminus (N(v) \cup N(v_i))$ for $i = 1, \dots, d$, and $G_{(v)} = G_{\{v_1, \dots, v_d\}} \cup \{e \in E_G \mid e \text{ is incident to one of } v_1, \dots, v_d \text{ but not } v\}$. By Theorem 4.6 (and Corollary 4.3) we have that

$$\beta_{i,j}(\mathcal{I}(G)) = \beta_{i,j}(J) + \beta_{i,j}(K) + \beta_{i-1,j}(L) \tag{4.2}$$

where $J = (vv_1, \dots, vv_d)$, $K = \mathcal{I}(G \setminus \{v\})$ and $L = J \cap K = v\mathcal{I}(G_{(v)}) + \sum_{i=1}^d vv_i\mathcal{I}(G_i)$.

It follows from (4.2) that $\mathcal{I}(G)$ satisfies property $N_{2,p}$ if and only if J and K satisfy property $N_{2,p}$, and L satisfies property $N_{3,p-1}$. Observe further that L satisfies property $N_{3,p-1}$ if and only if $L = v\mathcal{I}(G_{(v)})$ and $\mathcal{I}(G_{(v)})$ satisfies property $N_{2,p-1}$. Since J has a linear minimal free resolution, J always satisfies property $N_{2,p}$. By the induction hypothesis, K satisfies property $N_{2,p}$ if and only if every minimal cycle of $(G \setminus \{v\})^c$ has length $\geq p+3$. It can be seen that $(G \setminus \{v\})^c = G^c \setminus \{v\}$. Thus, it remains to prove that $L = v\mathcal{I}(G_{(v)})$ and $\mathcal{I}(G_{(v)})$ satisfies property $N_{2,p-1}$ if and only if every minimal cycle of G^c containing v has length $\geq p+3$.

Suppose first that $L = v\mathcal{I}(G_{(v)})$ and $\mathcal{I}(G_{(v)})$ satisfies property $N_{2,p-1}$. Consider $C = (vx_1 \dots x_l v)$ an arbitrary minimal cycle in G^c containing v (and thus, $l \geq 3$). We shall show that C has length $\geq p+3$. Since C is a minimal cycle, we have $vx_2, vx_3, \dots, vx_{l-1} \notin G^c$. This implies that $vx_2, \dots, vx_{l-1} \in G$. Thus, $\{x_2, \dots, x_{l-1}\} \subseteq \{v_1, \dots, v_d\}$. Also, since $vx_1, vx_l \in G^c$, we have $x_1, x_l \notin \{v_1, \dots, v_d\}$. This implies that $x_1x_l \notin G_{(v)}$. Therefore, x_1, \dots, x_l form either a minimal cycle or a triangle in $G_{(v)}^c$. Consider the case when $l \geq 4$. If $p = 2$, then clearly C has length $\geq p+3$. If $p > 2$ then by the induction hypothesis, since $G_{(v)}$ does not contain v and $\mathcal{I}(G_{(v)}^c)$ satisfies property $N_{2,p-1}$, every minimal cycle in $G_{(v)}^c$ must have length $\geq p+2$. Hence, $l \geq p+2$, whence C has length $\geq p+3$. It remains to consider the case when $l = 3$. Since C is a minimal cycle,

x_1x_3 is not a chord of C . This means that $x_1x_3 \in G$. Furthermore, as shown, $x_1, x_3 \notin \{v_1, \dots, v_d\}$ and $x_2x_1, x_2x_3 \in G^c$. This implies that $vx_2x_1x_3 \in J \cap K = L$ and $vx_2x_1x_3 \notin v\mathcal{I}(G_{(v)})$, a contradiction to the fact that $L = v\mathcal{I}(G_{(v)})$.

Conversely, suppose that every minimal cycle of G^c containing v has length $\geq p+3$. We need to prove: (a) $L = v\mathcal{I}(G_{(v)})$, and (b) $\mathcal{I}(G_{(v)})$ has property $N_{2,p-1}$.

To prove (a) we observe that if $v\mathcal{I}(G_{(v)}) \subsetneq L$ then there exists an edge $e = uw \in G$ such that $u, w \notin \{v, v_1, \dots, v_d\}$. But then (vuv_iw) forms a minimal cycle of length 4 in G^c , contradicting the assumption that every minimal cycle of G^c has length $\geq p+3$ for $p > 1$.

To prove (b) we observe that $\mathcal{I}(G_{(v)})$ is generated by quadratics, so (b) is true for $p = 2$. Assume that $p > 2$. By induction, we only need to show that every minimal cycle of $G_{(v)}^c$ has length $\geq p+2$. Consider an arbitrary minimal cycle $D = (x_1x_2 \dots x_lx_1)$ in $G_{(v)}^c$. Let $\{w_1, \dots, w_s\}$ be the set of vertices of $G \setminus \{v, v_1, \dots, v_d\}$ which are adjacent to at least one of the vertices $\{v_1, \dots, v_d\}$. If there exist $1 \leq i \neq j \leq s$ such that $w_i, w_j \in \{x_1, \dots, x_l\}$, then since $w_iw_j \notin G_{(v)}$ (by definition of $G_{(v)}$), w_iw_j must be an edge of D (otherwise D would have a chord). Without loss of generality, suppose $w_i = x_1$ and $w_j = x_l$. In this case, $(x_1x_2 \dots x_lvx_1)$ is a minimal cycle of G^c , which implies that $l+1 \geq p+3$, i.e., $l \geq p+2$. If there is at most one of $\{w_1, \dots, w_s\}$ belonging to $\{v_1, \dots, v_d\}$, then it is easy to see that D is a minimal cycle in G^c . This implies that $l \geq p+3 > p+2$. The result is proved. \square

Fröberg's [11] main theorem is a special case of Corollary 4.7. Recall that $\mathcal{I}(G)$ has a *linear resolution* if $\beta_{i,j}(\mathcal{I}(G)) = 0$ for all $j \neq i+2$. We say that G is *chordal* if every cycle of length > 3 has a chord; in other words, G has no minimal cycles.

Corollary 4.8 (see [11]). *Let G be a graph with edge ideal $\mathcal{I}(G)$. Then $\mathcal{I}(G)$ has a linear resolution if and only if G^c is a chordal graph.*

As a consequence of Theorem 4.6, we can give a new proof for the formula of [21] for the linear strand of $\mathcal{I}(G)$ when G contains no minimal cycle of length 4.

Corollary 4.9 (see [21]). *Let G be a graph with no minimal cycle of length 4. Let $k_{i+2}(G)$ denote the number of $(i+2)$ -cliques in G . Then, for any $i \geq 0$,*

$$\beta_{i,i+2}(\mathcal{I}(G)) = \sum_{u \in V_G} \binom{\deg u}{i+1} - k_{i+2}(G).$$

Proof. We have $\sum_{u \in V_G} \deg u = 2|E_G| = 2k_2(G)$. Thus, our statement is true for $i = 0$. Assume that $i \geq 1$. We shall use induction on $n = |V_G|$.

It is easy to verify the statement for $n \leq 3$. Assume that $n \geq 4$. The statement for the complete bipartite graph $\mathcal{K}_{1,n-1}$ follows by Theorem 2.7, so we may assume that G is not the complete bipartite graph $\mathcal{K}_{1,n-1}$. This guarantees that G contains a splitting vertex, say v . As before, let $N(v) = \{v_1, \dots, v_d\}$, let $G_i = G \setminus (N(v) \cup N(v_i))$ for $i = 1, \dots, d$, and let $G_{(v)} = G_{\{v_1, \dots, v_d\}} \cup \{e \in E_G \mid e \text{ incident to } v_i \text{ for some } i = 1, \dots, d \text{ but not } v\}$. By Theorem 4.6 (and Corollary 4.3), we have $\beta_{i,i+2}(\mathcal{I}(G)) = \beta_{i,i+2}(J) + \beta_{i,i+2}(K) + \beta_{i-1,i+2}(L)$ where $J = (vv_1, \dots, vv_d)$, $K = \mathcal{I}(G \setminus \{v\})$, and $L = v\mathcal{I}(G_{(v)}) + \sum_{i=1}^d vv_i\mathcal{I}(G_i)$. For each $i = 1, \dots, d$, the ideal $vv_i\mathcal{I}(G_i)$ is generated by monomials of degree 4. Thus the linear strand of L is the same as that of $v\mathcal{I}(G_{(v)})$ (or equivalently, that of $\mathcal{I}(G_{(v)})$).

Thus, we have

$$\beta_{i,i+2}(\mathcal{I}(G)) = \beta_{i,i+2}(J) + \beta_{i,i+2}(K) + \beta_{i-1,i+1}(\mathcal{I}(G_{(v)})). \quad (4.3)$$

For simplicity, let $G' = G \setminus \{v\}$ and $G'' = G_{(v)}$. Let $W = \{w_1, \dots, w_s\}$ be the set of vertices of $G \setminus \{v, v_1, \dots, v_d\}$ which are adjacent to at least one of the vertices of $N(v) = \{v_1, \dots, v_d\}$. Let $\{v_{j_1}, \dots, v_{j_{l_j}}\}$ be the set of vertices of $\{v_1, \dots, v_d\}$ that are adjacent to w_j , for $j = 1, \dots, s$. By the induction hypothesis we have

$$\begin{aligned} \beta_{i-1,i+1}(\mathcal{I}(G'')) &= \sum_{u \in N(v)} \binom{\deg_{G''} u}{i} + \sum_{u \in W} \binom{\deg_{G''} u}{i} - k_{i+1}(G'') \\ &= \sum_{u \in N(v)} \binom{\deg_G u - 1}{i} + \sum_{j=1}^s \binom{l_j}{i} - k'_{i+1}(G'') - k''_{i+1}(G'') \end{aligned} \quad (4.4)$$

where $k'_{i+1}(G'')$ denotes the number of $(i+1)$ -cliques G'' not containing any of the vertices in W and $k''_{i+1}(G'')$ denotes the number of $(i+1)$ -cliques of G'' containing at least one vertex in W . Observe that for each $j = 1, \dots, s$, since G contains no minimal cycle of length 4 and $vw_j \notin E_G$, we must have $v_{jt_1}v_{jt_2} \in E_G$ for any $1 \leq t_1 \neq t_2 \leq l_j$. This implies that $G_{\{v_{j_1}, \dots, v_{j_{l_j}}\}}$ is the complete graph on l_j vertices for any $j = 1, \dots, s$. Moreover, $w_l w_t \notin E_{G''}$ for any $1 \leq l \neq t \leq s$. Therefore, each $(i+1)$ -clique of G'' containing some vertices in W contains exactly one. We must have $k''_{i+1}(G'') = \sum_{j=1}^s \binom{l_j}{i}$. This, together with (4.4), gives

$$\beta_{i-1,i+1}(\mathcal{I}(G'')) = \sum_{u \in N(v)} \binom{\deg_G u - 1}{i} - k_{i+2}(G_{\{v, v_1, \dots, v_d\}}). \quad (4.5)$$

By induction we also have

$$\beta_{i,i+2}(K) = \sum_{u \in V_G \setminus \{v, v_1, \dots, v_d\}} \binom{\deg_G u}{i+1} + \sum_{u \in N(v)} \binom{\deg_G u - 1}{i+1} - k_{i+2}(G'). \quad (4.6)$$

It can further be seen that

$$\beta_{i,i+2}(J) = \binom{\deg_G v}{i+1}. \quad (4.7)$$

Now (4.3), (4.5), (4.6), and (4.7) combine to give us

$$\beta_{i,i+2}(\mathcal{I}(G)) = \sum_{u \in V_G} \binom{\deg_G u}{i+1} - k_{i+2}(G') - k_{i+2}(G_{\{v, v_1, \dots, v_d\}}).$$

Observe that an $(i+2)$ -clique in G either contains v (so it is an $(i+2)$ -clique of $G_{\{v, v_1, \dots, v_d\}}$) or is a $(i+2)$ -clique of $G' = G \setminus \{v\}$. Hence,

$$k_{i+2}(G) = k_{i+2}(G_{\{v, v_1, \dots, v_d\}}) + k_{i+2}(G')$$

and thus the result is proved. \square

5. FACET IDEALS AND SPLITTING FACETS

In this section we extend our method to the study of arbitrary square-free monomial ideals. Let Δ be a simplicial complex on the vertex set $V_\Delta = \{x_1, \dots, x_n\}$. Let $\mathcal{I}(\Delta)$ be the facet ideal of Δ in $R = k[x_1, \dots, x_n]$. Recall that, by abuse of notation, we will use F to denote a facet of Δ and the monomial $\prod_{x \in F} x$ in $\mathcal{I}(\Delta)$.

Definition 5.1. Let F be a facet of Δ . The *connected component* of F in Δ , denoted $\text{conn}_\Delta(F)$, is defined to be the connected component of Δ containing F . If $\text{conn}_\Delta(F) \setminus F = \langle G_1, \dots, G_p \rangle$, then we define the *reduced connected component* of F in Δ , denoted by $\overline{\text{conn}}_\Delta(F)$, to be the simplicial complex whose facets are given by $G_1 \setminus F, \dots, G_p \setminus F$, where if there exist G_i and G_j such that $\emptyset \neq G_i \setminus F \subseteq G_j \setminus F$, then we shall disregard the bigger facet $G_j \setminus F$ in $\overline{\text{conn}}_\Delta(F)$.

Let F be a facet of Δ . Let $\Delta' = \Delta \setminus F$ be the simplicial complex whose facet set is $\mathcal{F}(\Delta) \setminus F$. Let $J = (F)$ and $K = \mathcal{I}(\Delta')$. Note that $\mathcal{G}(\mathcal{I}(\Delta))$ is the disjoint union of $\mathcal{G}(J)$ and $\mathcal{G}(K)$. We are interested in finding F such that $\mathcal{I}(\Delta) = J + K$ gives a splitting for $\mathcal{I}(\Delta)$.

Definition 5.2. With the above notation, we shall call F a *splitting facet* of Δ if $\mathcal{I}(\Delta) = J + K$ is a splitting of $\mathcal{I}(\Delta)$.

Lemma 5.3. *With the above notation, we have*

$$J \cap K = (F)(\mathcal{I}(\overline{\text{conn}}_\Delta(F)) + \mathcal{I}(\Omega))$$

where Ω denotes the simplicial complex $\Delta \setminus \text{conn}_\Delta(F)$.

Proof. Since both J and K are monomial ideals, we have

$$J \cap K = (\{\text{lcm}(F, G) \mid G \in \mathcal{G}(K)\}).$$

It is easy to see that if H is a facet of $\overline{\text{conn}}_\Delta(F)$ and if G is a facet of $\text{conn}_\Delta(F)$ such that $G \setminus F = H$, then $G \neq F$ and $FH = \text{lcm}(F, G) \in J \cap K$. Thus, $(F)\mathcal{I}(\overline{\text{conn}}_\Delta(F)) \subseteq J \cap K$. Also, for any facet H of Ω , $FH = \text{lcm}(F, H) \in J \cap K$. Thus, $(F)\mathcal{I}(\Omega) \subseteq J \cap K$, and hence

$$(F)(\mathcal{I}(\overline{\text{conn}}_\Delta(F)) + \mathcal{I}(\Omega)) \subseteq J \cap K.$$

For the other inclusion, note that each $G \in \mathcal{G}(K)$ corresponds to a facet G of Δ' . There are two possibilities for this facet: (1) $G \in \text{conn}_\Delta(F)$, or (2) $G \notin \text{conn}_\Delta(F)$. It now follows from the construction of $\overline{\text{conn}}_\Delta(F)$ and Ω that case (1) leads to $\text{lcm}(F, G) \in (F)\mathcal{I}(\overline{\text{conn}}_\Delta(F))$ and case (2) results in $\text{lcm}(F, G) \in (F)\mathcal{I}(\Omega)$. \square

Lemma 5.4. *With the same notation as in Lemma 5.3, for all $i \geq 1$ and all $j \geq 0$*

$$\beta_{i-1, j}(J \cap K) = \sum_{l_1=0}^i \sum_{l_2=0}^{j-|F|} \beta_{l_1-1, l_2}(\mathcal{I}(\overline{\text{conn}}_\Delta(F))) \beta_{i-l_1-1, j-|F|-l_2}(\mathcal{I}(\Omega)).$$

Proof. Let $L = \mathcal{I}(\overline{\text{conn}}_\Delta(F)) + \mathcal{I}(\Omega)$. It follows from Lemma 5.3 that $J \cap K = FL$. Since none of the variables in F are present in L , F is not a zero-divisor of R/L . As a consequence, we have for all $i \geq 1$

$$\beta_{i-1, j}(FL) = \beta_{i-1, j-|F|}(L) = \beta_{i, j-|F|}(R/L).$$

Now we notice that $\overline{\text{conn}}_\Delta(F)$ and Ω do not share any common vertices. The statement, therefore, follows by applying Remark 2.5. \square

The following result gives a recursive like formula for the graded Betti numbers of the facet ideal of a simplicial complex in terms of the Betti numbers of facet ideals of subcomplexes. This result is a higher dimension analogue of Theorem 3.6.

Theorem 5.5. *Let F be a splitting facet of a simplicial complex Δ . Let $\Delta' = \Delta \setminus F$ and $\Omega = \Delta \setminus \text{conn}_\Delta(F)$. Then, for all $i \geq 1$ and $j \geq 0$,*

$$\beta_{i,j}(\mathcal{I}(\Delta)) = \beta_{i,j}(\mathcal{I}(\Delta')) + \sum_{l_1=0}^i \sum_{l_2=0}^{j-|F|} \beta_{l_1-1,l_2}(\mathcal{I}(\overline{\text{conn}}_\Delta(F))) \beta_{i-l_1-1,j-|F|-l_2}(\mathcal{I}(\Omega)).$$

Proof. By definition, $\mathcal{I}(\Delta) = J + K$ is a splitting of $\mathcal{I}(\Delta)$. The conclusion now follows from Theorem 2.2 and Lemma 5.4 and the fact that $\beta_{i,j}(J) = 0$ if $i \geq 1$. \square

We will now show that our formula in Theorem 5.5 is recursive when G is a forest. To do so, we first show that a leaf of Δ is a splitting facet. Recall that if F is a leaf of Δ , then F must have a vertex that does not belong to any other facet of the simplicial complex (see Remark 2.3 of [6]).

Theorem 5.6. *If F is a leaf of Δ , then F is a splitting facet of Δ .*

Proof. We need to show that if F is a leaf of Δ , then $\mathcal{I}(\Delta) = J + K$ with $J = (F)$ and $K = \mathcal{I}(\Delta \setminus F)$ is a splitting of $\mathcal{I}(\Delta)$. Without loss of generality, we may assume that $F = \{x_1, \dots, x_l\}$. We shall construct a splitting function $s : \mathcal{G}(J \cap K) \rightarrow \mathcal{G}(J) \times \mathcal{G}(K)$ for $\mathcal{I}(\Delta)$.

Suppose $L \in \mathcal{G}(J \cap K)$. Let $\mathcal{M}_L = \{G \in \mathcal{G}(K) \mid \text{lcm}(F, G) = L\}$. For each $G \in \mathcal{M}_L$, we order the elements of $G \cap F$ by the increasing order of their indexes and view $G \cap F$ as an ordered word of the alphabet $\{x_1, \dots, x_l\}$. Let $G_L \in \mathcal{M}_L$ be such that $G_L \cap F$ is minimal with respect to the lexicographic word ordering. Clearly, G_L is uniquely determined by L . Our splitting function s is defined as follows. For each $L \in \mathcal{G}(J \cap K)$,

$$s(L) = (\phi(L), \psi(L)) = (F, G_L).$$

We need to verify that s satisfies conditions (a) and (b) of Definition 2.1. Indeed, condition (a) follows obviously from the definition of the function s . Suppose $S \subseteq \mathcal{G}(J \cap K)$. Since F is a leaf of Δ , there exists a vertex $u \in F$ such that u is not in any other facet of Δ . This implies that u does not divide $\text{lcm}(\psi(S))$. Yet, since u is in F , u divides $\text{lcm}(S)$. Thus, $\text{lcm}(\psi(S))$ strictly divides $\text{lcm}(S)$. On the other hand, it is also clear that for any $G \in \mathcal{G}(K)$, F strictly divides $\text{lcm}(F, G)$, so $\text{lcm}(\phi(S)) = F$ strictly divides $\text{lcm}(S)$. The result is proved. \square

Because $\text{conn}_\Delta(F)$, $\Delta \setminus F$ and $\Delta \setminus \text{conn}_\Delta(F)$ are subcomplexes of Δ , it follows directly from the definition that if Δ is a forest, then so are $\text{conn}_\Delta(F)$, $\Delta \setminus F$ and $\Delta \setminus \text{conn}_\Delta(F)$. Thus, to show that our formula in Theorem 5.5 is recursive when G is a forest, it suffices to show that $\overline{\text{conn}}_\Delta(F)$ is also a forest.

Lemma 5.7. *Let F be a facet of a forest Δ . Then $\overline{\text{conn}}_\Delta(F)$ is a forest.*

Proof. Suppose $\Xi = \langle G_1, \dots, G_l \rangle$ is a connected component of $\overline{\text{conn}}_\Delta(F)$, where $G_i = F_i \setminus F$ and F_i is a facet of $\text{conn}_\Delta(F)$ for all $i = 1, \dots, l$. We shall show that Ξ has a leaf. Indeed, it is easy to see that $\Theta = \langle F_1, \dots, F_l \rangle$ is a connected subcomplex of $\text{conn}_\Delta(F)$. As observed, since Δ is a forest, so is $\text{conn}_\Delta(F)$. Thus, Θ has a leaf. Without loss of generality, assume that F_1 is a leaf of Θ . That is, either $l = 1$ or there exists another facet of Θ , say F_2 , such that $F_1 \cap H \subseteq F_1 \cap F_2$ for any facet $H \neq F_1$ of Θ . If $l = 1$, then clearly G_1 is a leaf of Ξ . Suppose $l > 1$. It is easy to see that $G_1 \cap (H \setminus F) = (F_1 \setminus F) \cap (H \setminus F) = (F_1 \cap H) \setminus F \subseteq (F_1 \cap F_2) \setminus F =$

$(F_1 \setminus F) \cap (F_2 \setminus F) = G_1 \cap G_2$. Thus, G_1 is also a leaf of Ξ . We have just shown that Ξ has a leaf in any case. The lemma is proved. \square

We can generalize Corollary 3.9 by giving a recursive formula for simplicial trees.

Theorem 5.8. *Let Δ be a simplicial forest. For any leaf F of Δ , $\Delta' = \Delta \setminus F$, $\Omega = \Delta \setminus \text{conn}_\Delta(F)$, and $\overline{\text{conn}}_\Delta(F)$ are also simplicial forests. Furthermore, the numbers $\beta_{i,j}(\mathcal{I}(\Delta))$ for all $i \geq 1$ and $j \geq 0$ can be computed recursively using the formula*

$$\beta_{i,j}(\mathcal{I}(\Delta)) = \beta_{i,j}(\mathcal{I}(\Delta')) + \sum_{l_1=0}^i \sum_{l_2=0}^{j-|F|} \beta_{l_1-1,l_2}(\mathcal{I}(\overline{\text{conn}}_\Delta(F))) \beta_{i-l_1-1,j-|F|-l_2}(\mathcal{I}(\Omega)).$$

Proof. Lemma 5.7 and the discussion before this lemma imply the first statement. The second statement follows from Theorems 5.5 and 5.6 because Δ' , Ω , and $\overline{\text{conn}}_\Delta(F)$ all have must have a leaf, which implies their facet ideals can also be split using Theorem 5.5. \square

Theorem 5.8 can be used to find a nice formula for the linear strand of facet ideals of pure forests, generalizing a result of Zheng [27, Proposition 3.3] and Corollary 4.9. Recall that a simplicial complex Δ is said to be pure $(d-1)$ -dimensional if $\dim F = d-1$, i.e., $|F| = d$, for any facet F of Δ . For a face G of dimension $d-2$ of a pure $(d-1)$ -dimensional simplicial complex Δ we define the *degree* of G in Δ , written $\deg_\Delta(G)$, to be the cardinality of the set $\{F \in \mathcal{F}(\Delta) \mid G \subseteq F\}$. Let $\mathcal{A}(\Delta)$ denote the set of $(d-2)$ -dimensional faces of Δ .

Theorem 5.9. *Let Δ be a pure $(d-1)$ -dimensional forest (for some $d \geq 2$). Then*

$$\beta_{i,i+d}(\mathcal{I}(\Delta)) = \begin{cases} |\mathcal{F}(\Delta)| & \text{if } i = 0 \\ \sum_{G \in \mathcal{A}(\Delta)} \binom{\deg_\Delta(G)}{i+1} & \text{if } i \geq 1. \end{cases}$$

Proof. The assertion is clear for $i = 0$. Suppose $i \geq 1$. Let $m = |\mathcal{F}(\Delta)|$ be the number of facets of Δ . We shall use induction on m . For $m = 1$, the assertion is obviously true. Suppose that $m > 1$. Let F be a leaf of Δ , and let $\Delta' = \Delta \setminus F$ and $\Omega = \Delta \setminus \text{conn}_\Delta(F)$. By Theorem 5.8, for $i \geq 1$ we have

$$\beta_{i,i+d}(\mathcal{I}(\Delta)) = \beta_{i,i+d}(\mathcal{I}(\Delta')) + \sum_{l_1=0}^i \sum_{l_2=0}^i \beta_{l_1-1,l_2}(\mathcal{I}(\overline{\text{conn}}_\Delta(F))) \beta_{i-l_1-1,i-l_2}(\mathcal{I}(\Omega)).$$

Observe that since $d \geq 2$, $l_1 - 1 \geq l_1 + 1 - d$. Thus, for any $l_2 = 0, \dots, i$, we have either $l_2 \leq l_1 - 1$ or $l_2 > l_1 + 1 - d$. If $l_2 \leq l_1 - 1$, then clearly $\beta_{l_1-1,l_2}(\mathcal{I}(\overline{\text{conn}}_\Delta(F))) = 0$. If $l_2 > l_1 + 1 - d$, then $i - l_2 < (i - l_1 - 1) + d$. This and the fact that $\Omega = \Delta \setminus \text{conn}_\Delta(F)$ is also a pure $(d-1)$ -dimensional forest imply that $\beta_{i-l_1-1,i-l_2}(\mathcal{I}(\Omega)) = 0$ unless $l_1 = l_2 = i$ (in which case $\beta_{i-l_1-1,i-l_2}(\mathcal{I}(\Omega)) = \beta_{-1,0}(\mathcal{I}(\Omega)) = 1$). Hence, we have

$$\beta_{i,i+d}(\mathcal{I}(\Delta)) = \beta_{i,i+d}(\mathcal{I}(\Delta')) + \beta_{i-1,i}(\mathcal{I}(\overline{\text{conn}}_\Delta(F))). \quad (5.1)$$

Clearly, $\beta_{i-1,i}(\mathcal{I}(\overline{\text{conn}}_\Delta(F)))$ forms the linear strand of $\mathcal{I}(\overline{\text{conn}}_\Delta(F))$ and is given by $\binom{s}{i}$ for any $i \geq 1$, where s is the number of isolated vertices of $\overline{\text{conn}}_\Delta(F)$. Since F is a leaf of Δ , there must exist a vertex $u \in F$ such that u is not in any other

facet of Δ . Let $H = F \setminus \{u\}$. Observe that $\{x\}$ (for some $x \neq u$) is an isolated vertex of $\overline{\text{conn}}_{\Delta}(F)$ if and only if $H \cup \{x\}$ is a facet of Δ . This implies that $s = \deg_{\Delta}(H) - 1$ (since $H = F \setminus \{u\}$ is not in $\overline{\text{conn}}_{\Delta}(F)$). This, together with the induction hypothesis, now gives

$$\begin{aligned} \beta_{i,i+d}(\mathcal{I}(\Delta)) &= \sum_{G \in \mathcal{A}(\Delta')} \binom{\deg_{\Delta'}(G)}{i+1} + \binom{\deg_{\Delta}(H) - 1}{i} \\ &= \sum_{G \in \mathcal{A}(\Delta) \setminus \{H\}} \binom{\deg_{\Delta}(G)}{i+1} + \binom{\deg_{\Delta}(H) - 1}{i+1} + \binom{\deg_{\Delta}(H) - 1}{i} \\ &= \sum_{G \in \mathcal{A}(\Delta)} \binom{\deg_{\Delta}(G)}{i+1}. \end{aligned}$$

The theorem is proved. \square

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