# WHISKERS AND SEQUENTIALLY COHEN-MACAULAY GRAPHS

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ABSTRACT. Let $G$ be a simple (i.e., no loops and no multiple edges) graph. We investigate the question of how to modify $G$ combinatorially to obtain a sequentially Cohen-Macaulay graph. We focus on modifications given by adding configurations of whiskers to $G$, where to add a whisker one adds a new vertex and an edge connecting this vertex to an existing vertex in $G$. We give various sufficient conditions and necessary conditions on a subset $S$ of the vertices of $G$ so that the graph $G \cup W(S)$, obtained from $G$ by adding a whisker to each vertex in $S$, is a sequentially Cohen-Macaulay graph. For instance, we show that if $S$ is a vertex cover of $G$, then $G \cup W(S)$ is a sequentially Cohen-Macaulay graph. On the other hand, we show that if $G \backslash S$ is not sequentially Cohen-Macaulay, then $G \cup W(S)$ is not a sequentially Cohen-Macaulay graph. Our work is inspired by and generalizes a result of Villarreal on the use of whiskers to get Cohen-Macaulay graphs.

## 1. Introduction

Let $G = (V_G, E_G)$ be a simple graph with vertex set $V_G = \{x_1, \ldots, x_n\}$ and edge set $E_G$. We can associate to $G$ a square-free monomial ideal

$$\mathcal{I}(G) = \langle x_i x_j \mid \{x_i, x_j\} \in E_G \rangle \subset R = k[x_1, \ldots, x_n]$$

(by abusing language, we shall use $x_i$s to denote both the vertices of $G$ and the corresponding variables in a polynomial ring). The ideal $\mathcal{I}(G)$ is usually referred to as the **edge ideal** of $G$.

In recent years there has been a flurry of work investigating how the combinatorial data of $G$ appears in algebraic invariants and properties of $\mathcal{I}(G)$. We mention, for example, the works [2, 5, 7, 8, 10, 12, 13, 14, 17, 19, 20]. In this paper, we examine how a particular structure of $G$ affects the Cohen-Macaulayness and sequentially Cohen-Macaulayness of its edge ideal. The property of being sequentially Cohen-Macaulay was first introduced by Stanley [18] as a generalization of the well-known property of being Cohen-Macaulay. We shall now recall the definition of sequentially Cohen-Macaulay modules over a polynomial ring.

**Definition 1.1.** Let $M$ be a graded module over $R = k[x_1, \ldots, x_n]$. We say that $M$ is **sequentially Cohen-Macaulay** if there exists a filtration

$$0 = M_0 \subset M_1 \subset \cdots \subset M_r = M$$

of $M$ by graded $R$-modules such that $M_i/M_{i-1}$ is Cohen-Macaulay for all $i$, and $\dim M_i/M_{i-1} < \dim M_{i+1}/M_i$ for all $i$, where dim denotes Krull dimension.

*Key words and phrases.* edge ideals of graphs, Alexander duality, sequential Cohen-Macaulayness.
Version: May 23, 2006.
The second author is partially supported by the Louisiana Board of Regents Enhancement Grant.

A graph $G$ is said to be **(sequentially) Cohen-Macaulay** if $R/\mathcal{I}(G)$ is a (sequentially) Cohen-Macaulay module over $R$. The problem of classifying Cohen-Macaulay or sequentially Cohen-Macaulay graphs is intractable, and thus it is natural to pay attention to some special classes of graphs. Of particular interest is the class of *trees* and *forests*, or slightly more generally, the class of *chordal* graphs. For example, Faridi [4] showed that simplicial trees are sequentially Cohen-Macaulay; the first author and Van Tuyl [5] extended this property in the case of graphs to the class of all chordal graphs; and, on the other hand, Herzog, Hibi, and Zheng [10] proved that a chordal graph is Cohen-Macaulay if and only if it is *unmixed*.

Our focus in this paper complements these work. We ask the following question: Given an arbitrary graph $G$, how can one modify $G$ to obtain a graph that is sequentially Cohen-Macaulay? Motivated by Villarreal's work in [19], we investigate the effect of adding "whiskers" to a graph. To add a **whisker** at a vertex $y$ to $G$, one adds a new vertex $x$ and the edge connecting $x$ and $y$ to $G$. We denote by $G \cup W(y)$ the graph obtained from $G$ by adding a whisker at $y$. More generally, if $S \subset V_G$ is a subset of the vertices of $G$, then we denote by $G \cup W(S)$ the graph obtained from $G$ by adding a whisker at each vertex in $S$. The origin of the name is clear from a picture of a whisker added to each vertex of a cycle, and the terminology appears in [17]. Our work is inspired by the following theorem of Villarreal (who cites the contributions of Vasconcelos, Herzog, and Fröberg as well) from [19]; see also [17, Theorem 2.1].

**Theorem 1.2.** *Let $G$ be a graph, and let $H$ be the graph $G$ with a whisker added to each vertex of $G$. Then $H$ is Cohen-Macaulay.*

This result is sharp; if a whisker is added to all but one vertex of $G$, then the conclusion of Theorem 1.2 is no longer true. For example, if we consider the graph $G$ with $V_G = \{z, y_1, y_2\}$ and $E_G = \{zy_1, zy_2\}$ and let $S = \{y_1, y_2\}$, then adding whiskers $x_1y_1$ and $x_2y_2$ to vertices in $S$ will result in $\mathcal{I}(G \cup W(S)) = (zy_1, zy_2, x_1y_1, x_2y_2) \subset R = k[z, y_1, y_2, x_1, x_2]$. In this case, $R/\mathcal{I}(G \cup W(S))$ is not a Cohen-Macaulay ring; i.e., $G \cup W(S)$ is not a Cohen-Macaulay graph. More surprisingly, perhaps, is the fact that adding whiskers can actually destroy the Cohen-Macaulay property of a graph. For instance, if $G$ consists of two vertices $\{y, z\}$ and a single edge $yz$, then $G$ is a Cohen-Macaulay graph, but $G \cup W(y)$ is not.

The goal of this paper is to explore how adding different configurations of whiskers to a graph $G$ affects the weaker property of a graph being sequentially Cohen-Macaulay. It turns out to be considerably easier to create a sequentially Cohen-Macaulay graph from an arbitrary graph by adding whiskers despite the fact that sequential Cohen-Macaulayness is still a strong property. Our first main result is stated as follows.

**Theorem 1.3** (Theorem 3.5)**.** *Let $G$ be a simple graph and let $S \subset V_G$. Suppose that $G \backslash S$ is a chordal graph or a five-cycle $C_5$. Then $G \cup W(S)$ is a sequentially Cohen-Macaulay graph.*

Theorem 1.3 gives us a number of interesting consequences. For example, Corollary 3.6 says that if $S \subset V_G$ is a *vertex cover*, i.e., a subset of vertices of $G$ so that every edge of $G$ is incident to at least one of these vertices, then $G \cup W(S)$ is sequentially Cohen-Macaulay. This shows that to create sequentially Cohen-Macaulay graphs by adding whiskers, the number of whiskers is not as important as their configuration. On the other hand, Corollary 3.9 says that if $|S| \geq |V_G| - 3$, then $G \cup W(S)$ is sequentially Cohen-Macaulay. This gives a bound on the number of vertices so that adding this many whiskers, regardless of their configuration, always results in a sequentially Cohen-Macaulay graph. Furthermore, Theorem 1.2 could also be recovered as a consequence of our Theorem 1.3 (Corollary 3.10).

The proof of Theorem 1.3 is based upon examining the Alexander dual of involved edge ideals; in particular, of $\mathcal{I}(G\backslash S)$ and $\mathcal{I}(G \cup W(S))$. We, in fact, study a slightly stronger property than being sequentially Cohen-Macaulay. This property requires that the ideal generated by all elements of degree $d$ of the Alexander dual has *linear quotients* (and, in particular, has a linear free resolution) for any $d \in \mathbb{N}$. This property is independent of the characteristic of the ground field $k$ and thus could be investigated using combinatorial means.

Our next main result addresses the converse of Theorem 1.3. We give a necessary condition on $G\backslash S$ for $G \cup W(S)$ to be sequentially Cohen-Macaulay.

**Theorem 1.4** (Theorem 4.1)**.** *Let $G$ be a simple graph and let $S \subset V_G$. If $G\backslash S$ is not a sequentially Cohen-Macaulay graph, then $G \cup W(S)$ is not sequentially Cohen-Macaulay.*

To prove Theorem 1.4, we examine syzygies of the Alexander dual of $\mathcal{I}(G \cup W(S))$ via simplicial homology and *upper Koszul simplicial complexes* associated to a square-free monomial ideals. Our arguments are inspired by [5].

Our paper is structured in the following way. In Section 2, we gather some preliminary notation on graphs and discuss some techniques from Alexander duality. In Section 3, we explore sufficient conditions on $S$ and $G\backslash S$ so that adding a whisker to each vertex in $S$ results in a sequentially Cohen-Macaulay graph. In particular, we prove Theorem 1.3 (Theorem 3.5) and give many interesting corollaries. In Section 4, the converse question is studied. We give a necessary condition on $S$ and $G\backslash S$ for $G \cup W(S)$ to be sequentially Cohen-Macaulay. Theorem 1.4 (Theorem 4.1) is proved in this section.

**Acknowledgement:** The authors would like to thank Adam Van Tuyl for many stimulating discussions on the topic.

## 2. Preliminaries

In this section, we fix some notation for graphs and discuss results relating to Alexander duality that we shall use in the rest of the paper.

Throughout the paper, $G = (V_G, E_G)$ will denote a simple graph, which is a graph without any loops or multiple edges. Often, we shall simply write $G$ and not specify its vertex and edge sets. For a vertex $x$ of $G$ we use $N(x)$ to denote the set of neighbors of $x$, which are the vertices connected to $x$ by an edge of $G$. An **induced subgraph** of $G$ is a subgraph $H$ of $G$ with the property that if $\{z_1, z_2\} \subset V_H$ and $z_1z_2 \in E_G$ then $z_1z_2 \in E_H$.

**Notation 2.1.** Let $G$ be a graph, and suppose $\{z_1, \ldots, z_r\}$ is a subset of the vertices of $G$. We use $G\backslash\{z_1, \ldots, z_r\}$ to mean the subgraph obtained from $G$ by removing the vertices $z_1, \ldots, z_r$ and all edges incident to at least one of these vertices.

Recall that a **whisker** at a vertex $y$ of $G$ refers to a new vertex $x$ and the edge $xy$ being added to $G$.

**Notation 2.2.** Let $G$ be a simple graph, and let $S = \{y_1, \ldots, y_n\}$ be a subset of vertices of $G$. By $G \cup W(S)$ we mean the graph with whiskers $x_iy_i$, for each $1 \leq i \leq n$, attached to $G$. For simplicity, we shall use $\{x_1y_1, \ldots, x_ny_n\}$ to denote $W(S)$ in this case.

**Notation 2.3.** A **cycle** in a simple graph $G$ is an alternating sequence of distinct vertices and edges $C = v_1e_1v_2e_2 \ldots v_{n-1}e_{n-1}v_ne_nv_1$ in which the edge $e_i$ connects the vertices $v_i$ and $v_{i+1}$ ($v_{n+1} \cong v_1$) for all $i$. In this case, $C$ is said to have **length** $n$ and is often referred to as an $n$-cycle. We shall also usually use $C_n$ to denote an $n$-cycle.

Of our particular interest is the class of chordal graphs. This class includes all trees and forests and has been the object of much study in recent years.

**Definition 2.4.** We call a graph $G$ **chordal** if for all $n \geq 4$, every $n$-cycle in $G$ has a **chord**, which is an edge connecting two non-consecutive vertices of the cycle.

For example, a four-cycle is not chordal, but a four-cycle together with a diagonal edge is chordal.

**Definition 2.5.** A **vertex cover** of a graph $G$ is a subset $V \subset V_G$ of the vertices of $G$ such that every edge of $G$ is incident to at least one vertex of $V$ (in particular, isolated vertices need not appear in a vertex cover). If $V$ is a vertex cover of $G$, then it is a **minimal vertex cover** of $G$ if no proper subset of $V$ is a vertex cover of $G$. For simplicity, we write vertex covers as monomials, so $\{z_1, \ldots, z_r\}$ will be written as $z_1 \cdots z_r$. A graph is said to be **unmixed** if every minimal vertex cover has the same cardinality.

Since our primary interest is edge ideals, which are square-free monomial ideals, we have a number of techniques available to analyze their properties, especially tools from Alexander duality theory.

Let $\Delta$ be a simplicial complex on vertices $[n] = \{1, \ldots, n\}$. The **Alexander dual** of $\Delta$ is the simplicial complex

$$\Delta^* = \{[n] \backslash F \mid F \notin \Delta\}.$$

One can define Alexander duality for square-free monomial ideals (in fact, for arbitrary monomial ideals; see [16, Section 5.2]) without reference to simplicial complexes by a simple operation. If

$$I = (x_{i_{11}} \cdots x_{i_{1s_1}}, \ldots, x_{i_{r1}} \cdots x_{i_{rs_r}})$$

is a square-free monomial ideal then its Alexander dual is the square-free monomial ideal

$$I^\vee = (x_{i_{11}}, \ldots, x_{i_{1s_1}}) \cap \cdots \cap (x_{i_{r1}}, \ldots, x_{i_{rs_r}}).$$

In other words, Alexander duality takes generators to primary components. For example, the Alexander dual of $I = (x_1x_2x_4, x_1x_4x_5, x_3x_4)$ is the ideal

$$I^\vee = (x_1, x_2, x_4) \cap (x_1, x_4, x_5) \cap (x_3, x_4).$$

This duality is particularly useful in Stanley-Reisner theory, but we shall view it somewhat differently in this paper. The edge ideal associated to a graph is not the graph's Stanley-Reisner ideal but instead almost a facet ideal (just leaving out isolated vertices). Therefore when we take the Alexander dual of an edge ideal, we are thinking of it as an algebraic transformation. Dualizing the edge ideal of a graph $G$ yields an unmixed, height two ideal, and its minimal generators are exactly the minimal vertex covers of $G$ (when we write the vertex covers as monomials).

A powerful feature of Alexander duality is that it allows us to link the Cohen-Macaulay property with homological features of the dual via the following theorem of Eagon and Reiner [3].

**Theorem 2.6.** *Let $I$ be a square-free monomial ideal in $R = k[x_1, \ldots, x_n]$ with Alexander dual $I^\vee$. Then $R/I$ is Cohen-Macaulay over $k$ if and only if $I^\vee$ has a linear resolution over $R$.*

In this paper, we are mostly interested in studying sequential Cohen-Macaulayness rather than Cohen-Macaulayness, and thus we need a slightly different result. Stanley introduced the notion of sequential Cohen-Macaulayness in connection with work of Björner and Wachs on nonpure shellability. Pure shellable complexes are Cohen-Macaulay, and Stanley identified sequential Cohen-Macaulayness as the appropriate analogue in the nonpure setting; that is, all nonpure shellable complexes are sequentially Cohen-Macaulay. Because sequential Cohen-Macaulayness is a weaker condition than Cohen-Macaulayness, the analogue to Theorem 2.6 must identify something weaker than the dual having a linear resolution. To this end, Herzog and Hibi [9] introduced the concept of *componentwise linearity.*

**Definition 2.7.** Let $I$ be a homogeneous ideal, and write $(I_d)$ for the ideal generated by all homogeneous degree $d$ elements of $I$. The ideal $I$ is **componentwise linear** if for all $d \in \mathbb{N}$, $(I_d)$ has a linear resolution.

All ideals with linear resolutions are componentwise linear, but so are all stable ideals and a number of others; see, for example, [6]. When $I$ is a square-free monomial ideal, Herzog and Hibi give the following useful criterion for $I$ to be componentwise linear. Write $(I_{[d]})$ for the ideal generated by all square-free monomials of degree $d$ in $I$.

**Theorem 2.8.** *Let $I$ be a square-free monomial ideal. $I$ is componentwise linear if and only if for all $d \in \mathbb{N}$, $(I_{[d]})$ has a linear resolution.*

Using componentwise linearity, Herzog and Hibi proved the following generalization of Theorem 2.6 in [9].

**Theorem 2.9.** *Let $I$ be a square-free monomial ideal in $R = k[x_1, \ldots, x_n]$ with Alexander dual $I^\vee$. $R/I$ is sequentially Cohen-Macaulay over $k$ if and only if $I^\vee$ is componentwise linear in $R$.*

Theorem 2.9 allows us to investigate the sequentially Cohen-Macaulay property of an ideal by determining when the Alexander dual is componentwise linear. Proving that a class of ideals is componentwise linear is generally difficult, but there are some good methods available. Herzog and Takayama's theory of linear quotients [11] gives one useful technique.

**Definition 2.10.** Let $I$ be a monomial ideal. $I$ is said to have **linear quotients** if $I$ has a system of minimal generators $\{u_1, \ldots, u_r\}$ with $\deg u_1 \leq \cdots \leq \deg u_r$ such that for all $1 \leq i \leq r-1$, $(u_1, \ldots, u_i) : (u_{i+1})$ is generated by linear forms.

**Remark 2.11.** It is easy to see that if $I$ is an ideal generated in a single degree, and $I$ has linear quotients, then $I$ has a linear resolution. We shall use this observation frequently: If $(I_d)$ has linear quotients for all $d$, then $I$ is componentwise linear. In particular, if $I$ is a square-free monomial ideal, and for all $d$, $(I_{[d]})$ has a linear resolution, then $I$ is componentwise linear.

Note that having linear quotients is independent of the characteristic of the field $k$. In general, an ideal may have a linear resolution when the base field has some characteristics but a nonlinear one over other characteristics; showing an ideal has linear quotients proves that the ideal has a linear resolution regardless of the characteristic.

**Definition 2.12.** We say that a graph $G$ has **dual linear quotients** if for each degree $d \in \mathbb{N}$, $(\mathcal{I}(G)^\vee_{[d]})$ has linear quotients.

**Remark 2.13.** As in Remark 2.11, if $G$ has dual linear quotients, then $G$ is sequentially Cohen-Macaulay (over a field of any characteristic). This approach has been used by Faridi [4] in proving that simplicial trees are sequentially Cohen-Macaulay, and by the first author and Van Tuyl [5] in showing that chordal graphs are sequentially Cohen-Macaulay.

As a byproduct of the arguments of [5] we also get:

**Theorem 2.14.** *Let $G$ be a chordal graph, and let $H$ be an arbitrary induced subgraph of $G$. Then $H$ has dual linear quotients.*

*Proof.* Since $H$ is an induced subgraph of $G$, $H$ is also chordal. The conclusion now follows from the arguments of [5, Theorem 3.2]. □

We shall use Theorem 2.14 in inductive arguments in the next section.

Using Alexander duality, one can describe how being Cohen-Macaulay differs from being sequentially Cohen-Macaulay in the square-free monomial ideal case. The next result is surely known, but we were unable to find a reference when writing this paper.

**Lemma 2.15.** *Let $I$ be a square-free monomial ideal in $R = k[x_1, \dots, x_n]$. Then $R/I$ is Cohen-Macaulay if and only if $R/I$ is sequentially Cohen-Macaulay, and $I$ is unmixed.*

*Proof.* If $R/I$ is Cohen-Macaulay, the result is trivial. Assume that $R/I$ is sequentially Cohen-Macaulay and that $I$ is unmixed. Let $I^\vee$ be the Alexander dual of $I$. Since $R/I$ is sequentially Cohen-Macaulay, $I^\vee$ is componentwise linear by Theorem 2.9. Because $I$ is unmixed, $I^\vee$ has all its generators in the same degree; hence since $I^\vee$ is also componentwise linear, $I^\vee$ has a linear resolution. Therefore, Theorem 2.6 implies that $R/I^{\vee\vee} = R/I$ is Cohen-Macaulay. □

Since the unmixedness of a graph (or a simplicial complex) is a combinatorial property (on the cardinality of minimal vertex covers), in light of Lemma 2.15, investigating the Cohen-Macaulayness of a graph (or a simplicial complex) reduces to studying the sequentially Cohen-Macaulayness of such a graph (or the simplicial complex). In particular, for a sequentially Cohen-Macaulay graph (or a sequentially Cohen-Macaulay simplicial complex), for example, a chordal graph or a simplicial tree, we know that it is Cohen-Macaulay if and only if it is unmixed.

## 3. Whiskers and sequentially Cohen-Macaulay graphs

In this section, we explore how to add a configuration of whiskers to an arbitrary graph to create a sequentially Cohen-Macaulay graph. The primary question in which we are interested is:

**Question 3.1.** Let $G$ be a graph and let $S \subset V_G$. Under what conditions on $S$ is $G \cup W(S)$ sequentially Cohen-Macaulay?

Because being sequentially Cohen-Macaulay is a weaker property than being Cohen-Macaulay, one expects that $S$ needs not be all of $V_G$ to ensure that $G \cup W(S)$ is sequentially Cohen-Macaulay. The focus of this section is on sufficient conditions that guarantee the sequential Cohen-Macaulayness of $G \cup W(S)$. Our results shall also recover Theorem 1.2 as a consequence.

Because the vertex covers of $G$ are the generators of the Alexander dual of $\mathcal{I}(G)$, we are often interested in ways to partition the set of vertex covers of $G$ of a particular cardinality. For any graph $G$ and vertex $x \in V_G$ with $N(x) = \{y_1, \ldots, y_t\}$, we can decompose the set of vertex covers of $G$ of size $d$ in the following way: Any vertex cover of $G$ of size $d$ is either $x$ times a vertex cover of $G \backslash \{x\}$ of size $d-1$, or it is $y_1 \cdots y_t$ times a vertex cover of $G \backslash \{x, y_1, \ldots, y_t\}$ of size $d - t$. For our purposes, we frequently consider the case in which $G$ contains a whisker $xy$, where $x$ is the vertex of degree one. In this case, the vertex covers of $G$ are decomposed based on covers of $G \backslash \{x\}$ and covers of $G \backslash \{x, y\}$. In particular, the set of vertex covers of $G$ of size $d$ is the union of $x$ times the vertex covers of $G \backslash \{x\}$ of size $d-1$ and $y$ times the vertex covers of $G \backslash \{x, y\}$ of size $d-1$.

The next theorem is the first step in exploring how to add whiskers to a graph to make it sequentially Cohen-Macaulay. Recall that $H$ is a **induced subgraph** of a graph $G$ if $H$ is a subgraph of $G$, and if $z_1$ and $z_2$ are vertices of $H$, and $z_1 z_2$ is an edge of $G$, then $z_1 z_2$ is an edge of $H$.

**Theorem 3.2.** *Let $G'$ be a simple graph and let $S = \{y_1, \ldots, y_n\} \subset V_{G'}$ be a subset of the vertices of $G'$. Let $\{x_1 y_1, \ldots, x_n y_n\}$ be whiskers of $G'$ at $S$ and let $G = G' \cup W(S)$. Suppose that if $H \subset G$ is an induced subgraph of $G$ such that both*

*(i) $\{x_1, \ldots, x_{n-1}\} \subset V_H$, and*

*(ii) $x_n \notin H$ and $y_n \notin H$*

*hold, then $H$ has dual linear quotients. Then all induced subgraphs $K \subset G$ such that $\{x_1, \ldots, x_n\} \subset V_K$ have dual linear quotients.*

*Proof.* For simplicity of notation, let $\{z_1, \ldots, z_r\} = V_{G'} \backslash S$. Fix an induced subgraph $K \subset G$ as in the statement of the theorem. Consider first the case in which $y_n \notin K$. Then $x_n$ is an isolated vertex of $K$. Let $H$ be the graph obtained from $K$ by removing the isolated vertex $x_n$. Clearly, $H$ satisfies properties $(i)$ and $(ii)$ of the hypothesis. Thus, $H$ has dual linear quotients, i.e., $(\mathcal{I}(H)^{\vee}_{[d]})$ has linear quotients for all $d \in \mathbb{N}$. Since the only edge to which $x_n$ is incident in $G$ is $x_n y_n$, the minimal generating set of $\mathcal{I}(K)$ is the same as the minimal generating set of $\mathcal{I}(H)$; thus the minimal generating sets of $\mathcal{I}(K)^{\vee}$ and $\mathcal{I}(H)^{\vee}$ are the same, though the first is an ideal of $k[x_1, \ldots, x_n, y_1, \ldots, y_{n-1}, z_1, \ldots, z_r]$, and the second is an ideal of $k[x_1, \ldots, x_{n-1}, y_1, \ldots, y_{n-1}, z_1, \ldots, z_r]$. Thus for all $d \in \mathbb{N}$, by [6, Lemma 2.9], since $(\mathcal{I}(H)^{\vee}_{[d]})$ has linear quotients, so does $(\mathcal{I}(K)^{\vee}_{[d]})$.

Consider instead the case in which $y_n \in K$. Let $H$ be the subgraph of $K$ obtained by removing $x_n$, $y_n$, and all edges incident to $x_n$ or $y_n$. Again, $H$ satisfies $(i)$ and $(ii)$ of the hypotheses; and thus, $H$ has dual linear quotients. Fix a degree $d \in \mathbb{N}$. Let $A_1, \dots, A_a$ be the monomials that represent all vertex covers of $K \backslash \{x_n\}$ of size $d-1$, and let $B_1, \dots, B_b$ be the monomials that represent all vertex covers of $H = K \backslash \{x_n, y_n\}$ of size $d-1$; that is, $(B_1, \dots, B_b) = (\mathcal{I}(H)^\vee_{[d-1]})$. We have

$$(\mathcal{I}(K)^\vee_{[d]}) = x_n(A_1, \dots, A_a) + y_n(B_1, \dots, B_b).$$

The $A_i$s are monomials in the variables $x_1, \dots, x_{n-1}, y_1, \dots, y_n, z_1, \dots, z_r$, and the $B_i$s are monomials in the variables $x_1, \dots, x_{n-1}, y_1, \dots, y_{n-1}, z_1, \dots, z_r$. Since $H$ has dual linear quotients, $(\mathcal{I}(H)^\vee_{[d]})$ has linear quotients for all $d$. We may assume that the $B_i$s are indexed in the order that gives linear quotients (that is, $(B_1, \dots, B_{i-1}) : B_i$ is generated by a subset of the variables for all $i$).

We wish to show that the ideal

$$(y_nB_1, \dots, y_nB_b, x_nA_1, \dots, x_nA_a)$$

has linear quotients. Since $(B_1, \dots, B_b)$ has linear quotients (in that order), it suffices to show that for all $j$,

$$(y_nB_1, \dots, y_nB_b, x_nA_1, \dots, x_nA_{j-1}) : x_nA_j$$

is generated by a subset of the variables. To this end, we consider two possibilities.

Suppose first that $y_n$ divides $A_j$. Then $x_nA_j = x_ny_nC$, where $C$ is a vertex cover of $H = K \backslash \{x_n, y_n\}$ of size $d-2$. Let $T$ be the set of variables in $\{x_1, \dots, x_{n-1}, y_1, \dots, y_{n-1}, z_1, \dots, z_r\}$ which are not in the support of $C$, and suppose $u \in T$. Then $uC$ is a vertex cover of $H$ of size $d-1$, so it is one of the $B_i$. Therefore for any $u \in T$, $ux_nA_j \in (y_nB_1, \dots, y_nB_b)$. Moreover, note that $(y_nB_1, \dots, y_nB_b, x_nA_1, \dots, x_nA_{j-1})$ is a square-free monomial ideal; thus, if $m$ is a minimal monomial generator of

$$(y_nB_1, \dots, y_nB_b, x_nA_1, \dots, x_nA_{j-1}) : x_nA_j,$$

then $m$ is square-free. This implies that $x_n$, $y_n$, and variables in the support of $C$ do not divide $m$. Thus

$$(y_nB_1, \dots, y_nB_b, x_nA_1, \dots, x_nA_{j-1}) : x_nA_j = (\text{all variables } u \in T).$$

Next we assume that $y_n$ does not divide $A_j$. Note that any $A_j$ that is not divisible by $y_n$ is one of the $B_i$s because a cover of $K \backslash \{x_n\}$ not containing $y_n$ is a cover of $H$. Thus $A_j = B_{i_j}$ for some $i_j$. Consider a monomial $m$ for which $mx_nA_j \in (y_nB_1, \dots, y_nB_b)$. Then since $y_n$ does not divide $A_j$, $y_n$ must divide $m$. But $y_nx_nA_j = y_nx_nB_{i_j} \in (y_nB_1, \dots, y_nB_b)$, so $y_nx_nA_j \in (y_nB_1, \dots, y_nB_b)$. Hence $(y_nB_1, \dots, y_nB_b) : x_nA_j = (y_n)$.

The last remaining situation is when $y_n$ does not divide $A_j$, and $m$ is a monomial such that $mx_nA_j$ lands in the ideal $(x_nA_1, \dots, x_nA_{j-1})$. This case requires a bit more work. We need to specify an order for the $A_i$ monomials. Note that, so far we have not used any feature of the ordering of the $A_i$s. We may pick an order so that all the $A_i$s not divisible by $y_n$ are indexed first, and those that are divisible by $y_n$ are last. Suppose that $\{A_1, \dots, A_t\}$ are all the $A_i$s that are not divisible by $y_n$, and $A_l = B_{i_l}$ for $l = 1, \dots, t$. For each $1 \le l \le t$, $A_l = B_{i_l}$ is a vertex cover of $K \backslash \{x_n\}$ not containing $y_n$, so it is divisible by all variables in $N(y_n) \backslash \{x_n\}$. Let $D$ be the monomial given by the variables in $N(y_n)\backslash\{x_n\}$. For $1 \leq l \leq t$, let $C_l = A_l/D$. Then, clearly, $\{C_1, \ldots, C_t\}$ are the vertex covers of $L = K\backslash\{x_n, y_n, N(y_n)\}$ of size $d-1-u$, where $u = |N(y_n)| - 1$. Conversely, if $C$ is a vertex cover of $L$, then $CD$ is a vertex cover of $K\backslash\{x_n\}$ not containing $y_n$. Thus $\{C_1, \ldots, C_t\}$ are all vertex covers of $L$ of size $d-1-u$. Since $x_n \in N(y_n)$ but $x_j \notin N(y_n)$ for $j \neq n$, it is easy to see that $L$ satisfies $(i)$ and $(ii)$ of the hypotheses. Therefore, $L$ has dual linear quotients. This implies that the ideal $(C_1, \ldots, C_t)$ has linear quotients. We shall reindex $\{A_1, \ldots, A_t\}$ so that $C_1, \ldots, C_t$ is the order of the generators in which $(C_1, \ldots, C_t)$ has linear quotients.

Now suppose that $m$ is a monomial so that $mx_nA_j \in (x_nA_1, \ldots, x_nA_{j-1})$. Dividing by the monomial given by $N(y_n)$ (including $x_n$), we have $mC_j \in (C_1, \ldots, C_{j-1})$. Since $(C_1, \ldots, C_t)$ has linear quotients, $(C_1, \ldots, C_{j-1}) : C_j = (x_{p_1}, \ldots, x_{p_v})$ for some subset of the variables. Thus if $mx_nA_j \in (x_nA_1, \ldots, x_nA_{j-1})$, then some variable $x_{p_w}$ divides $m$.

We have shown that $(y_nB_1, \ldots, y_nB_b, x_nA_1, \ldots, x_nA_a)$ has linear quotients. This is true for any $d \in \mathbb{N}$. Hence, the conclusion follows. □

We are now ready to prove our first main result.

**Theorem 3.3.** *Let $G$ be a simple graph. Let $S \subset V_G$ be such that $G\backslash S$ is a chordal graph. Then $G \cup W(S)$ is a sequentially Cohen-Macaulay graph.*

*Proof.* Let $S = \{y_1, \ldots, y_n\}$ and $W(S) = \{x_1y_1, \ldots, x_ny_n\}$. By Remark 2.13, it suffices to show that $G \cup W(S)$ has dual linear quotients.

We shall first construct a class of subgraphs of $G$ as follows. Let $G_0 = G\backslash S$. Let $G_1 = G_0 \cup \{x_1, y_1\} \cup \{\text{edges of } G \text{ incident to vertices of } V_{G_0} \cup \{x_1, y_1\}\}$. More generally, for $1 \leq i \leq n$, let $G_i = G_{i-1} \cup \{x_i, y_i\} \cup \{\text{edges of } G \text{ incident to vertices of } V_{G_{i-1}} \cup \{x_i, y_i\}\}$. Observe that $G_n = G$. Now, the conclusion will follow if we can show that every induced subgraph $K$ of $G$ (in particular, $G$ itself) containing $\{x_1, \ldots, x_n\}$ has dual linear quotients. To this end, we shall use induction on $i$ to show that every induced subgraph $K$ of $G_i$ containing $\{x_1, \ldots, x_i\}$ has dual linear quotients for $i = 0, \ldots, n$.

Indeed, for $i = 0$, the assertion follows from Theorem 2.14. Suppose $i \geq 1$. Consider an arbitrary induced subgraph $H$ of $G_i$ such that $\{x_1, \ldots, x_{i-1}\} \subset V_H$, $x_i \notin V_H$ and $y_i \notin V_H$. It is easy to see that $H$ is also an induced subgraph of $G_{i-1} = G_i\backslash\{x_i, y_i\}$. Thus, by induction $H$ has dual linear quotients. It now follows from Theorem 3.2 that every induced subgraph $K$ of $G_i$ with $\{x_1, \ldots, x_i\} \subset V_K$ has dual linear quotients. The theorem is proved. □

**Lemma 3.4.** *Let $C_n$ be an $n$-cycle. If $n \neq 3$ and $n \neq 5$, then $C_n$ is not sequentially Cohen-Macaulay. If $n = 3$ or $n = 5$, then $C_n$ has dual linear quotients, as do all subgraphs of $C_n$.*

*Proof.* If $n$ is not three or five, then [5, Proposition 4.1] shows that $C_n$ is not sequentially Cohen-Macaulay. If $n = 3$ then the edge ideal is $\mathcal{I}(C_3) = (x_1x_2, x_1x_3, x_2x_3)$, which is its own Alexander dual and clearly $(\mathcal{I}(C_3)_{[d]})$ has linear quotients in each degree $d$. If $n = 5$ then

$$\mathcal{I}(C_5)^\vee = (x_1x_2x_4, x_1x_3x_4, x_1x_3x_5, x_2x_3x_5, x_2x_4x_5).$$

It is easy to check that this ideal has linear quotients in the given order, and $(\mathcal{I}(C_5)^\vee_{[4]})$ has linear quotients with respect to descending graded reverse-lex order.

If $H$ is a proper subgraph of $C_3$ or $C_5$, it is a forest, and thus it has dual linear quotients by Theorem 2.14. □

Lemma 3.4 allows us to extend Theorem 3.3 slightly.

**Theorem 3.5.** *Let $G$ be a simple graph and let $S \subset V_G$. Suppose $G \backslash S$ is a chordal graph or a five-cycle $C_5$. Then $G \cup W(S)$ is a sequentially Cohen-Macaulay graph.*

*Proof.* If $G \backslash S$ is a chordal graph, then the conclusion is Theorem 3.3. Suppose that $G \backslash S = C_5$. Observe that the inductive argument of Theorem 3.3 is based on the fact that every induced subgraph of $G_0 = G \backslash S$ has dual linear quotients. In our current situation, by Lemma 3.4, $G_0 = G \backslash S = C_5$ does have this property. Therefore arguments similar to those in Theorem 3.3 yield the assertion. □

Theorem 3.5 (and Theorem 3.3) gives many interesting corollaries about the configurations of whiskers that can be added to a graph to obtain a sequentially Cohen-Macaulay graph.

**Corollary 3.6.** *Let $G$ be a simple graph and let $S \subset V_G$ be a vertex cover of $G$. Then $G \cup W(S)$ is a sequentially Cohen-Macaulay graph.*

*Proof.* Observe that since $S$ is a vertex cover of $G$, $G \backslash S$ is a graph of isolated vertices. A graph without any edges is clearly a chordal graph. Thus, the assertion is a consequence of Theorem 3.3. □

**Corollary 3.7.** *Let $G$ be a simple graph. Let $S \subset V_G$ be such that $G \backslash S$ is a forest (i.e., each connected component of $G \backslash S$ is a tree). Then $G \cup W(S)$ is a sequentially Cohen-Macaulay graph.*

*Proof.* The assertion is a direct consequence of Theorem 3.3 since every forest is a chordal graph. □

Corollary 3.7 allows one to make a cycle into a sequentially Cohen-Macaulay graph quite easily; only one whisker is necessary. With Van Tuyl, we noticed this phenomenon after doing many computations in the computer algebra system Macaulay 2 [15], and it was a primary initial motivation for this paper.

**Corollary 3.8.** *Let $C$ be a cycle and let $y \in V_C$ be a vertex of $C$. Then $C \cup W(y)$ is a sequentially Cohen-Macaulay graph.*

*Proof.* Clearly, $C \backslash \{y\}$ is a tree. Thus, the conclusion follows from Corollary 3.7. □

Notice that Corollary 3.6 states that to obtain sequentially Cohen-Macaulay graphs, the number of whiskers is not as important as their configuration. Our next corollary complements Corollary 3.6 to give a bound on the number of whiskers to add to a graph, regardless of how they are picked, to obtain a sequentially Cohen-Macaulay graph.

**Corollary 3.9.** *Let $G$ be a simple graph and let $S \subset V_G$. Assume that $|S| \geq |V_G| - 3$. Then $G \cup W(S)$ is a sequentially Cohen-Macaulay graph.*

*Proof.* Since $|S| \geq |V_G| - 3$, $G \backslash S$ is a graph on at most 3 vertices. Thus, $G \backslash S$ is either a three-cycle, a tree, or a graph with isolated vertices. These are all chordal graphs, and hence, the conclusion follows from Theorem 3.3. □

We will see later (in Example 4.3) that the bound $|V_G| - 3$ in Corollary 3.9 is sharp. Corollary 3.9 further allows us to recover Theorem 1.2 as a consequence of our work.

**Corollary 3.10.** *Let $G$ be a simple graph with vertex set $V_G$. Then $G \cup W(V_G)$ is a Cohen-Macaulay graph.*

*Proof.* By Corollary 3.9 we know that $G' = G \cup W(V_G)$ is sequentially Cohen-Macaulay. In view of Lemma 2.15, it suffices to show that $G'$ is unmixed; i.e., all minimal vertex covers of $G'$ have the same cardinality. Suppose $V_G = \{y_1, \dots, y_n\}$ and $W(V_G) = \{x_1y_1, \dots, x_ny_n\}$. Let $V$ be an arbitrary minimal vertex cover of $G'$. Clearly, for each $i = 1, \dots, n$, $V$ has to contain one of the vertices $\{x_i, y_i\}$ (to cover the edge $x_iy_i$). Moreover, since $V$ is minimal, for each $i = 1, \dots, n$, $V$ contains exactly one of the vertices $\{x_i, y_i\}$. Hence, $|V| = n$. This is true for any minimal vertex cover $V$ of $G'$. Thus, the assertion follows. □

In the final theorem of this section, we isolate the condition from the proof of Theorem 3.3 that yields that result and its corollaries.

**Theorem 3.11.** *Let $G$ be a simple graph with $S$ a subset of its vertex set. Then all induced subgraphs of $G\backslash S$ have dual linear quotients if and only if all induced subgraphs of $G \cup W(S)$ have dual linear quotients.*

*Proof.* If all induced subgraphs of $G \cup W(S)$ have dual linear quotients, then so do all induced subgraphs of $G\backslash S$ since any induced subgraph of $G\backslash S$ is an induced subgraph of $G \cup W(S)$. Assume instead that all induced subgraphs of $G\backslash S$ have dual linear quotients. Then an argument identical to the proof of Theorem 3.3 shows that all induced subgraphs of $G \cup W(S)$ have dual linear quotients. □

We now give two examples to show that the hypotheses of Theorem 3.11 cannot easily be weakened.

**Example 3.12.** Let $G$ be a simple graph with $S$ a subset of the vertex set of $G$, and assume that $G\backslash S$ has dual linear quotients. In this example, we show that if there exists a subgraph of $G\backslash S$ that does not have dual linear quotients, then $G \cup W(S)$ may fail to be sequentially Cohen-Macaulay.

Let $G$ be the graph on the vertex set $V_G = \{x_1, \dots, x_6\}$ together with edge set $E_G = \{x_1x_2, x_2x_3, x_3x_4, x_1x_4, x_3x_5, x_4x_5, x_5x_6\}$. Let $S = \{x_6\}$, so $G \cup W(S)$ is the graph $G$ along with a new vertex $x_7$ and edge $x_6x_7$. Then

$$\mathcal{I}(G\backslash S)^\vee = (x_1x_3x_4, x_2x_3x_4, x_1x_3x_5, x_2x_4x_5),$$

which has linear quotients in the order in which the generators are listed. Hence the graph $G\backslash S$ has dual linear quotients. Note, however, that not all induced subgraphs of $G\backslash S$ have dual linear quotients; the four-cycle comprised of the vertices $\{x_1, \dots, x_4\}$ is not even sequentially Cohen-Macaulay.

Now we consider $G \cup W(S)$. We have

$$\mathcal{I}(G \cup W(S))^\vee = (x_1x_3x_4x_6, x_2x_3x_4x_6, x_1x_3x_5x_6, x_2x_4x_5x_6, x_1x_3x_5x_7, x_2x_4x_5x_7).$$

The minimal graded free resolution of $\mathcal{I}(G \cup W(S))^\vee$ is

$$0 \longrightarrow R(-7) \longrightarrow R(-5)^5 \oplus R(-6) \longrightarrow R(-4)^6 \longrightarrow \mathcal{I}(G \cup W(S))^\vee \longrightarrow 0,$$

where $R = k[x_1, \ldots, x_7]$. Therefore $\mathcal{I}(G \cup W(S))^\vee$ is not componentwise linear because of the syzygies in degrees six and seven. Hence $G \cup W(S)$ is not sequentially Cohen-Macaulay.

**Example 3.13.** Again we assume that $G$ is a simple graph with $S \subset V_G$ such that $G \backslash S$ has dual linear quotients. Now we show that even if there exists a subgraph of $G \backslash S$ that does not have dual linear quotients (and, in fact, is not sequentially Cohen-Macaulay), $G \cup W(S)$ may itself have dual linear quotients.

Let $G$ be the graph with $V_G = \{x_1, \ldots, x_6\}$ and edge set

$$E_G = \{x_1x_2, x_2x_3, x_3x_4, x_1x_4, x_3x_5, x_4x_5, x_2x_6, x_3x_6\}.$$

Let $S = \{x_6\}$. Then $\mathcal{I}(G \backslash S)^\vee$ is the same as in Example 3.12, and hence $G \backslash S$ has dual linear quotients. Note that the induced subgraph on the vertices $\{x_1, \ldots, x_4\}$ is a four-cycle, which is not sequentially Cohen-Macaulay.

We want to show that $G \cup W(S)$ has dual linear quotients. The dual of the edge ideal of this graph is

$$\mathcal{I}(G \cup W(S))^\vee = (x_1x_3x_4x_6, x_2x_3x_4x_6, x_1x_3x_5x_6, x_2x_4x_5x_6, x_2x_3x_4x_7, x_1x_2x_3x_5x_7).$$

One can check that this ideal has linear quotients with respect to the order in which the generators are listed. Therefore $G \cup W(S)$ is sequentially Cohen-Macaulay.

Consequently, if $G \backslash S$ has dual linear quotients, but there exists a subgraph of $G \backslash S$ without dual linear quotients, then $G \cup W(S)$ may or may not be sequentially Cohen-Macaulay. This is the primary reason for our techniques in the proofs in Section 3; in our inductive approach, we assume that *all* induced subgraphs of a certain type have dual linear quotients to avoid cases like Example 3.12.

We conclude this section by remarking that it is difficult to find results analogous to Theorem 3.11 for Cohen-Macaulay graphs. One is tempted to conjecture that if $G$ is a simple graph, and $S$ is a subset of $V_G$ such that all induced subgraphs of $G \backslash S$ have dual linear quotients and are unmixed, then $G \cup W(S)$ is Cohen-Macaulay. Unfortunately, this is false. An easy counterexample is the case in which $G$ is the graph on two vertices, $y_1$ and $y_2$, with an edge connecting them. Let $S = \{y_1\}$; then $G \backslash S$ trivially has dual linear quotients and is unmixed. However, $k[x, y_1, y_2]/(xy_1, y_1y_2)$ is not Cohen-Macaulay. (There exist less degenerate examples as well; this is just the simplest counterexample.) The difficulty in searching for the appropriate analogue to Theorem 3.11 is guaranteeing the unmixedness of $G \cup W(S)$.

## 4. Whiskers and non-sequentially Cohen-Macaulay graphs

In the previous section, we have given sufficient conditions for getting sequentially Cohen-Macaulay graphs by adding whiskers. In this section, the converse problem is addressed. Our primary interest is necessary conditions on a graph $G$ and $S \subset V_G$ so that $G \cup W(S)$ has a chance to be sequentially Cohen-Macaulay.

To show that certain graphs are not sequentially Cohen-Macaulay, we exploit Alexander duality and show that the dual of the edge ideal is not componentwise linear. This requires investigating the syzygies of the dual, and to do that, we use simplicial homology.

Define a square-free vector to be a vector with its entries in $\{0,1\}$. For any monomial ideal $M$, we define the **upper Koszul simplicial complex of $M$**:

$$K^{\mathbf{b}}(M) = \{\text{square-free vectors } \mathbf{a} \text{ such that } \frac{x^{\mathbf{b}}}{x^{\mathbf{a}}} \in M\}.$$

See, e.g., [16]. Using the relation

$$\beta_{i,\mathbf{b}}(M) = \dim_k \tilde{H}_{i-1}(K^{\mathbf{b}}(M), k),$$

which is [16, Theorem 1.34], we can compute the $\mathbb{N}^n$-graded Betti numbers of $M$. We use this technique in the following theorem.

**Theorem 4.1.** *Let $G$ be a simple graph and let $S \subset V_G$. If $G\backslash S$ is not sequentially Cohen-Macaulay, then $G \cup W(S)$ is not sequentially Cohen-Macaulay.*

*Proof.* Let $\{z_1, \ldots, z_r\} = V_G \backslash S$. Because $G\backslash S$ is not sequentially Cohen-Macaulay, there exists $d$ such that $I := (\mathcal{I}(G\backslash S)^{\vee}_{[d]})$ has a nonlinear $i$th syzygy in its minimal free resolution. Suppose this nonlinear syzygy occurs in the multi-degree $\mathbf{b}$ corresponding to the square-free monomial $z_{i_1} \ldots z_{i_l}$ for some $l > d + i$. Then $\dim_k \tilde{H}_{i-1}(K^{\mathbf{b}}(I), k) \neq 0$.

Let $\{y_1, \ldots, y_n\} = S$ and let $W(S) = \{x_1y_1, \ldots, x_ny_n\}$. Let $J = (\mathcal{I}(G \cup W(S))^{\vee}_{[d+n]})$. Let $\mathbf{c}$ be the multi-degree of the square-free monomial $z_{i_1} \ldots z_{i_l} y_1 \ldots y_n$. We claim that the simplicial complexes $K^{\mathbf{b}}(I)$ and $K^{\mathbf{c}}(J)$ are the same. By definition, a square-free vector $\mathbf{a}$ is in $K^{\mathbf{c}}(J)$ if and only if $\frac{x^{\mathbf{c}}}{x^{\mathbf{a}}} \in (\mathcal{I}(G \cup W(S))^{\vee}_{[d+n]})$. In other words, a square-free vector $\mathbf{a}$ is in $K^{\mathbf{c}}(J)$ if and only if the square-free monomial corresponding to $\mathbf{c} - \mathbf{a}$ gives a vertex cover of $G \cup W(S)$. Since $\mathbf{c}$ has 0 in entries corresponding to $\{x_1, \ldots, x_n\}$, if $\mathbf{c} - \mathbf{a}$ gives a vertex cover of $G \cup W(S)$ then $\mathbf{c} - \mathbf{a}$ must have 1 in all entries corresponding to $\{y_1, \ldots, y_n\}$, and $\mathbf{a}$ must have 0 in all entries corresponding to $\{x_1, \ldots, x_n\}$. Therefore, the only places in which $\mathbf{a}$ may be nonzero are in entries corresponding to the $z_i$s. It follows that the vectors $\mathbf{a}$ such that $\mathbf{c} - \mathbf{a}$ gives a vertex cover of $G \cup W(S)$ are exactly the same as the vectors $(\mathbf{a}', \mathbf{0})$, where $\mathbf{0} = (0, \ldots, 0) \in \mathbb{Z}^{2n}$ appears in the entries corresponding to $\{x_1, \ldots, x_n, y_1, \ldots, y_n\}$, so that $\mathbf{b} - \mathbf{a}'$ gives a vertex cover of $G\backslash S$. Hence the $\mathbf{a}'$ in $K^{\mathbf{c}}(J)$ are exactly the $\mathbf{a}$ in $K^{\mathbf{b}}(I)$. This obviously implies that the simplicial complexes $K^{\mathbf{b}}(I)$ and $K^{\mathbf{c}}(J)$ are the same.

We now have $\dim_k \tilde{H}_{i-1}(K^{\mathbf{c}}(J), k) = \dim_k \tilde{H}_{i-1}(K^{\mathbf{b}}(I), k) \neq 0$. This gives a nonlinear $i$th syzygy of $J$ since $l > d + i$ implies $l + n > i + d + n$, and $J$ is generated in degree $d + n$. Hence, $J$ does not have a linear resolution, and thus $G \cup W(S)$ is not sequentially Cohen-Macaulay by Theorem 2.9. □

As a corollary, we can identify certain vertex sets to which adding whiskers does not yield a sequentially Cohen-Macaulay graph.

**Corollary 4.2.** *Let $G$ be a simple graph and let $S \subset V_G$. Suppose that $G\backslash S = C_n$, where $n$ is neither three nor five. Then $G \cup W(S)$ is not sequentially Cohen-Macaulay.*

*Proof.* Since $n$ is neither three nor five, by Lemma 3.4, $C_n$ is not sequentially Cohen-Macaulay. The assertion is a consequence of Theorem 4.1. □

Corollary 4.2 helps show the sharpness of Corollary 3.9. The following example illustrates how one can have a graph $G$ and a vertex subset $S$ containing all but four of the vertices of $G$, and $G \cup W(S)$ is not sequentially Cohen-Macaulay. (Of course, a four-cycle itself with $S$ being empty is a degenerate example.)

**Example 4.3.** Let $G$ be a four-cycle with vertices $\{y_1, \ldots, y_4\}$ and an edge $y_1y$ attached at $y_1$, so

$$\mathcal{I}(G) = (y_1y_2, y_2y_3, y_3y_4, y_1y_4, y_1y).$$

Let $S = \{y\}$, and add a whisker $xy$ to $G$ to form $G \cup W(S)$. Then

$$\mathcal{I}(G \cup W(S))^\vee = (y_1y_3y, y_2y_4y, y_1y_3x, y_1y_2y_4x) \subset R = k[y_1, \ldots, y_4, y, x].$$

Note that $(\mathcal{I}(G \cup W(S))^\vee_3)$ has minimal graded free resolution

$$0 \longrightarrow R(-4) \oplus R(-5) \longrightarrow R(-3)^3 \longrightarrow I \longrightarrow 0,$$

which is not a linear resolution. Hence the Alexander dual of $\mathcal{I}(G \cup W(S))$ is not componentwise linear, and therefore $G \cup W(S)$ is not sequentially Cohen-Macaulay.

The primary case we have not considered in our paper is when a graph is sequentially Cohen-Macaulay but does not have dual linear quotients. This case will require different techniques since sequential Cohen-Macaulayness can depend on the underlying field $k$, but having dual linear quotients is independent of the field. We give an example of this phenomenon.

**Example 4.4.** Let $\Delta$ be a minimal triangulation of the real projective plane. Then $\Delta$ is Cohen-Macaulay over a field $k$ if and only if the characteristic of $k$ is not two [1]. From this, we can construct an example of a graph $G$ that is Cohen-Macaulay if and only if the base field does not have characteristic two, using the method described in [10]. Let $P$ be the face poset of $\Delta$; then the order complex of $P$ has the property that all its minimal nonfaces are subsets of cardinality two, so the associated Stanley-Reisner ideal is generated by degree two monomials and hence is an edge ideal (with a large number of generators). The polynomial ring modulo this edge ideal is Cohen-Macaulay if and only if the ground field does not have characteristic two, just like the original simplicial complex $\Delta$.

We know of no example of a graph $G$ with a small number of vertices that is (sequentially) Cohen-Macaulay over fields of some characteristics but not others, and it would be interesting to know of small examples if they exist.

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