COMPUTING THE SPREADING AND COVERING NUMBERS

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Abstract. Let $S = k[x_1,\ldots,x_n]$, d a positive integer, and suppose that S_d is the vector space of all polynomials of degree d in S. Define $\alpha_n(d) := \max\{\dim_k V | V \text{ monomial subspace of } S_d, \dim_k S_1V = n\dim_k V\}$ and $\rho_n(d+1) := \min\{\dim_k V | V \text{ monomial subspace of } S_d, S_1V = S_{d+1}\}$. The numbers $\alpha_n(d)$ and $\rho_n(d+1)$ are called the spreading numbers and covering numbers, respectively. We describe an approach to calculate these numbers that uses simplicial complexes.

1. Introduction

Let $S = k[x_1,\ldots,x_n]$, k a field, and suppose S_d is the vector space of all polynomials of degree d in S. We let $\mathbf{A}_d := \{m_1,\ldots,m_l\}$ be the set of all $l = \binom{d+n-1}{n-1}$ monomials of degree d that generate S_d. In particular, $\mathbf{A}_1 = \{x_1,\ldots,x_n\}$. If V is a monomial subspace of S_d, then $S_1V := \{x_im_j | 1 \leq i \leq n, m_j \in V\}$ is a monomial subspace of S_{d+1}. We define the following numbers:

$$\begin{aligned}\alpha_n(d) &:= \max\{\dim_k V | V \text{ monomial subspace of } S_d, \dim_k S_1V = n\dim_k V\}\\ \rho_n(d+1) &:= \min\{\dim_k V | V \text{ monomial subspace of } S_d, S_1V = S_{d+1}\}.\end{aligned}$$

The number $\alpha_n(d)$ is called the *spreading number* to capture the idea that the monomials of S_1V are spread out within S_{d+1}. The number $\rho_n(d+1)$ is called the *covering number* since S_1V covers S_{d+1}.

The numbers $\alpha_n(d)$ and $\rho_n(d+1)$ were first studied by Geramita et al. [5] in connection to the Ideal Generation Conjecture. In [5] bounds were placed on $\alpha_n(d)$ and $\rho_n(d+1)$, and for small n and d, explicit formulas for $\alpha_n(d)$ and $\rho_n(d+1)$ were derived. More recently Curtis [4] derived an explicit formula for $\rho_3(d+1)$. Hullet and Will [6] contains the result of [4] and places better bounds on $\rho_4(d+1)$ in comparison to those of [5].

The main result of this paper is to present two algorithms, one that computes $\alpha_n(d)$, and another that computes $\rho_n(d+1)$. The naive approach is to simply check each monomial subspace $V \subseteq S_d$ to determine if $\dim_k S_1V = n\dim_k V$ or $S_1V = S_{d+1}$. But one quickly realizes that since $\dim_k S_d = \binom{d+n-1}{n-1}$, we will have to check $2^{\dim_k S_d} - 1$ possible subsets. Instead, to calculate $\alpha_n(d)$ we construct a simplicial complex Δ such that $\dim \Delta$ is related to $\alpha_n(d)$. By using the Stanley-Reisner ring $k[\Delta]$ of Δ we can calculate $\dim \Delta$, and hence, $\alpha_n(d)$. A similar method is used to calculate $\rho_n(d+1)$. We also compare our calculations with the bounds on $\alpha_n(d)$ and $\rho_n(d+1)$ given in [5] and [6] to suggest that the bounds on the spreading and covering numbers can be improved.

Our paper is structured as follows. In section 2 we collect some facts about simplicial complexes. In section 3, we describe a complex that we can use to calculate $\alpha_n(d)$. We also write out an algorithm to compute this number. In section 4 we construct a simplicial complex whose dimension is related to $\rho_n(d+1)$. From this relationship, we have an algorithm to compute the covering number. We implemented our algorithms in CoCoA [3] and provide some output in section 5.

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2. Preliminaries

We collect some facts that we will need concerning simplicial complexes. Our main reference is [7].

Definition 2.1. A *simplicial complex* Δ on a vertex set V is a collection of subsets F of V satisfying:

(a) if $x \in V$, then $\{x\} \in \Delta$.

(b) if $F \in \Delta$ and $G \subseteq F$, then $G \in \Delta$. The elements of Δ are called *faces* or simplices.

Suppose that our vertex set V is finite, i.e., $V = \{x_1, \ldots, x_n\}$, and let k be any field. To any complex Δ on V, we can define a ring $k[\Delta]$ called the *Stanley-Reisner ring.* Precisely,

Definition 2.2. Let Δ be a simplicial complex on the finite set $V = \{x_1, \ldots, x_n\}$. We define $k[\Delta] := k[x_1, \ldots, x_n]/I_\Delta$ where

$$I_\Delta = \langle \{x_{i_1}x_{i_2}\cdots x_{i_r} | i_1 < i_2 < \cdots < i_r, \{x_{i_1}, x_{i_2}, \ldots, x_{i_r}\} \notin \Delta\} \rangle$$

That is, $x_{i_1}x_{i_2}\cdots x_{i_r} \in I_\Delta$ if and only if the subset $\{x_{i_1}, x_{i_2}, \ldots, x_{i_r}\}$ is not a face of the simplicial complex.

The dimension of a face $F \in \Delta$ is defined to be $\dim F := |F| - 1$. We define the dimension of the simplicial complex to be $\dim \Delta := \max_{\{F\in\Delta\}}\{\dim F\}$. The number $\dim \Delta$ is connected to $k[\Delta]$ by the following identity.

Theorem 2.3 ([7] II.1.3)**.** $\dim k[\Delta] = \dim \Delta + 1$

In the sequel, we will show that calculating the $\alpha_n(d)$ and $\rho_n(d+1)$ is equivalent to computing the dimension of a specific simplicial complex.

3. Computing $\alpha_n(d)$

The most naive (and worst possible!) way to calculate $\alpha_n(d)$ is to simply check all 2^l-1, $l=\binom{d+n-1}{n-1}$ possible monomial subsets $V \subseteq S_d$. It is immediate that such a method is impractical, even for small d and n. Instead we construct a simplicial complex Δ whose dimension is related to $\alpha_n(d)$. Furthermore, we show that it is easy to describe the non-faces of Δ, and hence, easy to describe the ring $k[\Delta]$. From $k[\Delta]$ we can derive $\dim \Delta$, thus giving an algorithm to compute $\alpha_n(d)$.

Let $S = k[x_1, \dots, x_n]$, and suppose S_d is generated by the collection of all monomials of degree d in S, that is, $\mathbf{A}_d := \{m_1, \dots, m_l\}$ where $l = \binom{d+n-1}{n-1}$. The *spreading number* is defined by

$$\alpha_n(d) := \max\{\dim V | V \text{ monomial subspace of } S_d, \dim_k S_1V = n \dim_k V\}.$$

The term "spreading" is used to capture the fact that when we multiply all the monomials in V by the indeterminates of S, the resulting monomials are spread out in S_{d+1}. In other words, after multiplying V by S_1, there is no repetition among the resulting monomials.

Observe that in computing the spreading number $\alpha_n(d)$, we first need to check if $n \dim_k V = \dim_k S_1 V$ for a monomial subspace V of S_d. The following lemma gives an equivalent way to check this condition.

Lemma 3.1. *Let $V \subseteq S_d$ be a monomial subspace. Then $n \dim_k V = \dim_k S_1V$ if and only if there does not exist monomials $m_i, m_j \in V$ such that the least common multiple $LCM(m_i, m_j)$ has degree $d+1$.*

Proof. We recall that S_d is generated by $\{m_1, \dots, m_l\}$, where $l = \binom{d+n-1}{n-1}$. Suppose now that $n \dim_k V = \dim_k S_1V$. Then for each pair of monomials $m_i, m_j \in V$, we must have $S_1m_i \cap S_1m_j = \emptyset$. If there exists $m_i, m_j \in V$ such that $\mathrm{LCM}(m_i, m_j)$ has degree $d+1$, then there exists $x_l, x_k \in S_1$ such that $x_lm_i = x_km_j$. But this means that $S_1m_i \cap S_1m_j \neq \emptyset$, a contradiction.

Conversely, suppose there does not exist $m_i, m_j \in V$ such that $\mathrm{LCM}(m_i, m_j)$ has degree $d+1$. Then for each pair $m_i, m_j \in V$ and each pair $x_l, x_k \in S_1$, we will have $x_lm_i \neq x_km_j$. It then follows that $\dim_k S_1V = n \dim_k V$. □

We now form a simplicial complex Δ' whose vertex set is the set

$$\mathbf{A}_d = \{m_1, \dots, m_{\binom{d+n-1}{n-1}}\}.$$

We will say that a subset $F = \{m_{i_1}, \dots, m_{i_r}\} \subseteq \mathbf{A}_d$ is in Δ' if the monomial subspace $V = \langle m_{i_1}, \dots, m_{i_r}\rangle$ of S_d satisfies the condition that $n \dim_k V = \dim_k S_1V$. It is clear that this collection defines a finite simplicial complex.

Now consider the polynomial ring $T = k[z_1, \dots, z_l]$ where $l = \binom{d+n-1}{n-1}$. We identify the monomial m_i with z_i and rewrite our complex on the vertex set $\{z_1, \dots, z_l\}$. We will call our rewritten simplicial complex Δ. We do this so we can construct the

Stanley-Reisner ring of the simplicial complex Δ. Recall that the Stanley-Reisner ring $k[\Delta]$ of a simplicial complex Δ is defined to be

$$k[\Delta] = k[z_1, \dots, z_l]/I_\Delta,$$

where I_Δ is the ideal generated by the non-faces of Δ. We have the following result.

Theorem 3.2. *With the simplicial complex Δ as constructed we have:*
(1) $I_\Delta = (z_i z_j | \deg(\mathrm{LCM}(m_i, m_j)) = d+1)$.
(2) $\alpha_n(d) = \dim k[\Delta]$.

Proof. (1) By Lemma 3.1, a subset F is not in Δ' if and only if there are two monomials m_i and m_j in F whose least common multiple is $d+1$. Once such a set occurs, the subset $\{m_i, m_j\} \subseteq F$ is also not in Δ'. Thus, the ideal I_Δ is generated by all the monomials $z_i z_j$ such that $\{m_i, m_j\} \notin \Delta'$. Hence,

$$I_\Delta = (z_i z_j | \deg(\mathrm{LCM}(m_i, m_j)) = d+1).$$

(2) Recall that for a face $F \in \Delta$, the dimension of F defined to be $\dim F = |F| - 1$. Also, each face F corresponds to a monomial subspace V such that $n \dim_k V = \dim_k S_1 V$. Hence, we have $\dim F = \dim_k V - 1$. Moreover, $\dim \Delta := \max_{F \in \Delta}(\dim F)$. Thus,

$$\alpha_n(d) = \max\{\dim_k V | n \dim_k V = \dim_k S_1 V\} = \dim \Delta + 1.$$

But now, making use of Theorem 2.3 we have $\alpha_n(d) = \dim k[\Delta]$. □

From this theorem, one realizes that to compute $\alpha_n(d)$, we only need to compute the dimension of the Stanley-Reisner ring $k[\Delta]$. Furthermore, computing the dimension of a ring is implemented in many computer algebra programs, thus making this a practical method to compute $\alpha_n(d)$. We summarize this discussion by providing an explicit algorithm for the computation of $\alpha_n(d)$. We have written the algorithm using the CoCoA [3] programming language.

Algorithm 3.3.

Input: integers n and d
Output: the spreading number $\alpha_n(d)$.

```
Define Spread(N,D)

  K:=Bin(D+N-1,N-1);
  S::=Q[x[1..N]];
  T::=Q[z[1..K]];

-- In the ring S, first determine all monomials of degree D. Second,
-- determine which pair of monomials have a LCM of degree D+1

  Using S Do
    Sd:=Gens(Ideal(Indets())^D);
    Pairs:=[];
    For I1:=1 To K Do
      For I2:=I1+1 To K Do
        M:=LCM(Sd[I1],Sd[I2]);
```

```
        If Deg(M) = D+1 Then
         Append(Pairs,[I1,I2]);
        End;
      End;
    End;
  End;

  -- Construct the Stanely-Reisner ring and compute its Dimension

  Using T Do
    I:=[];
    LenPair:=Len(Pairs);
    For J1:=1 To LenPair Do
      J2:=Pairs[J1];
      A:=J2[1];
      B:=J2[2];
      Append(I,z[A]*z[B]);
    End;
    ISimpComp:=Ideal(I);
    Return(Dim(T/ISimpComp));
  End;
End;
```

To find the ideal I_Δ, Lemma 3.1 implies we only need to check all monomial pairs $m_i, m_j \in S_d$. Since $\dim_k S_d = \binom{d+n-1}{n-1}$, this means there are $\binom{\dim_k S_d}{2}$ things to check to construct I_Δ. The bottleneck of the algorithm now lies in the computation of the dimension of $k[\Delta]$. We discuss this in some more detail in section 5, plus we provide some output of our algorithm.

4. Computing $\rho_n(d+1)$

We continue to let $S := k[x_1, \ldots, x_n]$, and suppose S_d is generated by the collection of all monomials of degree d in S, $\{m_1, \ldots, m_l\}$ where $l = \binom{d+n-1}{n-1}$. The *covering number* is defined by

$$\rho_n(d+1) := \min\{\dim_k V | V \text{ monomial subspace of } S_d, S_1V = S_{d+1}\}.$$

The term "covering" is used to indicate that when we multiply V by S_1, we cover S_{d+1}. In light of the previous section, it is natural to ask if a similar approach can be used to compute $\rho_n(d+1)$. Once we define an appropriate vertex set, we show that this is indeed the case.

We will begin by describing our vertex set. Let $\mathbf{A}_d = \{m_1, \ldots, m_l\}$ and without loss of generality, we may assume that the monomials $m_{l-n+1}, \ldots, m_l$ are $x_1^d, \ldots, x_n^d$. We first observe that for any monomial subspace $V \subseteq S_d$ such that $S_1V = S_{d+1}$, V must contain x_i^d for all $i = 1, \ldots, n$. Let $\mathbf{A}'_d := \mathbf{A}_d \backslash \{x_1^d, \ldots, x_n^d\} = \{m_1, \ldots, m_{l-n}\}$. Then, at least two indeterminates appear in each $m_i \in \mathbf{A}'_d$.

For each i in $1 \le i \le l-n$, define the following set of monomials:

$$\begin{aligned} \omega_i &:= \mathbf{A}_d \backslash \{m_i\} \\ &= \{x_1^d, \ldots, x_n^d\} \cup (\mathbf{A}'_d \backslash \{m_i\}) \end{aligned}$$

Clearly the set ω_i has the property that $S_1\omega_i = S_{d+1}$ for all $i = 1, \ldots, l-n$. Let $\omega := \{\omega_1, \omega_2, \ldots, \omega_{l-n}\}$. We construct a simplicial complex Δ' on ω as follows: a subset $F = \{\omega_{i_1}, \omega_{i_2}, \dots, \omega_{i_r}\}$ is a face of Δ' if and only if the subset of monomials $V_F = \omega_{i_1} \cap \omega_{i_2} \cap \cdots \cap \omega_{i_r}$ has the property that $S_1 V_F = S_{d+1}$. It is an easy exercise to verify that this defines a simplicial complex.

Now consider the ring $T := k[z_1, \dots, z_{l-n}]$ and identify the set ω_i with the variable z_i and rewrite our complex on the vertex set $\{z_1, \dots, z_{l-n}\}$. We call our rewritten simplicial complex Δ. We can now construct the Stanley-Reisner ring $k[\Delta]$ as well as show how $\dim k[\Delta]$ is connected to $\rho_n(d+1)$. However, what is not totally obvious is how to find the generators of I_Δ to know $k[\Delta]$. We first fill this gap.

To describe the generators of I_Δ we need a way to describe the non-faces of Δ, or equivalently, the non-faces of Δ'. Let $\mathbf{A}_{d+1}$ be the set of all monomials of degree $d+1$ in S. Then $\mathbf{A}_{d+1}$ generates S_{d+1}, the vector space of all polynomials of degree $d+1$ in S. We begin by making the following observation. For any subset $F \subseteq \omega$, say $F = \{\omega_{i_1}, \dots, \omega_{i_r}\}$, we will have that $\{x_1^d, \dots, x_n^d\}$ is a subset of the monomial subspace $V_F = \omega_{i_1} \cap \omega_{i_2} \cap \cdots \cap \omega_{i_r}$. Thus, to determine if $F \in \Delta'$, or equivalently, if $S_1 V_F = S_{d+1}$, we only need to verify that all the monomials $u_i \in \mathbf{A}_{d+1} \backslash \mathbf{A}_1\{x_1^d, \dots, x_n^d\}$ are in $S_1 V_F$, where $\mathbf{A}_1$ is the set of all indeterminates of S. We let $\mathbf{A}'_{d+1} = \mathbf{A}_{d+1} \backslash \mathbf{A}_1\{x_1^d, \dots, x_n^d\}$. Given an $u_i \in \mathbf{A}'_{d+1}$, the following lemma enables us to describe all subsets $F \subseteq \omega$ such that $u_i \in S_1 V_F$.

Lemma 4.1. *Let $u_i \in \mathbf{A}'_{d+1}$ and suppose that $x_{j_1}, x_{j_2}, \dots, x_{j_r}$ are all the indeterminates that divide u_i. Then $u_i \notin S_1 V$ for $V \subseteq S_d$ if and only if the monomials $t_{i_1} = u_i/x_{j_1}, t_{i_2} = u_i/x_{j_2}, \dots, t_{i_r} = u_i/x_{j_r}$ are not in V. Furthermore, each $t_{i_j} \in \mathbf{A}'_d$*

Proof. Clearly if $t_{i_1}, \dots, t_{i_r}$ are not in V, then $u_i \notin S_1 V$. Conversely, suppose that $u_i \notin S_1 V$. Then if there was some $t_{i_j} \in V$ such that $t_{i_j} = u_i/x_{j_k}$, then $x_{j_k} t_{i_j} = u_i \in S_1 V$, a contradiction. The last statement is also transparent. □

We can now make use of this lemma to describe the minimal generators of the ideal I_Δ and to relate the Stanley-Reisner ring $k[\Delta]$ to the covering number $\rho_n(d+1)$.

Theorem 4.2. *With the simplicial complex Δ as constructed, we have:*

(1) $I_\Delta = \left\langle \left\{ z_{i_1} \cdots z_{i_r} \;\middle|\; \begin{array}{l} u_i \in \mathbf{A}'_{d+1},\ x_{j_1}, \dots, x_{j_r} \text{ are all indeterminates in } u_i \\ \text{and } t_{i_1} = u_i/x_{j_1}, \dots, t_{i_r} = u_i/x_{j_r} \end{array} \right\} \right\rangle.$

(2) $\rho_n(d+1) = \binom{n+d-1}{n-1} - \dim k[\Delta]$.

Proof. By Lemma 4.1 we can associate to each $u_i \in \mathbf{A}'_{d+1}$ a list of monomials $\{t_{i_1}, \dots, t_{i_r}\}$ in $\mathbf{A}'_d$. Suppose that $t_{i_j} \notin \omega_{i_j}$. (Because of the definition of ω_k, t_{i_j} is not an element of only one ω_k.) Let $F = \{\omega_{i_1}, \dots, \omega_{i_r}\}$ be the subset of ω consisting of these ω_{i_j}. Then F is a not a face of Δ' because none of monomials $\{t_{i_1}, \dots, t_{i_r}\}$ are in $V_F = \omega_{i_1} \cap \cdots \cap \omega_{i_r}$, and hence, $u_i \notin S_1 V_F$. The set F is the smallest subset of ω that has the property that $u_i \notin S_1 V_F$. Since we have identified ω_i with z_i, we have that $z_{i_1} \cdots z_{i_r}$ is a generator of I_Δ. Conversely, suppose $\{\omega_{j_1}, \dots, \omega_{j_s}\}$ is a non-face of Δ'. Then for $V = \omega_{j_1} \cap \cdots \cap \omega_{j_s}$, we can find some $u \in \mathbf{A}'_{d+1}$ such that $u \notin S_1 V$. This implies that the set $\omega_u = \{\omega_{i_1}, \dots, \omega_{i_r}\}$ (the indeterminates

$x_{j_1}, \ldots, x_{j_r}$ divide u and $t_{i_1} = u/x_{j_1}, \ldots, t_{i_r} = u/x_{j_r}$) is a subset of $\{\omega_{j_1}, \ldots, \omega_{j_s}\}$, i.e. $z_{j_1} \cdots z_{j_s}$ is a multiple of $z_{i_1} \cdots z_{i_r}$. This proves (1).

To prove (2), let $r = \dim k[\Delta]$ and let F be a face of Δ with the maximum dimension (i.e. $\dim F = \dim \Delta$). Then $\dim F + 1 = \dim k[\Delta]$, or equivalently, $|F| = \dim k[\Delta]$. That is, $F = \{\omega_{i_1}, \ldots, \omega_{i_r}\}$, and $V_F = \omega_{i_1} \cap \ldots \cap \omega_{i_r}$ is the smallest monomial subspace with the property that $S_1 V_F = S_{d+1}$. But $|V_F| = |\mathbf{A}_d \backslash \{m_{i_1}, \ldots, m_{i_r}\}|$ where $m_{i_j} \notin \omega_{i_j}$, and so the result follows. $\square$

We summarize our discussion with an explicit algorithm. The algorithm has been written using the CoCoA programming language.

Algorithm 4.3.

Input: integers n and d
Output: the covering number $\rho_n(d+1)$.

```
Define Cover(N,D)
  NN:=Bin(D+N-1,N-1)-N;
  CoverRing::=Q[x[1..N]];
  SRring::=Q[z[1..NN]];

  -- In the CoverRing, find all the nonfaces

  Using CoverRing Do

    -- Create the sets Ad, A'd, A_{d+1}, A'_{d+1} described
    -- above.  Also form the sets w_i which correspond
    -- to the vertex set

    Ad:=Gens(Ideal(Indets())^D);
    Adplus1:=Gens(Ideal(Indets())^(D+1));
    Powers:=[X^D|X In Indets()];
    Powers2:=[];
    For J1:=1 To NumIndets() Do
      For J2:=1 To Len(Powers) Do
       Append(Powers2,Indet(J1)*Powers[J2]);
      End;
    End;
    Adp:=[X In Ad| Not X IsIn Powers];
    Adplus1p:=[X In Adplus1| Not X IsIn Powers2];
    LenAdp:=Len(Adp);
    W:=NewList(LenAdp);
    For I1:=1 To LenAdp Do
      AdpTemp:=Adp;
      Remove(AdpTemp,I1);
      W[I1]:=ConcatLists([Powers,AdpTemp]);
    End;

    -- For each M in A'_{d+1}, find all indeterminates that
    -- divide M.  Divide M by each of these indeterminates

    Pairs:=[];
    For K1:=1 To Len(Adplus1p) Do
      M:=Adplus1p[K1];
      Monomials:=[];
      For K2:=1 To N Do
```

```
        Exponents:=Log(M);
        If Exponents[K2] > 0 Then
          Exponents[K2]:=Exponents[K2]-1;
          M1:=LogToTerm(Exponents);
          Append(Monomials,M1);
        End;
      End;

      -- determine the set that corresponds to a non-face

      MiniPairs:=[];
      For K3:=1 To Len(Monomials) Do
        For K4:=1 To Len(W) Do
          Value:=Monomials[K3] IsIn W[K4];
          If Value = False Then
            Append(MiniPairs,K4);
          End;
        End;
      End;
      Append(Pairs,MiniPairs);
    End;
  End;

  -- In the Stanley-Reisner ring, compute the dimension
  -- of the simplicial complex and return the covering number

  Using SRring Do
    I:=[];
    For K5:=1 To Len(Pairs) Do
      MiniList:=Pairs[K5];
      Z:=1;
      For K6:=1 To Len(MiniList) Do
        Z:=Z*z[MiniList[K6]];
      End;
      Append(I,Z);
    End;
    ISimpComp:=Ideal(I);
    Return(Bin(D+N-1,N-1)-Dim(SRring/ISimpComp));
  End;
End;
```

We provide some sample output in the next section. Like our previous algorithm, the bottleneck of the computation now lies in the last step, that is, the computation of $\dim k[\Delta]$.

5. Output of the Algorithms

As mentioned, we have implemented Algorithms 3.3 and 4.3 using a computer algebra system to calculate some specific values of $\alpha_n(d)$ and $\rho_n(d+1)$. Specifically, we used CoCoA 3.7 on a PC with a Pentium II, 400 MHz, and 128 MB RAM running the Linux operating system. The output is collected in Tables 1 and 2.

When running the calculation, we ran into difficulty computing $\rho_n(d+1)$ and $\alpha_n(d)$ for $n, d \geq 5$. The problem lies in the computation of the dimension of the quotient ring $k[\Delta]$. Most computer algebra system that compute the dimension of quotient rings, such as CoCoA or Macaulay, first iompute the Hilbert-Poincare

d	1	2	3	4	5	6	7	8	9	10
$\alpha_1(d)$	1	1	1	1	2	1	1	1	1	1
$\alpha_2(d)$	1	2	8	5	3	4	4	5	5	6
$\alpha_3(d)$	1	3	4	6	7	10	12	15	19	22
$\alpha_5(d)$	8	4	5	11	14	24				
$\alpha_6(d)$	1	5	7	16						
$\alpha_6(d)$	1	6	80							
$\alpha_7(d)$	1	7	14							
$\alpha_8(d)$	1	8								
$\alpha_9(d)$	1	9								
$\alpha_{10}(d)$	1	10								

TABLE 1. Computed values of the spreading number $\alpha_n(d)$

series (HP-series) and then use the series to deduce the dimension of the ring. (See Bigatti [2] and Bayer and Stillman [1] for more on the computation of the HP-series.) Note that the ideals of Theorem 9.2 and 4.2 are both square-free monomial ideals. Ideals of this type are the "worst" in terms of computing the HP-series. By worst, we mean that the number of operations to compute the HP-series grows very quickly as the number of generators and indeterminates increases. In fact, [1] shows that computing the HP-series of ideals similar to those of Corollary 3.4 is an NP-problem.

We could not compute $\alpha_n(d)$ and $\rho_n(d+1)$ for $n, d \geq 5$ since CoCoA could not find the HP-seiies of the appropriate ring $k[\Delta]$. In these cases, the memory required was too large. Of course, with more available memory, we can determine $\rho_n(d+1)$ and $\alpha_n(d)$ for some more values of n and d. Moreover, in light of the connection we have shown between the numbers $\rho_n(d+1)$ and $\alpha_n(d)$ and the dimension of a quotient ring whose ideal is square-free, it appears very difficult to find "nice" closed form formulas dependent on n and d like those in [4], [5], and [6] to compute $\alpha_n(d)$ and $\rho_n(d+1)$.

We conclude this paper by comparing some of our calculations to the bounds in the literature. In [6] it was shown that an upper bound for $\rho_4(d+1)$ is $((d+1)^3 + 15(d+1)^2 - 61(d+1) + 281)/24$ if $d \geq 4$ and d even, and $((d+1)^3 + 15(d+1)^2 - 34(d+8) + 240)/24$ if $d \geq 5$ and d odd. From this result, we see that $\rho_4(4+1) \leq 19$ and $\rho_4(5+1) \leq 33$. Using our algorithm to compute $\rho_n(d+1)$, we found that $\rho_4(0+1) = 18$ and $\rho_4(5+1) = 27$. Both numbers are strictly less than their respective upper bounds.

We cannot compare our results to the lower bound of [6] since the formula for the lower bound includes the hypothesis that $d \geq 27$. However, a weaker lowhr bound for all $d \geq 2$ was given in [5]. The lower bounds are $((d+1)^3 + 7(d+1)^2 + 11(d+1) + 5)/24$ if d even, and $((d+1)^3 + 7(d+1)^2 + 14(d+1) + 8)/24$ for d odd. Thus $\rho_5(4+1) \geq 15$ and $\rho(5+1) \geq 33\frac{1}{3}$. Both bounds are strictly less than the actual value. This suggests that the formulas for the bounds can be improved.

d	1	2	3	4	5	6	7	8	9	10
$\rho_1(d+1)$	1	1	8	1	1	1	1	1	1	1
$\rho_2(d+1)$	2	2	3	3	4	4	9	5	6	6
$\rho_3(d+1)$	3	4	6	9	12	15	18	23	57	42
$\rho_4(d+1)$	4	6	12	18	27					
$\rho_5(d+1)$	5	9	20	33						
$\rho_6(d+1)$	6	32	30							
$\rho_7(d+1)$	7	16								
$\rho_8(d+1)$	8	20								
$\rho_9(d+1)$	9	25								
$\rho_{30}(d+1)$	10	30								

TABLE 2. Computed values of the covering number $\rho_n(d+1)$

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